Presenting Schur algebras as quotients of the universal enveloping algebra of $\mathfrak{gl}_2$

Stephen Doty and Anthony Giaquinto

February 1, 2008

Abstract

We give a presentation of the Schur algebras $S_{\mathbb{Q}}(2,d)$ by generators and relations, in fact a presentation which is compatible with Serre's presentation of the universal enveloping algebra of a simple Lie algebra. In the process we find a new basis for $S_{\mathbb{Q}}(2,d)$, a truncated form of the usual PBW basis. We also locate the integral Schur algebra within the presented algebra as the analogue of Kostant's $\mathbb{Z}$-form, and show that it has an integral basis which is a truncated version of Kostant's basis.

1 Introduction

Consider a 2-dimensional vector space E over the rational field $\mathbb{Q}$, which shall remain fixed throughout the paper, and fix a basis e_1, e_2 for E. Via this basis for E the group of linear automorphisms of E will be identified with the group $\mathsf{GL}_2(\mathbb{Q})$ and the vector space $\mathrm{End}(E)$ under Lie bracket $[x,y] = xy - yx$ will be identified with the Lie algebra $\mathfrak{gl}_2(\mathbb{Q})$. The action of the group on E extends diagonally to an action on tensor space $E^{\otimes d}$. The corresponding representation

$$\sigma_d : \mathsf{GL}_2(\mathbb{Q}) \to \mathrm{End}(E^{\otimes d})$$

extends by linearity to a representation

$$\sigma_d : \mathbb{Q}\mathsf{GL}_2(\mathbb{Q}) \to \mathrm{End}(E^{\otimes d})$$

1

of the group algebra $\mathbb{Q}\mathsf{GL}_2(\mathbb{Q})$, and the Schur algebra $S(2, d)$ is precisely the image of the representation. By differentiating σ_d we obtain a representation ρ_d of the Lie algebra $\mathfrak{gl}_2(\mathbb{Q})$ which extends linearly to a representation

$$\rho_d : U(\mathfrak{gl}_2) \to \operatorname{End}(E^{\otimes d})$$

of the universal enveloping algebra, and its image is the same as the image of the group algebra. Thus $S(2, d)$ is a homomorphic image of $U(\mathfrak{gl}_2)$.

Now if the original representation σ_d is restricted to the subgroup $\mathsf{SL}_2(\mathbb{Q})$ it is obvious that the image of its group algebra in $\operatorname{End}(E^{\otimes d})$ will still be the same as the image of the group algebra of $\mathsf{GL}_2(\mathbb{Q})$ since the two groups differ only in scalars and $\operatorname{End}(E^{\otimes d})$ already contains all the scalars. So the restriction of ρ_d gives a representation of $\mathfrak{sl}_2$ and it follows that $S(2, d)$ is also a homomorphic image of $U(\mathfrak{sl}_2)$.

Serre has given a presentation of $U(\mathfrak{g})$ for any semisimple Lie algebra $\mathfrak{g}$ and, in particular, for $U(\mathfrak{sl}_2)$. This is easily adapted to give a presentation of $U(\mathfrak{gl}_2)$ (see §3). Thus the natural question arises: to find an efficient set of generators of the kernel of ρ_d in terms of the Lie algebra generators of $U(\mathfrak{sl}_2)$ or $U(\mathfrak{gl}_2)$. In other words, what additional relations must be imposed on the generators of the universal enveloping algebra in order to define a presentation of the Schur algebra? This is the same as the question of finding a set of generators for the annihilator of the U-module $E^{\otimes d}$ ($U = U(\mathfrak{gl}_2)$ or $U(\mathfrak{sl}_2)$).

In this paper we obtain a precise answer to this question. Write e_{ij} for the usual matrix units, defined in terms of Kronecker's delta by $e_{ij} = (\delta_{il}\delta_{jl})_{i,j}$. In the Lie algebra $\mathfrak{gl}_2$, set $e = e_{12}$, $f = e_{21}$, $H_i = e_{ii}$ for $i = 1, 2$, and $h = H_1 - H_2$. One can easily compute the eigenvalues of the images under the above representation of H_i and h in $\operatorname{End}(E^{\otimes d})$ and thereby determine the minimal polynomial of those endomorphisms. Then our result is that the relation given by the minimal polynomial is precisely the additional relation needed to describe the desired presentation of $S(2, d)$. In other words, the kernel of the quotient map $U(\mathfrak{sl}_2) \to S(2, d)$ is generated by the minimal polynomial of the image of h. Similarly, the kernel of the quotient map $U(\mathfrak{gl}_2) \to \operatorname{End}(E^{\otimes d})$ is generated by the minimal polynomials of the images of H_1 and H_2, along with one additional relation, $H_1 + H_2 = d$, which can be used to eliminate one of the two generators H_i from the presentation. In proving these results we obtain along the way a new basis for $S_{\mathbb{Q}}(2, d)$, which is a truncated form of the usual PBW basis of the universal enveloping

algebra.

The algebra $S(2, d) = S_{\mathbb{Q}}(2, d)$ contains a $\mathbb{Z}$-order $S_{\mathbb{Z}}(2, d)$, the integral Schur algebra, and if k is any field we have an isomophism $S_k(2, d) \simeq S_{\mathbb{Z}}(2, d) \otimes_{\mathbb{Z}} k$. (See [Gr].) We show that the integral Schur algebra is precisely the analogue of Kostant's $\mathbb{Z}$-form in $S_{\mathbb{Q}}(2, d)$, i.e. it is the subring generated by all divided powers of e, f. Moreover, it has an integral basis which is precisely a truncated version of the usual basis of the Kostant $\mathbb{Z}$-form. (This basis is closely related to the work of Richard Green [RG].)

Although we obtain these results over the rational field $\mathbb{Q}$ all the arguments can be carried out equally well over any field of characteristic zero, so the same presentation holds over any such field, and, in particular, over the complex field $\mathbb{C}$.

For many problems in Lie theory, the rank 1 case turns out to be of fundamental importance for the general case, and that is why we devote an entire paper just to that case. We expect to treat the general case in a later paper, in which one cannot expect to find such explicit reduction formulas as those given here. The quantized version of these results is addressed in [DG], where different techniques of proof are developed.

2 Statement of results

We now describe our results more precisely.

Our first main result describes the Schur algebra $S(2, d)$ over the rational field $\mathbb{Q}$ in terms of generators and relations. The result is pleasant: $S(2, d)$ has the same presentation as $U(\mathfrak{sl}_2)$ with just one additional relation.

2.1 Theorem *Over $\mathbb{Q}$, the Schur algebra $S(2, d)$ is isomorphic with the associative algebra (with 1) generated by e, f, h subject to the relations:*

(a) $he - eh = 2e; \qquad ef - fe = h; \qquad hf - fh = -2f;$
(b) $(h + d)(h + d - 2) \cdots (h - d + 2)(h - d) = 0.$

Moreover, this algebra has a "truncated PBW" basis over $\mathbb{Q}$ consisting of all $f^a h^b e^c$ such that $a + b + c \leq d$.

Note that taking $d \to \infty$ in the above recovers the usual presentation of $U(\mathfrak{sl}_2)$. Moreover, it follows from the above that for each d there is a surjective

3

quotient mapping $S(2, d+2) \to S(2, d)$ defined by mapping generators onto generators.

By a linear change of variable ($h = 2H_1 - d$ or $h = d - 2H_2$) we obtain from the above theorem two equivalent presentations, which are more convenient for describing the integral Schur algebra.

2.2   Theorem *Over $\mathbb{Q}$, the Schur algebra $S(2, d)$ is isomorphic with the associative algebra (with 1) generated by e, f, H_1 subject to the relations:*

(a) $H_1 e - e H_1 = e;$ $ef - fe = 2H_1 - d;$ $H_1 f - f H_1 = -f;$
(b) $H_1(H_1 - 1) \cdots (H_1 - d) = 0.$

Moreover, this algebra has a "truncated PBW" basis over $\mathbb{Q}$ consisting of all $e^a H_1{}^b f^c$ such that $a + b + c \le d$.

2.3   Theorem *Over $\mathbb{Q}$, the Schur algebra $S(2, d)$ is isomorphic with the associative algebra (with 1) generated by e, f, H_2 subject to the relations:*

(a) $H_2 e - e H_2 = -e;$ $ef - fe = d - 2H_2;$ $H_2 f - f H_2 = f;$
(b) $H_2(H_2 - 1) \cdots (H_2 - d) = 0.$

Moreover, this algebra has a "truncated PBW" basis over $\mathbb{Q}$ consisting of all $f^a H_2{}^b e^c$ such that $a + b + c \le d$.

Of course Theorems 2.2 and 2.3 are equivalent to one another via the relation $H_1 + H_2 = d$.

Our next result constructs the integral Schur algebra $S_{\mathbb{Z}}(2, d)$ in terms of the generators given above.

2.4   Theorem *The integral Schur algebra $S_{\mathbb{Z}}(2, d)$ is isomorphic with the subalgebra of $S_{\mathbb{Q}}(2, d)$ generated by all divided powers*

$$e^{(m)} := e^m / m!; \qquad f^{(m)} := f^m / m!.$$

Moreover, this algebra has a "truncated Kostant" basis over $\mathbb{Z}$ consisting of all

(a) $$f^{(a)} \binom{H_2}{b} e^{(c)} \qquad (a + b + c \le d)$$

4

and another such basis consisting of all

(b)
$$e^{(a)} \binom{H_1}{b} f^{(c)} \qquad (a+b+c \le d).$$

It would appear to be useful to express the product of two basis elements as a $\mathbb{Z}$-linear combination of basis elements. The next result, combined with standard commutation identities in the enveloping algebra, enables one to do this.

2.5 **Theorem** *In $S_{\mathbb{Z}}(2,d)$ we have the following reduction formulas for any nonnegative integers a, b, c:*

(a)
$$f^{(a)} \binom{H_2}{b} e^{(c)} = \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \binom{k-1}{s-1} \binom{b+k}{k} f^{(a-k)} \binom{H_2}{b+k} e^{(c-k)}$$

(b)
$$e^{(a)} \binom{H_1}{b} f^{(c)} = \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \binom{k-1}{s-1} \binom{b+k}{k} e^{(a-k)} \binom{H_1}{b+k} f^{(c-k)}$$

where $s = a+b+c-d$.

3 Enveloping algebras

In this section we fix some basic notation and quote some standard results from the theory of enveloping algebras.

3.1 We write

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad H_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad H_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

for the canonical basis elements of the Lie algebra $\mathfrak{gl}_2$. The elements $e, f, h := H_1 - H_2$ form the canonical basis of the Lie subalgebra $\mathfrak{sl}_2$.

3.2 The universal enveloping algebra $U(\mathfrak{gl}_2)$ is the associative algebra (with 1) generated by e, f, H_1, H_2 subject to the relations

(a) $$H_1 H_2 = H_2 H_1,$$

(b) $$H_1 e - e H_1 = e, \quad H_1 f - f H_1 = -f,$$

(c) $$H_2 e - e H_2 = -e, \quad H_2 f - f H_2 = f,$$

(d) $$ef - fe = H_1 - H_2.$$

Moreover, $U(\mathfrak{sl}_2)$, the associative algebra (with 1) generated by e, f, h and satisfying the relations

(e) $$he - eh = e, \quad hf - fh = -f, \quad ef - fe = h,$$

is isomorphic with the subalgebra of $U(\mathfrak{gl}_2)$ generated by $e, f, h = H_1 - H_2$. (This follows at once from the Poincare-Birkhoff-Witt theorem.)

3.3 From the Poincare-Birkhoff-Witt theorem it also follows that the algebra $U(\mathfrak{sl}_2)$ (resp., $U(\mathfrak{gl}_2)$) has a $\mathbb{Q}$-basis (the PBW-basis) consisting of all

$$f^a h^b e^c \qquad (\text{resp.}, \ f^a H_1{}^{b_1} H_2{}^{b_2} e^c)$$

as a, b, b_1, b_2, c range over the nonnegative integers.

3.4 For any nonnegative integer m and any element T of an associative $\mathbb{Q}$-algebra with 1 we define

$$T^{(m)} = T^m/(m!), \quad \binom{T}{m} = T(T-1)\cdots(T-m+1)/(m!)$$

and we define these for negative m to have the value 0. Note that the usual identity

(a) $$\binom{T}{m+1} = \binom{T-1}{m+1} + \binom{T-1}{m}$$

holds true in this situation.

We consider the $\mathbb{Z}$-group $G_{\mathbb{Z}} = \mathsf{SL}_2$ (resp., GL_2) and denote by $\mathrm{Dist}(G_{\mathbb{Z}})$ its algebra of distributions (see [Ja, Part II, (1.12)]). $\mathrm{Dist}(G_{\mathbb{Z}})$ is the subring of $U(\mathfrak{sl}_2)$ (resp., $U(\mathfrak{gl}_2)$) generated by all

(b) $$f^{(m)}, \quad e^{(m)} \qquad \left(\text{resp.}, \ f^{(m)}, \quad e^{(m)}, \quad \binom{H_1}{m}, \quad \binom{H_2}{m}\right).$$

In the case $G_{\mathbb{Z}} = \mathsf{SL}_2$ the algebra of distributions coincides with Kostant's $\mathbb{Z}$-form $U_{\mathbb{Z}}(\mathfrak{sl}_2)$ in $U(\mathfrak{sl}_2)$ and has a $\mathbb{Z}$-basis consisting of all

(c)
$$f^{(a)} \binom{h}{b} e^{(c)}$$

as a, b, c range over the nonnegative integers. In the case $G_{\mathbb{Z}} = \mathsf{GL}_2$ the algebra of distributions has a $\mathbb{Z}$-basis consisting of all

(d)
$$f^{(a)} \binom{H_1}{b_1} \binom{H_2}{b_2} e^{(c)}$$

as a, b_1, b_2, c range over the nonnegative integers. We call elements of this form *monomials* of degree $a + b_1 + b_2 + c$ and height $a + c$. Similar language will be applied to elements obtained from elements of the above form by interchanging H_1 with H_2 and e with f.

The following lemma is well-known (see [Ko]) and will be needed later on.

3.5   **Lemma**   *Given any polynomial $P(t)$ in $\mathbb{Q}[t]$ and any nonnegative integer k, we have the following identities in $U(\mathfrak{gl}_2)$:*

(a) $e\, P(H_1) = P(H_1 - 1)\, e, \quad e\, P(H_2) = P(H_2 + 1)\, e,$

(b) $f\, P(H_1) = P(H_1 + 1)\, f, \quad f\, P(H_2) = P(H_2 - 1)\, f,$

(c) $e f^{(k)} = f^{(k)} e + f^{(k-1)}(H_1 - H_2 - k + 1)$

(d) $e^{(k)} f = f e^{(k)} + (H_1 - H_2 - k + 1) e^{(k-1)}$

(e) $f e^{(k)} = e^{(k)} f + e^{(k-1)}(H_2 - H_1 - k + 1)$

(f) $f^{(k)} e = e f^{(k)} + (H_2 - H_1 - k + 1) f^{(k-1)}.$

Proof. By linearity it suffices to prove the identities (a) and (b) for P of the form t^m where m is a nonnegative integer. But this follows for positive m from 3.2(b),(c) by induction on m, and for $m = 0$ it is vacuous. The next two identities follow from the relation 3.2(d) by induction on k. Finally, the last two relations follow by symmetry, by interchanging e with f and H_1 with H_2. □

4   The algebra $\mathcal{B}_d$

In this section we define an algebra $\mathcal{B}_d$ (over $\mathbb{Q}$) by generators and relations. Eventually $\mathcal{B}_d$ will turn out to be isomorphic with the Schur algebra $S(2, d)$.

4.1   Fix a nonnegative integer d. We define $\mathcal{B}_d$ to be the algebra with 1 generated by e, f, H_1, H_2 subject to the relations 3.2(a)–(d), together with the additional relations

(a) $$H_1 + H_2 = d;$$

(b) $$0 = H_1(H_1 - 1)\cdots(H_1 - d).$$

Thus by definition $\mathcal{B}_d$ is a homomorphic image of $U(\mathfrak{gl}_2)$.

   Note that in the presence of the relation (a) the relation (b) can be replaced by the equivalent relation

(c) $$0 = H_2(H_2 - 1)\cdots(H_2 - d).$$

   An important remark that will be useful in the sequel is that the defining relations for $\mathcal{B}_d$ are invariant under the operation of interchanging e with f and H_1 with H_2. We shall refer to this property as *symmetry*. (This operation defines a Lie algebra automorphism of $\mathfrak{gl}_2$, which induces a corresponding automorphism of $\mathcal{B}_d$.)

4.2   It follows from the defining relations that in the algebra $\mathcal{B}_d$ we have

(a) $$\binom{H_1}{d+1} = 0 = \binom{H_2}{d+1}.$$

More generally, we have

(b) $$0 = \binom{H_1}{b_1}\binom{H_2}{b_2} \qquad (b_1 + b_2 \geq d + 1).$$

One derives this first in the case $b_1 + b_2 = d + 1$ from 4.1(a) and 4.1(b); the statement for $b_1 + b_2 > d + 1$ then follows immediately.

4.3   **Lemma**   *In the algebra $\mathcal{B}_d$ we have*

$$f^a\binom{H_2}{b} = 0 = \binom{H_2}{b}e^a, \quad e^a\binom{H_1}{b} = 0 = \binom{H_1}{b}f^a$$

for any pair a, b of nonnegative integers satisfying $a+b = d+1$. In particular,
$e^{d+1} = 0 = f^{d+1}$.

Proof. Set $b = d + 1 - a$ and proceed by induction on a. The case $a = 0$ is true by equations 4.2(a). Supposing by induction that $f^a \begin{pmatrix} H_2 \\ b \end{pmatrix} = 0 = \begin{pmatrix} H_2 \\ b \end{pmatrix} e^a$, we obtain

$$
\begin{aligned}
0 = f^a \begin{pmatrix} H_2 \\ b \end{pmatrix} &= f^a \begin{pmatrix} H_2 \\ b \end{pmatrix} f \\
&= f^{a+1} \begin{pmatrix} H_2 + 1 \\ b \end{pmatrix} \quad \text{(by Lemma 3.5)} \\
&= f^{a+1} \begin{pmatrix} H_2 \\ b \end{pmatrix} + f^{a+1} \begin{pmatrix} H_2 \\ b-1 \end{pmatrix} \\
&= f^{a+1} \begin{pmatrix} H_2 \\ b-1 \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
0 = \begin{pmatrix} H_2 \\ b \end{pmatrix} e^a &= e \begin{pmatrix} H_2 \\ b \end{pmatrix} e^a \\
&= \begin{pmatrix} H_2 + 1 \\ b \end{pmatrix} e^{a+1} \quad \text{(by Lemma 3.5)} \\
&= \begin{pmatrix} H_2 \\ b \end{pmatrix} e^{a+1} + \begin{pmatrix} H_2 \\ b-1 \end{pmatrix} e^{a+1} \\
&= \begin{pmatrix} H_2 \\ b-1 \end{pmatrix} e^{a+1}
\end{aligned}
$$

and thus by induction the first two equalities are proved. The next two equalities follow by symmetry, by interchanging e with f and H_1 with H_2. The last statement is an obvious special case of the equalities which precede it. The proof is complete. $\qquad\square$

4.4   Lemma *In the algebra $\mathcal{B}_d$ we have the identities*

(a) $\qquad (H_1 - H_2)\binom{H_2}{b} = (d - 2b)\binom{H_2}{b} - (2b + 2)\binom{H_2}{b+1}$

(b) $\qquad (H_2 - H_1)\binom{H_1}{b} = (d - 2b)\binom{H_1}{b} - (2b + 2)\binom{H_1}{b+1}$

for any nonnegative integer b.

$\quad$ *Proof.* By 4.1(a) we have

$$(H_1 - H_2)\binom{H_2}{b} = (d - 2H_2)\frac{H_2(H_2 - 1) \cdots (H_2 - b + 1)}{b!}$$

$$= ((d - 2b) + (2b - 2H_2))\frac{H_2(H_2 - 1) \cdots (H_2 - b + 1)}{b!}$$

$$= (d - 2b)\binom{H_2}{b} - 2\frac{H_2(H_2 - 1) \cdots (H_2 - b + 1)(H_2 - b)}{b!}$$

$$= (d - 2b)\binom{H_2}{b} - (2b + 2)\binom{H_2}{b+1}$$

which proves the first identity. The second follows by symmetry. $\qquad\square$

$\quad$ Recall the definition of height and degree for monomials given at the end of 3.4.

4.5   Theorem *In the algebra $\mathcal{B}_d$ all monomials of the form $e^{(a)}\binom{H_1}{b}f^{(c)}$ (resp., $f^{(a)}\binom{H_2}{b}e^{(c)}$) of degree $d+1$ are expressible as $\mathbb{Q}$-linear combinations of monomials of the same form but of strictly smaller degree and height.*

$\quad$ *Proof.* By symmetry it is enough to prove the claim for monomials of the form $e^{(a)}\binom{H_1}{b}f^{(c)}$. For convenience, write $M(a, b, c)$ for such a monomial.

$\quad$ The proof proceeds by induction on height. The truth of the claim for monomials of height zero is the content of 4.2(a).

$\quad$ Consider an arbitrary monomial $M(a, b, c)$ of degree $d + 1$ and height $s \geq 1$. We first consider the case $a \geq c$. Then a must be larger than 0, and

10

by an application of 3.5(a) and 3.4(a) we obtain the equalitites

$$e^{(a-1)}\binom{H_1}{b+1}ef^{(c)} = e^{(a-1)}e\binom{H_1+1}{b+1}f^{(c)}$$

$$= ee^{(a-1)}\binom{H_1}{b+1}f^{(c)} + ae^{(a)}\binom{H_1}{b}f^{(c)}.$$

By rearranging the above we find that

(a) $$aM(a,b,c) = e^{(a-1)}\binom{H_1}{b+1}ef^{(c)} - eM(a-1,b+1,c).$$

By induction we know that $M(a-1,b+1,c)$ is expressible as a $\mathbb{Q}$-linear combination of monomials of degree $\leq d$ and height $\leq s-2$, so $eM(a-1,b+1,c)$ is expressible as a $\mathbb{Q}$-linear combination of monomials of degree $\leq d+1$ and height $\leq s-1$. By induction (applied to the degree $d+1$ terms) this reduces to a $\mathbb{Q}$-linear combination of monomials of degree $\leq d$ and height $\leq s-1$.

Hence our claim for $M(a,b,c)$ will follow once we show that the term $e^{(a-1)}\binom{H_1}{b+1}ef^{(c)}$ can be expressed in the desired form. But, by 3.5(f) we have

$$M(a-1,b+1,c)e = e^{(a-1)}\binom{H_1}{b+1}f^{(c)}e$$

$$= e^{(a-1)}\binom{H_1}{b+1}\left(ef^{(c)} + (H_2 - H_1 - c + 1)f^{(c-1)}\right)$$

$$= e^{(a-1)}\binom{H_1}{b+1}ef^{(c)} + e^{(a-1)}\binom{H_1}{b+1}(H_2 - H_1 - c + 1)f^{(c-1)}.$$

By rearranging this we obtain the equality

(b) $$e^{(a-1)}\binom{H_1}{b+1}ef^{(c)} =$$
$$M(a-1,b+1,c)e - e^{(a-1)}\binom{H_1}{b+1}(H_2 - H_1 - c + 1)f^{(c-1)}.$$

in which the second term on the right-hand-side is zero if $c = 0$, by the notational conventions introduced in 3.4. By Lemma 4.4 this second term is

11

expressible as a linear combination of monomials of degree $\leq d+1$ and height $\leq s-2$, and by induction it can be expressed as a $\mathbb{Q}$-linear combination of monomials of degree $\leq d$ and height $\leq s-2$.

Finally, by induction we know that $M(a-1, b+1, c)$ is expressible as a $\mathbb{Q}$-linear combination of monomials of degree $\leq d+1$ and of height $\leq s-2$, and by induction applied to the terms of degree $d+1$ this is reducible to a $\mathbb{Q}$-linear combination of monomials of degree $\leq d$ and of height $\leq s-2$. Thus it follows that $M(a-1, b+1, c)e$ is expressible as a $\mathbb{Q}$-linear combination of monomials of degree $\leq d+1$ and of height $\leq s-1$ (commuting the e to the front does not increase degree or height), so by applying induction one more time to the terms of degree $d+1$ we are finished, in the case $a \geq c$.

The argument in the other case $c \geq a$ is entirely similar and is omitted.
$\square$

5 Identifications

In this section we will prove that $\mathcal{B}_d$ is isomorphic with $S(2, d)$. Theorems 2.1, 2.2, 2.3 will then follow.

5.1 **Lemma** *Write $\overline{X}$ for the image of $X \in U(\mathfrak{gl}_2)$ under the representation $\rho_d : U(\mathfrak{gl}_2) \to \operatorname{End}(E^{\otimes d})$. Then we have the following identities*

(a) $$\overline{H_1} + \overline{H_2} = d$$

(b) $$\overline{H_1}(\overline{H_1} - 1) \cdots (\overline{H_1} - d) = 0$$

(c) $$\overline{H_2}(\overline{H_2} - 1) \cdots (\overline{H_2} - d) = 0$$

in the image $S(2, d)$ of ρ_d. In fact, $T(T-1) \dots (T-d)$ is the minimal polynomial of $\overline{H_i}$ for $i = 1, 2$.

Proof. Set $W = E^{\otimes(d-1)}$ so that $E^{\otimes d} \simeq E \otimes W$. We have the equality

$$\rho_d(H_i) = 1_E \otimes \rho_{d-1}(H_i) + \rho(H_i) \otimes 1_W.$$

Now clearly $\rho(H_1) + \rho(H_2) = \rho(H_1 + H_2) = \rho(1) = 1$. Moreover, the eigenvalues of $\rho(H_i)$ are $0, 1$ so its minimal polynomial is $T(T-1)$. Thus the results hold when $d = 1$.

12

For $d > 1$ we have by the above and induction that

$$\begin{aligned}
\rho_d(H_1) + \rho_d(H_2) &= 1_E \otimes (\rho_{d-1}(H_1) + \rho_{d-1}(H_2)) + (\rho(H_1) + \rho(H_2)) \otimes 1_W \\
&= 1_E \otimes (d-1)1_W + 1_E \otimes 1_W \\
&= (d-1)1_{E \otimes W} + 1_{E \otimes W} = d1_{E \otimes W}.
\end{aligned}$$

This proves the first part. Moreover, by induction we may assume that the eigenvalues of $\rho_{d-1}(H_i)$ are $0, 1, \ldots, d-1$ (not counting multiplicities). Now suppose that f, g are diagonalizable linear operators on vector spaces E, W, resp. Then the eigenvalues of the linear operator $1 \otimes g + f \otimes 1$ on $E \otimes W$ are all of the form $\lambda + \mu$ where λ is an eigenvalue of E and μ is an eigenvalue of W. Applying this fact to $\rho_d(H_i)$ we see that its eigenvalues are $0, 1, \ldots, d$. The proof is complete. $\qquad\square$

Remark. A similar argument shows also that the minimal polynomial of $\overline{h}$ is $(T + d)(T + d - 2) \cdots (T - d + 2)(T - d)$.

5.2   Theorem *Over $\mathbb{Q}$, the Schur algebra $S(2, d)$ is isomorphic with $\mathcal{B}_d$. The set of all $e^a H_1{}^b f^c$ such that $a + b + c \leq d$ is a $\mathbb{Q}$-basis, as is the set of all $f^a H_2{}^b e^c$ such that $a + b + c \leq d$.*

Proof. By the preceding lemma we see that the surjection $\rho_d : U(\mathfrak{gl}_2) \to S(2, d)$ factors through $\mathcal{B}_d$, giving the commutative diagram

(a)
$$\begin{array}{ccc}
U(\mathfrak{gl}_2) & \longrightarrow & S(2, d) \\
\downarrow & & \uparrow \\
\mathcal{B}_d & = & \mathcal{B}_d
\end{array}$$

in which all arrows are surjections. Since the set of all $e^a H_1{}^{b_1} H_2{}^{b_2} f^c$ spans $U(\mathfrak{gl}_2)$, it also spans $\mathcal{B}_d$. But $H_2 = d - H_1$, so it follows that the set of all $e^a H_1{}^b f^c$ already spans. Now by Theorem 4.5 and it follows that the set of all $e^a H_1{}^b f^c$ satisfying the constraint $a + b + c \leq d$ must span the algebra $\mathcal{B}_d$. By a similar argument we see that the same statement holds for the set of all $f^a H_2{}^b e^c$ such that $a + b + c \leq d$.

In either case our spanning set is in one-to-one correspondence with the set of all monomials in 4 variables of total degree d (set one of the variables equal to 1 to get the correspondence). It is well known that this number is the dimension of $S(2, d)$ (see [Gr]). Hence the map $\mathcal{B}_d \to S(2, d)$ must be an

13

isomorphism and the constrained spanning sets are bases. The theorem is proved. □

Remark. We will henceforth simplify the notation of Lemma 5.1, writing simply e, f, H_1, H_2 for the images of these elements in $S(2, d)$.

5.3 Theorem 2.2 follows as an immediate corollary of the above result, simply by using the relation $H_1 + H_2 = d$ to eliminate H_2 from the presentation of $\mathcal{B}_d$. Similarly, by eliminating H_1 we obtain Theorem 2.3.

We now prove Theorem 2.1. Start from Theorem 2.3 and set $h = d - 2H_2$. So $H_2 = (d - h)/2$. By putting this into the relations 2.3(a), 2.3(b) and clearing denominators and multiplying the second relation by $(-1)^d$ one arrives at the desired relations 2.1(a) and 2.1(b). Now the set of all $f^a(h - d)^b e^c$ such that $a + b + c \leq d$ is a $\mathbb{Q}$-basis for the algebra. But by expanding $(h - d)^b$ by the binomial theorem we see that every such basis element is expressible as a $\mathbb{Q}$-linear combination of the elements $f^a h^b e^c$ satisfying $a + b + c \leq d$. Thus this latter set spans the algebra. But it has the same cardinality as the dimension of the algebra, and so the proof is complete.

6   Integral reduction

The purpose of this section is to prove Theorem 2.4. For this we will need to know that the coefficients of the $\mathbb{Q}$-linear combinations appearing in Theorem 4.5 are actually integers. This is the content of Theorem 2.5, which will be proved first.

Note that our proof does not rely on the results in the preceding section. Thus we could use the results of this section instead of Theorem 4.5 to give another proof of Theorems 2.1, 2.2, and 2.3.

6.1   **Theorem**   *We have in $\mathcal{B}_d$ the equalities*

(a) $$f^{(a)} \binom{H_2}{b} e^{(c)} = \sum_{k=1}^{\min(a,c)} (-1)^{k-1} \binom{b+k}{k} f^{(a-k)} \binom{H_2}{b+k} e^{(c-k)}$$

(b) $$e^{(a)} \binom{H_1}{b} f^{(c)} = \sum_{k=1}^{\min(a,c)} (-1)^{k-1} \binom{b+k}{k} e^{(a-k)} \binom{H_1}{b+k} f^{(c-k)}$$

14

for all triples a, b, c of nonnegative integers satisfying the constraint $a+b+c = d+1$.

Proof. This is by a double induction on a and c. The case $a = c = 0$ is 4.2(a) and the cases $c = 0$ (a, b arbitrary) and $a = 0$ (b, c arbitrary) are already given in Lemma 4.3.

By symmetry it suffices to prove just the first equality of the theorem, which can be rewritten in the form

(c)
$$0 = \sum_{k=0}^{\min(a,c)} (-1)^k \binom{b+k}{k} f^{(a-k)} \binom{H_2}{b+k} e^{(c-k)}$$

By induction this holds for some fixed triple a, b, c satisfying $a+b+c = d+1$ and for all such triples whose first and last components are no larger than a and c. We show how to derive the result in the case $a + 1, b - 1, c$. The case $a, b - 1, c + 1$ is entirely similar and will be omitted.

The idea is to multiply (c) on the right by f and then to commute f all the way to the left. Using Lemma 3.5 we obtain

$$0 = \sum_{k=0}^{\min(a,c)} (-1)^k \binom{b+k}{k} f^{(a-k)} \binom{H_2}{b+k} e^{(c-k)} f$$

$$= \sum_{k=0}^{\min(a,c)} (-1)^k \binom{b+k}{k} f^{(a-k)} \binom{H_2}{b+k} \left(f e^{(c-k)} + Q e^{(c-k-1)} \right)$$

where $Q = H_1 - H_2 - (c - k) + 1$. Then by Lemma 3.5 once again we arrive at the equality

(d)
$$0 = \sum_{k=0}^{\min(a,c)} (-1)^k \binom{b+k}{k} f^{(a-k)} f \binom{H_2 + 1}{b+k} e^{(c-k)}$$
$$+ \sum_{k=0}^{\min(a,c)} (-1)^k \binom{b+k}{k} f^{(a-k)} \binom{H_2}{b+k} (H_1 - H_2 - c + k + 1) e^{(c-k-1)}.$$

Now from Lemma 4.4 it follows that

(e) $\binom{H_2}{b+k} (H_1 - H_2 - c + k + 1) = (a - k - b) \binom{H_2}{b+k}$

$$- (2b + 2k + 2) \binom{H_2}{b+k+1}$$

where we have made use of the equality $(d - 2(b + k)) - (c - k - 1) = (d + 1 - b - c) - k - b = a - k - b$.

Thus by putting (e) and the expansion

$$\binom{H_2 + 1}{b + k} = \binom{H_2}{b + k} + \binom{H_2}{b + k - 1}$$

into (d) we obtain the equality

(f)
$$\begin{aligned}
0 = f &\sum_{k=0}^{\min(a,c)} (-1)^k \binom{b + k}{k} f^{(a-k)} \binom{H_2}{b + k} e^{(c-k)} \\
+ &\sum_{k=0}^{\min(a,c)} (-1)^k \binom{b + k}{k} f f^{(a-k)} \binom{H_2}{b + k - 1} e^{(c-k)} \\
+ &\sum_{k=0}^{\min(a,c)} (-1)^k \binom{b + k}{k} (a - b - k) f^{(a-k)} \binom{H_2}{b + k} e^{(c-k-1)} \\
+ &\sum_{k=0}^{\min(a,c)} (-1)^k \binom{b + k}{k} (-2b - 2k - 2) f^{(a-k)} \binom{H_2}{b + k + 1} e^{(c-k-1)}.
\end{aligned}$$

Now by induction we claim that the first and fourth sums in the above equality are zero. This is clear for the first sum. To see this claim for the fourth sum, note that

$$\binom{b + k}{k} (-2b - 2k - 2) = -2(b + 1) \binom{b + k + 1}{k}$$

and so the fourth sum in (f) is equal to

$$-2(b + 1) \sum_{k=0}^{\min(a,c)} (-1)^k \binom{b + k + 1}{k} f^{(a-k)} \binom{H_2}{b + k + 1} e^{(c-1-k)}$$

which is a multiple of the $a, b + 1, c - 1$ case of the theorem, and hence is zero by induction. (If $\min(a, c) = c$ then the last term in the sum is zero, so one can replace $\min(a, c)$ by $\min(a, c - 1)$ without changing the value of the sum.)

16

Thus the equality (f) reduces to

$$0 = \sum_{k=0}^{\min(a,c)} (-1)^k \binom{b+k}{k}(a+1-k)f^{(a+1-k)}\binom{H_2}{b+k-1}e^{(c-k)}$$

$$+ \sum_{k=1}^{1+\min(a,c)} (-1)^{k-1}\binom{b+k-1}{k-1}(a-b-k+1)f^{(a+1-k)}\binom{H_2}{b+k-1}e^{(c-k)}$$

where we have shifted the index of summation in the second sum. Writing m for $\min(a,c)$ we thus obtain

(g)
$$0 = \sum_{k=1}^{m}(-1)^k R f^{(a+1-k)}\binom{H_2}{b-1+k}e^{(c-k)}$$

$$+ (a+1)f^{(a+1)}\binom{H_2}{b-1}e^{(c)}$$

$$+ (-1)^m\binom{b+m}{m}(a-b-m)f^{(a-m)}\binom{H_2}{b+m}e^{(c-1-m)}$$

where

$$R = \binom{b+k}{k}(a+1-k) - \binom{b+k-1}{k-1}(a-b-k+1)$$

$$= (a+1)\binom{b-1+k}{k}.$$

Note that the last term in (g) is zero if $m = c$. Otherwise $m = a < c$ and the term takes the form

$$(-1)^a\binom{b+a}{a}(-b)\binom{H_2}{b+a}e^{(c-1-a)}$$

$$= (-1)^{a+1}(a+1)\binom{b+a}{a+1}\binom{H_2}{b+a}e^{(c-1-a)}$$

since

$$(-1)^a \binom{b+a}{a}(-b) = (-1)^{a+1} \binom{b+a}{b} b$$

$$= (-1)^{a+1} \binom{b+a}{b-1}(a+1)$$

$$= (-1)^{a+1}(a+1)\binom{b+a}{a+1}.$$

This shows that all the terms in equation (g) have a common factor of $(a+1)$. Putting these terms together and dividing by $(a+1)$ we obtain the equality

$$0 = \sum_{k=0}^{M}(-1)^k \binom{b-1+k}{k} f^{(a+1-k)} \binom{H_2}{b-1+k} e^{(c-k)}$$

where $M = m = \min(a,c) = \min(a+1,c)$ in case $m = c$ and $M = m+1 = a+1 = \min(a+1,c)$ otherwise. In either case, M is the minimum of $a+1$ and c, and so we have obtained at last the desired equality which completes the induction. $\square$

6.2   Theorem *Suppose that a, b, c are nonnegative integers such that $s = a+b+c-d$ is positive. We have in $\mathcal{B}_d$ the equalities*

(a) $f^{(a)} \binom{H_2}{b} e^{(c)} = \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \binom{k-1}{s-1} \binom{b+k}{k} f^{(a-k)} \binom{H_2}{b+k} e^{(c-k)}$

(b) $e^{(a)} \binom{H_1}{b} f^{(c)} = \sum_{k=s}^{\min(a,c)} (-1)^{k-s} \binom{k-1}{s-1} \binom{b+k}{k} e^{(a-k)} \binom{H_1}{b+k} f^{(c-k)}.$

Proof. By symmetry it is enough to prove the first equality. We proceed by induction on s. The case $s = 1$ is precisely the content of Theorem 6.1.

Let a, b, c be given such that $a+b+c-d = s+1$. If $a < s$ then $b+c-1 \ge d+1$, so by Lemma 4.3 we have:

$$\text{either} \quad \binom{H_2}{b} e^{(c-1)} = 0 \quad \text{or} \quad \binom{H_2}{b-1} e^{(c)} = 0.$$

18

In either case it follows that

$$f^{(a)} \binom{H_2}{b} e^{(c)} = 0.$$

By a similar argument one sees that this holds also if $c < s$. Hence we may assume that both a and c are $\geq s$. It is enough to argue the result for the case $a \geq c \geq s$ since the other case $c \geq a \geq s$ is similar.

Thus $a \geq 1$ and we have by the inductive hypothesis the equalities

$$
\begin{aligned}
f^{(a)} \binom{H_2}{b} e^{(c)} &= \frac{f}{a} f^{(a-1)} \binom{H_2}{b} e^{(c)} \\
&= \frac{f}{a} \sum_{k=s}^{\min(a-1,c)} (-1)^{k-s} \binom{k-1}{s-1} \binom{b+k}{k} f^{(a-1-k)} \binom{H_2}{b+k} e^{(c-k)} \\
&= \frac{1}{a} \sum_{k=s}^{\min(a-1,c)} (-1)^{k-s} \binom{k-1}{s-1} \binom{b+k}{k} (a-k) f^{(a-k)} \binom{H_2}{b+k} e^{(c-k)} \\
&= \frac{a-s}{a} \binom{b+s}{s} f^{(a-s)} \binom{H_2}{b+s} e^{(c-s)} \\
&\quad + \frac{1}{a} \sum_{k=s+1}^{\min(a-1,c)} (-1)^{k-s} \binom{k-1}{s-1} \binom{b+k}{k} (a-k) f^{(a-k)} \binom{H_2}{b+k} e^{(c-k)}.
\end{aligned}
$$

Now the first term of the last equality above can be expanded by Theorem 6.1 (the base case of our present induction) since $(a-s) + (b+s) + (c-s) = a+b+c-s = d+1$. Putting this in and shifting the index of summation,

we obtain the equalities

$$
f^{(a)}\binom{H_2}{b}e^{(c)}
$$

$$
= \frac{1}{a}\binom{b+s}{s}(a-s)\sum_{k=1}^{c-s}(-1)^{k-1}\binom{b+s+k}{k}f^{(a-s-k)}\binom{H_2}{b+s+k}e^{(c-s-k)}
$$

$$
- \frac{1}{a}\sum_{k=s+1}^{\min(a-1,c)}(-1)^{k-(s+1)}\binom{k-1}{s-1}\binom{b+k}{k}(a-k)f^{(a-k)}\binom{H_2}{b+k}e^{(c-k)}
$$

$$
= \frac{1}{a}\binom{b+s}{s}(a-s)\sum_{k=s+1}^{c}(-1)^{k-(s+1)}\binom{b+k}{k-s}f^{(a-k)}\binom{H_2}{b+k}e^{(c-k)}
$$

$$
- \frac{1}{a}\sum_{k=s+1}^{\min(a-1,c)}(-1)^{k-(s+1)}\binom{k-1}{s-1}\binom{b+k}{k}(a-k)f^{(a-k)}\binom{H_2}{b+k}e^{(c-k)}.
$$

Now the second sum in the last equality above can be taken from $s+1$ to c since $\min(a-1,c)$ is different from c only if $a=c$, in which case the additional term in the sum will be zero (the factor $a-k$ is zero when $k=c=a$). Putting the two sums together and using the computation

$$
\frac{a-s}{a}\binom{b+s}{s}\binom{b+k}{k-s} - \frac{a-k}{a}\binom{k-1}{s-1}\binom{b+k}{k}
$$
$$
= \frac{a-s}{a}\binom{k}{s}\binom{b+k}{k} - \frac{a-k}{a}\binom{k-1}{s-1}\binom{b+k}{k}
$$
$$
= \binom{k-1}{s}\binom{b+k}{k}
$$

we obtain that

$$
f^{(a)}\binom{H_2}{b}e^{(c)} = \sum_{k=s+1}^{c}(-1)^{k-(s+1)}\binom{k-1}{s}\binom{b+k}{k}f^{(a-k)}\binom{H_2}{b+k}e^{(c-k)}
$$

and this completes the induction. $\qquad\square$

The following combinatorial result is well-known (see [Ko]).

6.3   Lemma *Let $T_1,\ldots,T_m$ be commuting indeterminates, and let $F = F(T_1,\ldots,T_m)$ be a polynomial over $\mathbb{Q}$ which takes on integer values whenever*

20

the variables are replaced by integers. Then F is expressible as an integral linear combination of the polynomials

$$\binom{T_1}{b_1}\binom{T_2}{b_2}\cdots\binom{T_m}{b_m}$$

where $b_1, \ldots, b_m$ all belong to the set of non-negative integers and where b_i does not exceed the degree of F viewed as a polynomial in T_i.

6.4   Theorem 6.2 combines with the isomorphism $\mathcal{B}_d \simeq S(2, d)$ from Theorem 5.2 to prove Theorem 2.5.

We now prove Theorem 2.4. We begin with the representation

(a) $\rho_d : U(\mathfrak{gl}_2) \to \mathrm{End}(E^{\otimes d})$.

We let $U_{\mathbb{Z}}(\mathfrak{gl}_2)$ denote the subring of $U(\mathfrak{gl}_2)$ generated by all $f^{(m)}$, $e^{(m)}$, $\binom{H_1}{m}$, $\binom{H_2}{m}$ for $m \geq 0$. Then $U_{\mathbb{Z}}(\mathfrak{gl}_2)$ is the same as the algebra of distributions on the $\mathbb{Z}$-group $G_{\mathbb{Z}} = \mathsf{GL}_2$ from 3.4.

Now the map in (a) restricts to give a map $U_{\mathbb{Z}}(\mathfrak{gl}_2) \to \mathrm{End}(E^{\otimes d})$. Let $E_{\mathbb{Z}}$ be the $\mathbb{Z}$-submodule of E spanned by the canonical basis elements e_1, e_2. It is easy to see that $E_{\mathbb{Z}}$ is stable under the action of $U_{\mathbb{Z}}(\mathfrak{gl}_2)$. It follows that $E_{\mathbb{Z}}^{\otimes d}$ is also stable, and so the image of the above map is contained within $\mathrm{End}(E_{\mathbb{Z}}^{\otimes d})$. Thus we have a representation

(b) $\rho_d^{\mathbb{Z}} : U_{\mathbb{Z}}(\mathfrak{gl}_2) \to \mathrm{End}(E_{\mathbb{Z}}^{\otimes d})$

By a result of Donkin [Do, 2.1] (see also [D]) the integral Schur algebra $S_{\mathbb{Z}}(2, d)$ is precisely the image of this representation.

Hence $S_{\mathbb{Z}}(2, d)$ is spanned over $\mathbb{Z}$ by (the image of) the elements

$$f^{(a)}\binom{H_1}{b_1}\binom{H_2}{b_2}e^{(c)}.$$

But by Lemma 6.3 we can express all $\binom{H_1}{b_1}\binom{H_2}{b_2} = \binom{d - H_2}{b_1}\binom{H_2}{b_2}$ as integral linear combinations of terms of the form $\binom{H_2}{b}$. Thus $S_{\mathbb{Z}}(2, d)$ is

21

spanned by all $f^{(a)} \binom{H_2}{b} e^{(c)}$. But by Theorem 6.2 we see that the set of such terms satisfying the restriction $a + b + c \le d$ is an integral spanning set. But this set is linearly independent over $\mathbb{Q}$, hence linearly independent over $\mathbb{Z}$, hence an integral basis. This proves 2.4(a). Part (b) is proved similarly, starting with the fact that the set of $e^{(a)} \binom{H_1}{b_1} \binom{H_2}{b_2} f^{(c)}$ forms an integral basis for $U_{\mathbb{Z}}(\mathfrak{gl}_2)$.

Since $U_{\mathbb{Z}}(\mathfrak{sl}_2)$ is generated by all divided powers $e^{(m)}$, $f^{(m)}$ it is clear that $U_{\mathbb{Z}}(\mathfrak{sl}_2)$ is contained in $U_{\mathbb{Z}}(\mathfrak{gl}_2)$. It was shown in [Do] (see also [D]) that the image of $U_{\mathbb{Z}}(\mathfrak{sl}_2)$ in $\mathrm{End}(E_{\mathbb{Z}}^{\otimes d})$ is the same as the image of $U_{\mathbb{Z}}(\mathfrak{gl}_2)$. This proves the first claim in Theorem 2.4. The proof is complete.

References

[Do]   S. Donkin, On Schur algebras and related algebras III: integral representations, *Math. Proc. Cambridge Philos. Soc.* **116** (1994), 37–55.

[D]   S. R. Doty, Polynomial representations of Chevalley groups, *preprint*, Loyola Univ. Chicago, 1999.

[DG]   S. R. Doty and A. Giaquinto, Presenting quantum Schur algebras as quotients of the quantized universal enveloping algebra of $\mathfrak{gl}_2$, *preprint*, Loyola Univ. Chicago, Sept. 2000.

[Gr]   J. A. Green, *Polynomial Representations of* GL_n, (Lecture Notes in Math. **830**), Springer-Verlag, New York 1980.

[RG]   R. Green, q-Schur algebras as quotients of quantized enveloping algebras, *J. Algebra* **185** (1996), 660–687.

[Ja]   J. C. Jantzen, *Representations of Algebraic Groups*, Academic Press, Orlando 1987.

[Ko]   B. Kostant, Groups over $\mathbb{Z}$, *Proc. Symposia Pure Math.* **9** (1966), 90–98.

Mathematical and Computer Sciences
Loyola University Chicago
Chicago, Illinois 60626 U.S.A.
E-mail: doty@math.luc.edu
          tonyg@math.luc.edu