NON–ABELIAN QUANTUM ALGEBRAIC TOPOLOGY: I. From Fundamental Concepts to Physical Realizations in Quantum Gravity, Spin Networks, Superconductivity and Quantum Phase Transitions *via* Local-to-Global Constructions.
v.20e

I. C. Baianu* et al.

ABSTRACT

Conceptual developments and a novel approach to Quantum theories are here proposed starting from existing Quantum Group Algebra (QGA), Algebraic Quantum Field Theories (AQFT), standard and effective Quantum Field Theories (QFT), as well as the refined 'machinery' of Non–Abelian Algebraic Topology (NAAT), Category Theory (CT) and Higher-Dimensional Algebra (HDA). The open question of building valid QST representations in Quantum Gravity (QG) is approached as a *Local-to-Global* (LG) mathematical construction problem in Quantum Algebraic Topology (QAT). QST representations are here proposed for quantum systems with either *finite* (Quantum Mechanics and Quantum Automata) or *infinite* degrees of freedom (Quantum Field Theories (QFT)). New possibilities for the application of fundamental theorems from Algebraic Topology to physical processes such as *Quantum Phase Transitions* (e.g., as in *superconductivity, colossal magnetoresistance and ferromagnetism*), spin network fluctuations, parallel transport and quantum tunneling are also formulated. Among such fundamental theorems with potential physical applications in QAT is the Generalized van Kampen theorem (GvKT); GvkT potential applications include the construction of 'global' QST representations in the physical context of relativistic quantum measurements or quantum gravity, and the precise embedding of smaller structures into larger ones in order to obtain exact solutions for the former. Several other applications of AT fundamental theorems are suggested for the formulation of QST non–Abelian structural approximations by algebraic– topological, as well as logical, means. The logical links between Quantum Operator Algebras and their corresponding,'dual' structure of the Quantum State Spaces are also investigated. An extensive review and critical evaluation of published and archived articles, as well as standard texts, on QA, QFT, AQFT, TQFT, Supersymmetry/Supergravity (SG) and Gauge theories (GT) is summarized and provides essential concepts relevant to the development of improved mathematical representations of Quantum Space–Time (QST) and Quantum State Spaces (QSS). Among such key concepts are: Quantum Group Algebras (QGAs)/Groupoids, Hopf and C*-algebras, Lie 'algebras', Quantization and Asymptotic Morphisms, Locally Topological Groupoids, Crossed Modules of Groups or Lie Double Groupoids, Lie Algebroids, Crossed Complexes over Groupoids, Holonomy and Gauge Transformation Groupoids, Quantum Principal Bundles and Sheaves, CW–complexes of Spin Networks and Quantum Spin 'Foams'. These important concepts are presented in a sequence tailored to aid the development of a *non–Abelian* algebraic–topological framework for QG theories of intense gravitational fields in curved, fluctuating QST. Furthermore, two novel related concepts of Quantum Fundamental Groupoid (QFG) and Gravitonic Homotopy Representation (GHR) of Generally Relativistic Quantum Space-Time (GR-QST) are proposed as indispensible mathematical tools for a Generally Relativistic Quantum Field Theory (GR-QFT) of *intense*, *quantized* gravitational fields. The concepts presented in this paper are intended to fill gaps between Quantum Field theories and General Relativity Theory, thus leading towards Generally Relativistic Quantum Gravity in a *non–Abelian* framework of *Fluctuating*, Quantum Space-Time.

KEYWORDS:

Quantum Algebraic Topology (QAT); Algebraic Topology of Quantum Systems; Relativistic Quantum Gravity (RQG); Gravitonic Homotopy Representation (GHR); non–Abelian Algebraic Topology; Quantum Fundamental Groupoid; Lattice Quantum Field Theories (LQFT), QED, QCD and Lattice Quantum Gravity; Poisson–Lie Manifolds and Dilaton QG Theories; Algebraic Quantum Field Theories (AQFT); Modular Approaches to AQFT; Generalized, Extended Topological Quantum Field Theories (GE–TQFT), TQFT/ TFT and Homotopy QFT; Quantum Algebras; Quantum Groups/Quantum Ring structures; Lie algebras and Lie–Algebroids; Hopf, Weak Hopf and Jordan algebras; C*–algebras; Compact Quantum Groupoids and Quantum 'Metrics'; Quantum Groupoid C*–algebras; Topological Groupoids; Gauge Transformations and Renormalization Groupoids in QG theories; Supergravity and Supersymmetry theories; Nuclear Frechet spaces; Gel'fand triples and GNS Representations of QSS; Quantum Automata, Nano-Automata and Quantum Algebraic Computation; Quantization; non–commutative Geometry; CW–complex approximations of 'Quantum Spin Foams' in QSS; Generalized van Kampen Theorem (GvKT) Applications to QSS and QST invariants; Fluctuating Quantum space–time in intense Gravitational Fields.

Initiating authors: I. C. Baianu*,J. F. Glazebrook**, and R. Brown***

[a4paper,11pt]article
amssymb,amsbsy,amsmath,
latexsym,theorem,eepic,enumerate
xypic,times,color

geometry
textwidth=17cm,textheight=22cm Example[sec@ tion] [example]Diagram
indent0mm

Non-Abelian Quantum Algebraic Topology: QST QSS Representations v.20– both parts: I II, Sections 1 to 9, complete; refs. list to be edited next. $*CONFIDENTIAL DOCUMENT*$

Contents

1. Introduction 5

2. Quantum Measurements and Quantum Logics. 10

2.1. Quantum Fields, Symmetry, Space-Time and Connections to General Relativity 10

2.2. Quantum Logics and A 'Global' Obstruction to Complete Quantum Measurements for QSS dimensions higher than *2* 13

3. Quantum Dynamics, Symmetry Groups and Observable Operator Representations in Quantum Group Algebras (QA) 14

3.1. Classical *vsus* Quantum Dynamics 14

3.2. Quantum Lorentz Transformations 18

3.3. Symmetry Groups, Quantum Group Algebras (Hopf Algebras) and Other Operator Algebras 18

3.4. General Quantization Procedures: The First Quantization 26

3.5. Quantum Problem Solving by Algebraic Methods: The Finite Lie 'Algebra' of Quantum Commutators and the corresponding, unique (*continuous*) Lie Groups. Applications to Chemical, Quantum–Molecular Structure and Quantum Physics. 28

3.6. Extensions of Quantum Mechanics to Infinite Systems and Electrodynamics 31

4. Generalizations of Fundamental Quantum Concepts required by Modern Quantum Theories: From Superconductivity to Quantum Gravity 36

4.1. Concrete Representations of C*–Algebras: The Gel'fand Triples 36

4.2. **Galois Extension Conjecture** 37

4.3. Quantum Statistical Mechanics of Phase Transitions in Macroscopically Coherent, Quantum Systems: Spontaneous Symmetry Breaking Mechanisms 37

4.4. Quantum Symmetry Groupoids and Pseodogroups. Quantum 6j–Groupoids of Type II_1 von Neumann Subfactor Paragroups. Christoffel symbols and ETQFT Symmetry Tetrahedron. 39

4.5. Quantum Field Configuration J-Groupoids of Type III von Neumann Subfactors. Locally Compact Quantum Groups and Groupoids. 39

4.6. Interacting Quantum Spin Groups and Quasi-Particles: Explicit Hamiltonians and Numerical Computations Compared with Experimental Results in Solid Electrolytes and Ferromagnetic Glasses. 39

5. Novel Representations of Global and Local Symmetry Breaking in Quantum Systems. Quantum Algebraic Topology of Space–Time Representations Consistent with the Symmetries of Quantum Fields: *Quantum Crossed Modules, Quantum 'Convolution' Groupoids and Quantum Cross Complexes over a Groupoid; Non-Abelian Gauge Theories in AQFT and Quantum Gravity.* 40

5.1. Quantum Algebraic Topology of Space–Time Representations Consistent with the Symmetries of Quantum Fields: *Local–to–Global (LG) Construction Principles based on AQFT and Quantum 'Axiomatics'.* 40

5.2. Quantizing Space–Time: Causal Sets 43

5.3. Quantizing Groupoids: Compact Quantum Groupoids (CQG), CQG–Algebras, Quantum Metrics and Quantum Principal Bundles 45

5.4. Assembling Groupoids: Crossed C-Modules and Crossed Complexes over a Groupoid. The Fundamental Groupoid of a Crossed Complex 49

5.5. The Quantum Fundamental Groupoid (QFG) in Quantum Space–Time Representations 50

5.6. The Holonomy Groupoid–A Unique Local to Global Construction based on Locally Topological Groupoids 51

5.7. Emergence of Extended 'Supersymmetry' by Assembling Quantum Topological Groupoids (QTGs): *Linking Quantum Metric Spaces to QTGs in QST Representations.* 54

5.8. 'Global' Non-Abelian Gauge Transformations. Quantum Crossed Modules (QCM), Quantum Convolution Groupoids (QCG) and Their Quantum Cross Complexes (QCC). 56

5.9. Lie Algebroids 56

5.10. Applications of 2–Lie Groups and Groupoids in Non-Abelian Gauge Theories, AQFT and Quantum Gravity. 56

5.11. Lattice Quantum Gravity (LQG), Gauge Theories and New Approaches to Many-Body Problems 57

6. Supergravity Theories. The Metric Superfield and Supersymmetry Algebras. 57

6.1. Supergravity Theories 57

6.2. The Metric Superfield 57

6.3. The Supersymmetry and Graded Lie Algebras 59

7. Fundamental Algebraic Topology Theorems with Applications to *Local-to-Global* Constructions of Quantum Space-Time and Quantum State Space Representations 61

7.1. The Fundamental Theorem of Hurewicz and The Whitehead Theorem: *Related Propositions and Their Applications* 61

7.2. The Cohomology Group Theorem 63

7.3. The Approximation Theorem for an Arbitrary Space: *The Colimit of a Sequence of CW–complexes* 63

7.4. The Quantum Algebraic Topology of CW–complex Representations: New QAT Theorems for Quantum State Spaces of Spin Networks and Quantum Spin 'Foams' based on CW n-Connected Models and AT fundamental theorems. 64

7.5. The Quantum Algebraic Topology of Loop Quantum Gravity (LQG) and Closed Superstrings in QST 66
7.6. Novel Homotopic Representations of GR-QST: The Quantum Gravitonic Homotopy (QGH) 66
8. The Generalized van Kampen Theorem and Its Possible Applications to Physical Representations in Quantum Algebraic Topology 66
8.1. The van Kampen Theorem, Its Generalizations and Applications 66
8.2. Applications of the Van Kampen Theorem to Crossed Complexes and Representations of Quantum Space-Time in terms of Quantum Crossed Complexes over a Quantum Groupoid 67
8.3. The Generalized van Kampen Theorem (GvKT) 68
8.4. Construction of The Homotopy Double Groupoid 70
8.5. The Homotopy Double Groupoid of a Hausdorff space. 72
8.6. The Generalized van Kampen Theorem (GvKT) in terms of Double Groupoids and Its Possible Applications in Quantum Algebraic Topology 76
8.7. Novel GvKT Applications and Related Conjectures 76
9. Conclusions and Discusion 80
References 81

1. Introduction

One of the most significant and difficult enterprises in modern theoretical physics involves the quest for a Quantum Gravity (QG) theory. This quest entails developing a Generally Relativistic Quantum Field Theory (GR-QFT) which can hopefully incorporate, in a consistent manner, both masses and fields into a unified Quantum Gravity theory. A fundamental issue in modern physics is, therefore, the correct realization/ *mathematical representation of the physical structure of space, or space–time,* generated by matter coupling to various quantum fields. Another outstanding problem of current theoretical physics concerns the mathematical representation of symmetry changes and spontaneous symmetry/ supersymmetry 'breaking'. *Symmetry changes* which are involved in phase transitions of *quantum-coherent* macroscopic systems such as liquid ^3He, superconductors of Type I and II, and a variety of ferromagnetic materials, require such novel mathematical treatments and *generalized symmetry representations*. On the other hand, the spontaneous supersymmetry 'breaking' is currently thought to play a very significant role in Supergravity/Quantum Gravity and other unification–motivated, physical theories (Weinberg, 2000; ref. [170]).

We define our main goal as the development of a consistent framework for mathematical representations of key physical concepts such as Space–Time, *Quantum State Space (QSS) and Quantum Space–Time (QST)* that will hopefully facilitate the emergence of a complete and logically consistent theory of Quantum Fields coupled with Matter in a 'covariant', Generally Relativistic Quantum Gravity. Our approach involves *local–to–'global'* constructions of QSS or QST, and it is, in essence, centered on applications of *Algebraic Topology* (and especially groupoids of various flavors, Category Theory (CT) and Higher Dimensional Algebra (HDA) to Quantum Systems, including *quantum fields.* This approach relies heavily on quantum groups/ groupoids, Quantum Algebra (QA), Homotopy, Homology and Cohomology as its key ingredients. Therefore, it is appropriate to define this area of fundamental, theoretical research as "Quantum Algebraic Topology" (QAT), and also specify its 'neighbors' in the fields of theoretical physics and 'physical mathematics'. We propose here to define *Quantum Algebraic Topology* as the area of theoretical physics concerned with the *applications of Algebraic Topology methods, results and constructions (including its extensions to Category/Topos Theory and Higher Dimensional Algebra) to fundamental quantum physics problems, such as the representations of Quantum Space-Times and Quantum State Spaces in Quantum Gravity, in arbitrary reference frames.* Clearly, the Homotopy and Cohomology of Quantum Field Configuration Spaces (HC-QFCS) as introduced and summarized by Steven Weinberg in 1995 (ref.[170]) is an integral part of QAT and may indeed be considered to be its main starting point. Perhaps the neighbor areas with which QAT overlaps significantly are: Algebraic Quantum Field Theories (AQFT)/Local Quantum Physics (LQP), Axiomatic QFT, Lattice QFT (LQFT) and Supersymmetry/Supergravity. One can also claim overlap with various Topological Field Theories (TFT), or Topological 'Quantum' Field Theories (TQFT), Homotopy QFT (HQFT), Dilaton, and Lattice Quantum Gravity (respectively, DQG and LQG) theories. Note, however, that quantization, relativistic concepts, physical relevance and interpretations seem to have mostly dropped out of the picture in strictly mathematical presentations which are common in either TFT or HQFT publications in the 'lower' dimensions (i.e., $n \leqslant 3d$), or even for $n = 4d$. Perhaps, this occurred as a result of selecting a minimum number of assumptions in the mathematical models and treatments implemented by TFT and HQFT which are concerned mostly with *'topological'* invariants in the *'lower'* dimensional spaces (i.e., $n \leqslant 3$) and partition functions or 'state sums'. This review and concept development paper is, therefore, focused on the essential aspects of Quantum Algebraic Topology (QAT) with emphasis on the *physical* interpretation of mathematical concepts in the context of quantum theory, as well as the recent advances made through higher dimensional algebra. Although Newton's injunction: "*I do not make hypotheses*! " has 'converted' many to his thinking, many such models that 'do not make hypotheses' are very restrictive, and their implicit assumptions may be harder to deal with than a few explicitly made hypotheses, physical postulates or 'axioms'.

We also include here an explanation of how several fundamental concepts and theorems of Algebraic Topology can be employed to solve problems based on quantum theories for a very wide range of systems. Such systems under consideration are: quantum lattices, quantum automata, quantum fields, very large masses coupled with quantum fields and quantized space–time, as well as gravitationally anomalous systems that are presently inaccessible to direct observation.

We are making from the outset a basic, logical consistency assumption in the form of the following conjecture.

Conjecture 1. *If there is an internally Consistent Theory of General Relativity, CTGR, which predicts correctly the large–scale structure and dynamics of all physical fields and matter, then there exists also a complete Quantum Gravity Theory (in n–dimensions, with $n \geqslant 4$, n–QGT) which includes CTGR and physical causality in a logically consistent manner by employing mathematical representations of Quantum Space–Time and Quantum State Space that characterize* both *their large and small scale structures.*

Note 1.1 The fundamental open questions whose resolution may lead to CTGR and a 'proof' of this conjecture are: *the consistency* of the null–cone from special relativity with CTGR, the *quantization* of space–time for dimensions $n \geqslant 4$ in QGT, *the dynamic equations of motion* for quantized CTGR in the presence of *intense* gravitational fields, and the related problem of valid *mathematical representations of QST and QSS* in QGT. The preferred candidate for the representation of space-time in CTGR is a 'globally hyperbolic' space-time in four dimensions (actually, 3+1, "4d-space"). The framework developed in this paper is designed to address the fourth of the open fundamental questions of modern physics listed above because it appears to us to be more amenable to a mathematical resolution than the three preceding ones.

There are several distinct 'programmes' aimed at developing a Quantum Gravity theory. These include–but are not limited to– the following:

• The Penrose, twistors programme applied to an open curved space-time (ref. [109]), (which is presumably a globally hyperbolic, relativistic space-time). This may also include the idea of developing a *'sheaf cohomology'* for twistors (ref. [109]) but still needs to justify the assumption in this approach of a charged, fundamental fermion of spin-3/2 of undefined mass and unitary 'homogeneity' (which has not been observed so far);

• The Weinberg, *supergravity* theory, which is consistent with supersymmetry and super-algebra, and utilizes *graded Lie algebras* and *matter-coupled superfields* in the presence of *weak* gravitational fields (that will be concisely summarized in Section 6);

• The Hawking, no boundary (closed), *continuous* space-time programme (ref. [109]) in quantum cosmology, concerned with singularities, such as black and 'white' holes; S. W. Hawking combines, joins, or 'glues' an initially flat Euclidean metric with convex Lorentzian metrics in the expanding, and then contracting, space-times with a very small value of Einstein's cosmological 'constant'. Such 'Hawking', double-pear shaped, space-times also have an initial Weyl tensor value close to zero and, ultimately, a largely fluctuating Weyl tensor during the 'final crunch' of our universe, presumed to determine the irreversible arrow of time; furthermore, an observer will always be able to access through measurements only *a limited part* of the global space-times in our universe;

• The TQFT/HQFT approach that aims at finding the 'topological' invariants of a manifold embedded in an abstract vector space related to the statistical mechanics problem of defining extensions of the partition function for many-particle quantum systems;

• The string and superstring theories/M-theory that 'live' in higher dimensional spaces (e.g., $n \geqslant 6$, preferred $n-dim = 11$), and can be considered to be topological representations of physical entities that vibrate, are quantized, interact, and that might also be able to 'predict' fundamental masses relevant to quantum 'particles';

• The Baez 'categorification' programme ([9], [10]) that aims to deal with Quantum Field and QG problems at the abstract level of categories and functors in what seems to be mostly a global approach;

and

• The 'monoidal category' and valuation approach initiated by Isham (ref. [119]) to the quantum measurement problem and its possible solution through local-to-global, finite constructions in small categories.

The presentation of this paper is divided in two parts: presentation of the essential concepts and the relevant background – concentrated mostly in the beginning part up to Section 4, and the derivation of new concepts and results in the second part containing sections 5 through 9. The following is a more detailed outline of the mainp sections in this paper.

In Section 2 we shall concisely discuss several different views on the *quantum measurement* problem and its links with *Quantum Logics*, as well as possible connections with AQFT and QG, that are pertinent to the problem of mathematical representations of QSS and QST. A concise summary of the relevant background concerning relativistic approaches to quantum fields and space–time will also be presented in this section together with its links to the next four sections. (The subjects of *Relativistic QFT* and *Quantum Gravity* are revisited in greater detail in sections 6 and 7.)

Section 3 introduces several essential concepts in Quantum Dynamics and the *Quantum Group Algebra of Quantum Observable Operators*, such as: Quantum Group Algebra/Hopf Algebra, Poisson, von Neumann and Jordan Algebras, as well as *Lie 'algebras' and C*-algebras.* The question of general quantization procedures is then presented with special emphasis on the Wigner–Weyl–Moyl analytic Quantization and asymptotic morphisms. The generalization to *quantizing groupoids* and other algebraic structures is presented in Section 4. Related generalizations of quantum fundamental concepts, such as those utilized in *Local Quantum Physics/AQFT, AXFT, Lattice QFT*, Topological QFT and Homotopy QFT are also outlined in Section 4. For this purpose, we are linking the basic concepts of *Quantum Algebra* defined in Section 3 with novel approaches and developments that involve *Algebraic Topology* for the construction of QSS and QST representations that lead us to *Quantum Algebraic Topology.* The question of 'quantizing' space–time is also addressed in Section 4 based on concrete representations of C*–algebras and quantum 'metric' that involve *Compact Quantum Groupoids, Quantum Groupoid C*–algebra and Quantum Principal Bundles.*

In Sections 4 and 5 we discuss *quantum phase transitions*, the occurrence of *spontaneous symmetry breaking* for a wide range of quantum systems and their important consequences at both quantum and macroscopic levels. These are, therefore, striking examples of *local-to-global* problems associated with global and/or local *symmetry changes in quantum systems* of considerable, scientific, technological and practical interest. The related physical phenomena are. therefore, being intensely studied both theoretically and experimentally. In Section 5 we are also introducing novel representations of global and local symmetry breaking in quantum systems. A related QAT problem that will be addressed in the beginning of Section 5 is the representation of quantum space-times in a manner which is consistent with the quantum field symmetries.

We discuss in some detail in Section 6 how *supersymmetry* allows the construction of a 'linearized' version of *quantum supergravity (QSG)* which is thought to be consistent with the Standard Model. According to the QFT textbook by Weinberg (2000) (*with the QSG theory presented in detail in vol.3*), the QSG theory is only *approximately* correct in the limit of *weak* gravitational fields and correspondingly small masses [170]. Linking QAT representations of QST with supersymmetry is then suggested as a strong potential candidate for a relativistic quantum field theory which includes *intense* gravitational fields and, therefore, curved, or *non–Abelian* algebraic topology representations of quantum space-time in QG (cf. discussion in Section 9).

The problem of *local–to–global* construction principles for QST representations will be addressed in Section 7 in terms of 'quantum' causal sets, Locally Topological Groupoids, Crossed Complexes over a Groupoid, the *Quantum Atlas and the Quantum Fundamental Groupoid (QFG)* of QSS and QST. Novel, concrete representations of QSS and QST in terms of Gel'fand triples will also be developed together with a *sheaf* representation of locally topological subgroupoids and *Locally Lie subgroupoids.* The latter concepts have immediate applications in the 'lower' dimensional Dilaton Quantum Gravity (DQG) theories (for $n \leqslant 4$) that involve *Poisson–Lie manifolds.* We also present in Section 7 several examples of *Algebraic Topology* (AT) fundamental theorem applications, such as the generalized Hurewicz, the J. H. Whitehead and the CW–approximation theorems to quantum systems relevant to quantum gravity models. We also present in Section 7 several new theorems concerning the mathematical representations of quantum spin networks and quantum 'foams' in QSS and QST. Further possibilities are explored in Section 8 for the application of fundamental AT theorems, i.e., the *generalized van Kampen (GvK)* to provide rigorous constructions of QSS and quantum space-time, as well as to derive key *algebraic invariants of QSS and QST.* Related, homotopy and homology/cohomology applications in QAT are 'Loop Quantum Gravity' and closed superstrings in QST, as well as the development of *'Gravitonic Homotopy Representations'(GHR)* of Relativistic QST. Such representations involve gauge transformations that were found previously to have a *'transformation' groupoid* structure, instead of the simpler, group symmetry. The latter applications are closely related to the *Lattice Quantum Gravity (LQG)* computations discussed in Section 5.7 and may lead to extensions of the *Supersymmetry and Supergravity* approach –introduced in Section 6 for weak gravitational fields–to intense gravitational fields in Quantum Gravity.

In Section 9 we summarize our conclusions by comparing various theoretical approaches to the problem of characterizing quantum space–time representations in Relativistic Quantum Gravity. We also propose several conjectures concerning the *Galois extensions* of the algebra underlying QSS, as well as future possible developments in higher dimensional algebra leading to higher-dimensional AQFT/ TQFT/ HQFT and Curved Quantum Space-Time 'models' of Relativistic Quantum Gravity. Last–but–not–least, we discuss in Section 8 the possibility of 'recovering' through QAT constructions the large–scale, global structure of a consistent General Relativity Theory through *quantum principal bundles*/fibrations and *sheaf constructions* based on the *quantum local net structures*, and/or quantum spin networks of AQFT, as well as possible *Grothendieck representations of quantum space-time topology* in quantum gravity. A major advantage of such *non–Abelian* algebraic topology representations is the absence of physically meaningless singularities, as well as the possibility of developing improved renormalization theories (for eliminating either UV or IR *anomalies*) that may employ our proposed QSS and quantum space-time representations.

2. Quantum Measurements and Quantum Logics.

Questions of measurement in quantum mechanics (QM) and quantum field theory (QFT) have been debated for over 80 years. The intellectual stakes are still dramatically high and the problem rattled the development of 20th (and 21st) century physics at the foundations. Up to 1955, Bohr's Copenhagen school dominated the terms and practice of quantum mechanics having reached (partially) eye–to–eye with Heisenberg on empirical grounds, although not the case with Einstein who was firmly opposed on grounds of incompleteness with respect to physical reality. Even to the present day, the 'hard' philosophy of this school is respected throughout most of theoretical physics. On the other hand, post 1955, the measurement problem adopted a new lease on life when von Neumann beautifully formulated QM in the mathematically rigorous context of separable Hilbert spaces for finite quantum systems. Measurement, it was argued, involved the influence of the Schrödinger equation for time evolution of the wave function ψ, so leading to the notion of entanglement of states and the indeterministic reduction of the wave packet. Once ψ is determined it is possible to compute the probability of measurable outcomes, at the same time modifying ψ relative to the probabilities of outcomes and observations eventually causes its collapse. The well–known paradox of Schrödinger's cat and the Einstein–Podolsky–Rosen (EPR) experiment are questions mooted once dependence on reduction of the wave packet is jettisoned, but then other interesting paradoxes have shown their faces. Consequently, QM opened the door to other interpretations such as the Bohm–deBroglie 'pilot-wave' quantum theory- and the Everett–Wheeler assigned measurement within different worlds, theories not without their respective advantages and potential shortcomings (see Krips 1999, and Wheeler and Zurek,1983).

2.1. Quantum Fields, Symmetry, Space-Time and Connections to General Relativity.

As the experimental findings in''high-energy' physics–coupled with theoretical studies– have revealed the presence of new fields and symmetries, there appeared the need in modern physics to develop systematic procedures for generalizing/generating space–times and Quantum State Space (QSS) representations that reflect the existence of such new fields and symmetries.

In the General Relativity (GR) formulation, the local structure of space–time – which is characterized by its tensors and curvature – incorporates the gravitational fields surrounding various masses. In Einstein's own representation, the 'physical space–time of GR' has the structure of a Riemann R^4 space over large distances, although the detailed local structure of space–time – as Einstein suggested – is likely to be significantly different.

On the other hand, there is a growing consensus in theoretical physics that a valid theory of Quantum Gravity requires a much deeper understanding of the small (est)–scale structure of Quantum Space–Time (QST) than currently developed. In Einstein's GR theory and his subsequent attempts at developing an unified field theory (as in the space concept advocated by Leibnitz), space-time does *not* have an *independent existence* from objects, matter or fields, but is instead an entity generated by the *continuous* transformations of fields [**?**] (Einstein, 1950, 1954; ref.x.x). Hence, the continuous nature of space–time adopted in GR and Einstein's subsequent field theoretical developments. Furthermore, the quantum, or 'quantized', versions of space-time, *QST*, are operationally defined through local quantum measurements in general reference frames that are prescribed by GR theory. Such a definition is therefore subject to the postulates of both GR theory and the axioms of Local Quantum Physics (that are briefly summarized in Subsection 3.3). We must empasize, however, that this is *not* the usual definition of position and time observables in 'standard' QM. Therefore, the general reference frame positioning in QST is itself subject to the Heisenberg uncertainty principle, and therefore it acquires through quantum measurements a certain 'fuzziness' at the Planck scale which is intrinsic to all microphysical quantum systems, as further explained in this section. Whereas Newton, Riemann, Einstein, Weyl, Hawking, Weinberg and many other exceptional theoreticians regarded the physical space as being represented by a *continuum*, there is an increasing number of proponents for a *discrete, 'quantized'* structure of space–time. The latter view is not without its problems and advantages. The biggest problem for any discrete, 'point-set' (or discrete topology), view of physical space–time is not only its immediate conflict with Einstein's General Relativity representation of space–time as a *continuous Riemann* space, but also the impossibility of carrying out quantum measurements to localize precisely either quantum events or masses at singular (in the sense of disconnected, or isolated), sharply defined, geometric points in space–time. One of the proposed resolutions of this problem is non–commutative Geometry (NCG), or 'Quantum Geometry', where QST has 'no points', in the sense of visualization of such a geometrical space as some kind of a distributive and commutative lattice of space-time 'points'. The quantum 'metric' of QST in NCG would be related to a certain, fundamental quantum field operator, or 'fundamental triplet (or quintet)' construction (Connes, 2004). Although quantization is standard in Quantum Mechanics (QM) for most of the quantum observables, it does run into major difficulties when applied to position and time. In standard QM, there are at least two implemented approaches to solve the problem, one of them designed 72 years ago by von Neumann (1933).

Another potential concern is the inadequacy of the long-standing model of space-time as a 4–dimensional manifold *with a Lorentz metric*. The hope of some of the earlier approaches to quantum gravity (QG) was to cope with extremely small length scales where a manifold structure may be justifiably foresaken (for instance, at the Planck length $L_p = (\frac{G\hbar}{c^3})^{\frac{1}{2}} \approx 10^{-35}m$). On the other hand, one needs to reconcile the discreteness versus continuum approach in view of space–time diffeomorphisms and that space–time may be suitably modeled as some type of 'combinatorial space' (such as a simplicial complex, a poset, or a spin network) The monumental difficulty is that to the present day, apart from a distinct lack of experimental evidence, there is no specific agreement on the kind of data, plus no agreement on the actual conceptual background to obtaining the data in the first place(!) This difficulty equates with how one can relate the approaches to QG to run the gauntlet of conceptual problems in QFT and (General Relativity) GR. To quote an example, the space–time metric tensor: $\gamma = (\gamma_{ab})$ is less a fundamental field than perhaps once thought since it leads to describing an essentially classical gravitational field. A case study in ref. [79] involves quantizing one side of Einstein's field equations by a quantum expectation value, so that a coupling of γ to quantized matter is given by an expression such as:

$$G_{\mu\nu}(\gamma) = \langle\ \psi\ |T_{\mu\nu}(g, \widehat{\phi}|\ \psi\ \rangle\ ,$$

where $|\psi\rangle$ denotes a state in the Hilbert space of quantized matter variables $\widehat{\phi}$, and the subsequent source of the gravitational field is given by the expectation of the corresponding energy–momentum tensor $T_{\mu\nu}$. Unfortunately, this expression is not without its ontological and 'physical' problems sufficiently serious to prevent the development of a complete QG theory that includes this expression. Three possible approaches were suggested by Butterfield and Isham in ref. [79] (cf. also an extensive survey article by Rovelli, 1997):

(1) to develop and test a quantized form of classical relativity theory;

(2) to recover GR as the low energy limit of a QFT approach which is not a quantization of a classical theory (e.g., *via quantum algebras/groups and their representations*);

(3) to develop a new theory, such as a 'quantization of topology' or 'causal' structures where, for instance, microphysical states provide amplitudes to the values of quantities whose norms squared define probabilities of occurrence for *physical*, quantum events.

We turn now to another facet of quantum measurement. Note first that QFT pure states resist description in terms of field configurations since the former are not always physically either observable or interpretable. Algebraic quantum field theory (AQFT) as expounded by Roberts (2004) points to various questions raised by considering theories of (unbounded) operator–valued distributions and *quantum field nets* of von Neumann algebras. Using in part a gauge theoretic approach, the idea is to regard two field theories as equivalent when their associated nets of observables are isomorphic. More specifically, AQFT considers taking *additive nets of quantum field algebras* over subsets of Minkowski space, which among other properties, enjoy Bose–Fermi commutation relations. There may be analogs with sheaf theory in this approach, even though these analogs appear to be limited. The typical AQFT net does not seem to give rise to a presheaf because the relevant morphism orientations are in reverse. Closer then is to regard a net as a precosheaf, but the additivity does not allow proceeding to a cosheaf structure. This may be a reflection of some deeper incompatibility of AQFT with those aspects of quantum gravity (QG) where the sheaf– theoretic/topos approaches are advocated (as, for example, in [79]Butterfield and Isham (1999)–(2004).

2.2. **Quantum Logics and A 'Global' Obstruction to Complete Quantum Measurements for QSS dimensions higher than *2*.** Arm–in–arm with the measurement problem goes a problem of 'the right logic', for quantum mechanical/complex biological systems and quantum gravity. It is well–known that classical Boolean truth–valued logics are patently inadequate for quantum theory. Logical theories founded on projections and self–adjoint operators on Hilbert space H do run into certain problems. One 'no–go' theorem is that of Kochen–Specker (KS) which for $\dim H > 2$, does not permit an evaluation (global) on a Boolean system of 'truth values'. In ref [79] (Butterfield and Isham (1999)–(2004)), self–adjoint operators on H with purely discrete spectrum were considered. The KS theorem is then interpreted as saying that a particular presheaf does not admit a global section. Partial valuations corresponding to local sections of this presheaf are introduced, and then generalized evaluations are defined. The latter enjoy the structure of a Heyting algebra and so comprise an intuitionistic logic. Truth values are describable in terms of sieve–valued maps, and the *generalized evaluations* are identified as *subobjects in a topos*. The further relationship with interval valuations motivates associating to the presheaf a von Neumann algebra where the supports of states on the algebra determines this relationship.

The above considerations lead directly to the organization of this paper in the next four sections that proceeds from defining in Section 3 the basic concepts of Quantum Algebra/Algebraic Quantum Field Theory (AQFT) which link quantum measurements with *Quantum Logics* and *QST construction* problems, to constructions of QST representations in Section 4 based on the existing QA, AQFT and Algebraic Topology concepts, as well as several new QAT concepts that are developed in this paper. The quantum algebras defined in Section 3 have corresponding, 'dual' quantum state spaces that are concisely discussed in Section 4. (For the QSS detailed properties, and also the rigorous proofs of such properties, the reader is referred to the recent book by Alfsen and Schultz (2003; ref[3]). Then, we utilize in Sections 7 and 8 a significant amount of recently developed results in Algebraic Topology (AT), such as for example, the *Generalized van Kampen theorem (GvKT)* in Section 8 to illustrate how constructions of QSS and QST, *non-Abelian* representations can be either generalized or extended on the basis of *GvKT*. We also employ the categorical form of the *CW–complex Approximation* (CWA) theorem) in Section 7 to both systematically construct such generalized representations of quantum space–time and provide, together with GvKT, the principle methods for determining the general form of the fundamental *algebraic invariants* of their *local or global*, topological structures. The algebraic invariant of Quantum Loop (such as, the graviton) Topology in QST is defined in Section 7.7 as the *Quantum Fundamental Groupoid (QFG)* of QST which can be then calculated– at least in principle – with the help of AT fundamental theorems, such as *GvKT*, especially for the relevant case of space–time representations with *non-Abelian* algebraic topology.

3. Quantum Dynamics, Symmetry Groups and Observable Operator Representations in Quantum Group Algebras (QA)

3.1. **Classical *vsus* Quantum Dynamics.** We shall introduce first the classical dynamic equations of motion and then compare such equations with the standard quantum dynamics which is subject to constraints through the Principles of Quantum Mechanics. Such constraints do not occur, of course, in the case of classical dynamics, and it is therefore all the more remarkable that certain classical results can be obtained from quantum mechanics in the limit of the Planck constant going to zero.

The analytical mechanics of a classical system characterizes the dynamics of a system of $n-$*particles*, or bodies, b_i, in terms of *dynamic variables* that are functions of the *generalized position*, q_i , and *momentum*, p_i, *coordinates* using Netwon' s equations applied to such a system with additional system constraints specified as the boundary and initial conditions. The dynamic states of such a classical system are completely specified by an $n-tuple$ of (q_i, p_i) values which define a point in the *configuration space* of the system.

3.1.1. *Equations of Motion for a Closed, Calssical System with a Finite Number of Degrees of Freedom.* The condition for energy conservation in the system is expressed in terms of a Hamiltonian function, $H(q_i, p_i)$, defined as:

$H(q_i; p_i) := \Sigma_{i=1,...,n}(p_i)^2/2m + V_i(q_i)$

where V_i' s are potential energy functions for the particles b_i.

Furthermore, with this state-function, the Lagrangian equations specified next can be derived and expressed in terms of partial derivatives with respect to q_i and p_i; these equations of motion can be shown to govern the dynamics of such a dynamical system.

Classical Lagrangian Mechanics Assume we have a system with holonomic constraints. Holonomic constraints are constraints of the form

$f(r1, r2, r3, ..., t) = 0.$

They reduce the number of degrees of freedom of the system. If the constraints are holonomic, then the forces of constraints do no virtual work. Assume the system has n independent generalized coordinates q_i. Assume that the generalized applied forces,

$Q_j = \Sigma_i F_i \cdot \frac{\partial r_i}{\partial q_j}$

are given by

$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}(\frac{\partial U}{\partial \dot{q_j}})$

with U some scalar function, i.e. the generalized applied forces are derived from a potential. Then the equations of motion may be obtained from the Lagrange equations:

$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q_j}})$ - $(\frac{\partial L}{\partial q_j}) = 0$

where *L*= T- U is *the Lagrangian* of the system. L is a function of the generalized coordinates and momenta.

Define the *generalized momentum* or conjugate momentum (also called 'canonical' momentum) as

$p_j = \frac{\partial L}{\partial \dot{q_j}}$.

If the Lagrangian does not contain a given coordinate q_j then the coordinate is said to be *cyclic* and the corresponding conjugate momentum p_j is conserved.

The Hamiltonian H of a system is given by :

$H\ (q,p,t) = \Sigma_i (p_i \cdot \dot{q_i}) - \mathbf{L}$.

H is , in general, a function of time, the generalized coordinates and momenta of the system.

The *equations of motion* can be obtained from Hamiltons equations,

$\dot{q} = \frac{\partial H}{\partial p_i}$

and

$\dot{p} = \frac{\partial H}{\partial q_i}$

If the generalized coordinates do not explicitly depend on time, then $H = E$, the *total energy* of the system.

The above can also be expressed in terms of a Poisson bracket as will see in more detail below in the quantum algebra subsection

$\{f, g\} := \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p^i}$.

As we shall see in the next sections, even though one can write formally Hamilton's equations of motion for a quantum system in a similar form to those shown above in terms of Hamiltonian derivatives with respect to positions and conjugate momenta, the physical and mathematical meaning is significantly different in the case of quantum dynamics; the Hamiltonian, as well as generalized position and momentum coordinates are *operators* that 'operate' on , or apply to, special functions of complex variables (the 'wave functions' or 'eigenvectors') that have no classical counterpart or a direct, (intuitive) physical picture. Such *wave functions* are subject to specific quantum equations of motion for specific quantum systems with specific boundary and initial conditions. In the simple case of a *stationary* quantum system whose Hamiltonian is independent of time, in the Schrödinger picture, in which the wave functions, $\Psi(q, p; t)$ are certain complex functions of time, one can write the stationary Schrödinger equation:

$H\Psi_i(q, p; t) = E_i \cdot \Psi_i(q, p; t)$

where E_i are the energy 'eigenvalues', or the stationary energy (quantum) levels of the system which are obtained by solving the above Schrödinger equations. Note that neither the stationary nor the more general, dynamic, quantum equations of motion specified next can be obtained, or 'derived', from the classical equations of motion. Moreover, the *quantum Hamiltonian operator* may depend on other dynamic variables than positions and momenta which have no classical analogue (or counterpart), as discussed in more detail with examples in the next subsection. In the general case, when the Hamiltonian depends on time in the Schrödinger representation one can write the quantum dynamic equations of motion of a finite quantum system as:

$$\frac{d\Psi(t)}{dt} = -(i/\hbar) \cdot H(t)\Psi(t),$$

with the required boundary and initial conditions. Furthermore, in either the stationary or dynamic quantum case one can no longer represent the state of the system as in classical dynamics by a 'point' in configuration space, that is completely specify the state by coordinates for all positions and momenta 'for that state'; this is because one cannot simultaneously measure both position and momentum in any quantum system, as measurement of one of the two ('conjugate') dynamic variables drastically alters the other (this will be discussed next in further detail under the 'Heisenberg Uncertainty Principle'). As a consequence, the 'configuration' of a quantum system in a certain quantum state is represented by a complex function in a Hilbert (vector) space, and cannot be specified by a 'point' whose coordinates are generalized positions and momenta as in classical dynamics. The quantum wave function will depend on both spatial coordinates and time, but it will not have a single value determined by any specific pair of position and momentum values, for example. In addition to amplitude (related to the square root of the probability of a quantum event), such a quantum wave function will have a *'phase'* that has no analogue in classical mechanics, but will be thought to play a similar role to that of the phase of a continuous wave, for example, as in classical wave optics/ electromagnetism.

3.1.2. *The Principles of Quantum Mechanics and Selected Examples of their Application to Quantum Systems.* On the other hand, even though the approach to the dynamics of a quantum mechanical system is still based on measurements of dynamical variables, precise values of both position and momentum cannot be simultaneously obtained for a quantum entity, such as an electron in an atom, or any quantum system in general. This fact is expressed as the Heisenberg Uncertainty Principle, which includes, however, not just the position and momentum but any two dynamical variables, or more precisely, observables, that are 'incompatible', in a certain precise, physical sense that can be stated mathematically as a *non-commutativity* relationship. The latter is specified in the next section in terms of non-commuting operators correspondinng to such 'incompatible' observables. As a consequence of this intrinsic measurement problem characteristic of all quantum systems (which will be discussed in further detail in Section 2), one cannot specify any longer a point in the quantum configuration space, or Quantum State Space (QSS), of any quantum system by specifying the position and momentum coordinate values because such values cannot be simultaneously, uniquely and repeatably determined through any experiment. Furthermore, there will be certain dynamic variables in a quantum system that have no classical analogue. An example of such a quantum observable with no classical analogue is the *'spin'* of quantum 'particles', such as: electrons, protons, neutrons, pions, quarks, nuclei, photons and gravitons, etc., which is an *intrinsic* property of such entities that has a *relativistic* origin according to Dirac and the standard model of modern physics (see Weinberg, 2001,[170] for example). In the case of massive quantum 'particles', such as electrons, protons or neutrons, the spin property becomes readily apparent in the dynamics of such quantum entities (quantum 'particles') when they are subjected to an external, static magnetic field. Therefore, such spin effects can be determined experimentally, very accurately, through spectroscopic, resonance measurements that are made in the presence of a magnetic field which is external to such massive quantum 'particles' that also possess an *intrinsic magnetic moment*. Thus, the spin property is directly linked to the presence of an intrinsic magnetic moment which is tiny, but not zero, in such massive, quantum 'particles'. This means also that the nuclei of radioactive isotopes with zero magnetic moment such as $^{12}C,^{16}O$ and ^{36}Cl have zero spin and, therefore, they do not exhibit any resonant absorption of radiofrequency energy for any value of the applied, external magnetic field (Abragam, 1968). An example of a radioactive isotope nucleus that has, however, an appreciable value of the intrinsic magnetic moment is tritium, ^{3}H, and it also has a spin value which is not zero, but 1/2. All quantum 'particles' with spin 1/2 are called *'fermions'*, they are subject to the Pauli exclusion principle, and also obey *Fermi statistics*. On the other hand, all quantum 'particles' with integer spin values are called *'bosons'*, they obey *Bose-Einstein statistics* and are not subject to the Pauli exclusion principle. This means that any number of bosons can occupy the same energy level, therefore leading to the formation at low temperatures of *Bose-Einstein condensates*. The exact opposite is valid for fermions: no two fermions can 'simultaneously' occupy the same energy level in the same quantum system.

In the case of 'massless' quanta ('with zero mass at rest') such as the photons and gravitons, the effects of the spin on dynamics are often determined indirectly through relativistic calculations (e.g., in Quantum Electrodynamics (QED) for the photon, as it will be further detailed in Section 3). Such striking differences between the behavior of fermions and bosons have their origin in the fundamentally *different quantum symmetries* associated with their different spin properties. The behavior and quantum statistics differences are very great between fermions and bosons even though the differences between the spin energy levels of a given fermion, such as the proton, for example, can be tiny, on the order of thermal energy at room temperature. Even more striking is the fact, that in certain systems, such as Type I superconducting materials, two fermions can couple indirectly via phonons forming a stable 'quasi-particle', or fermion-pair, with spin zero (i.e., become 'boson-like'), and therefore escape the Pauli exclusion principle and follow the Bose-Einstein statistics which leads to their condensation at temperatures below a certain critical value, T_c. In the case of electrons in a Type I superconductor, such electron pairs are called *'Cooper-pairs'*, and their formation changes the form of the quantum Hamiltonian and dynamic equations for superconductors as in the BCS theory (*the Bardeen–Cooper–Schaeffer* (1958x), (ref. [?]) mechanism of superconductivity). Furthermore, the underlying change in the quantum symmetry associated with the spin property in superconductors, called *global 'spontaneous symmetry breaking'*, is essential to understanding the mechanism of superconductivity (p.x12 of vol. 3, in Weinberg, 2001). Global, as well as dynamic, *spontaneous symmetry breaking is a fundamental process* of a much wider significance than superconductivity, that has a great many applications in modern theoretical physics, a variety of which are elegantly and clearly presented by Weinberg in a recent, 1,500–page textbook in three volumes (Weinberg, 2001; ref. [170]). Perhaps, the most significant of its consequences is *the electroweak unification theory* involving at least $SU(2) \times U(1)$ *symmetry breaking*, as explained in detail by Weinberg (Ch. 6 in vol.2 of ref. [170]). *SU(2)* is the standard notation for a special type of a very important symmetry group in QM and high-energy physics that has a representation in terms of 'special' 2×2 matrices called *Pauli, or spin, matrices*, as will be further specified in *Example 3.2.1*, and also at the end of Section 6; (in the latter section, we will discuss *supersymmetry* and its role(s) in *supergravity theories* (Ch. 9x in vol.3 of Weinberg, 2001, ref. [170]). Weinberg's electroweak unified theory also predicts the appearance (including absorption and emission) of so called 'Goldstone' bosons (which are defined as massless particles, often also with a zero spin), associated with the presence of global symmetry breaking. Other spin-related symmetry differences are between photons and gravitons that lead to quite markedly different behaviors in terms of couplings with matter between the electromagnetic and gravitational fields, as well as their generation and propagation through either matter or space-time. Lastly, according to GR, supported by several experimental verifications, intense graviton fields 'distort' the geometry of space-time whereas photons are thought to follow geodesics in space-time (which are determined by the gravitational fields present). gravitational fields is one such effect.

With the exception of observables that have no classical analogue, such as spin, one might expect to be able to 'recover' from Quantum Mechanics– artificially extrapolated– in the limit of the Planck constant $h \longrightarrow 0$, the classical dynamics for simple, finite systems based on either Newtonian dynamics/Galilean relativity or classical (special) relativistic dynamics. This is often called *'the Correspondence Principle'* and is quite helpful in gaining a simple understanding of quantum systems that have a classical analogue, such as is the case of the Harmonic Oscillator, a favorite test system for many a quantum development. On the other hand, it seems that the classical equations of motion are consistent with a 'correspondence principle' between classical and quantum dynamics only when Cartesian coordinate systems are employed to make the 'transfer(s)' between the two domains. Furthermore, a naive 'quantization' attempt that would simply multiply the classical Poisson bracket by the constant factor $(i/\hbar)$ would not allow one to produce the correct form of the quantum Hamiltonian. The imposition of further constraints such as boundary and initial conditions, plus a precise (and sophisticated, analytic quantization) procedure is essential to obtaining the correct form of the quantum Hamiltonian (p.x23 in Weinberg, 2001). Such quantization procedures will also be discussed in general in subsection 3.3.

3.2. **Quantum Lorentz Transformations.** [170](Weinberg,vol.2)

3.3. **Symmetry Groups, Quantum Group Algebras (Hopf Algebras) and Other Operator Algebras.** We shall proceed in this section to define the concept of Hopf algebra which is essential to building quantum group algebras or 'quantum groups'. Quantization procedures are then introduced either as a tool for generating quantized spaces with non–commutative geometry through 'Heisenberg deformations' (Connes,1994), or in a generalized form that involves the tangent groupoids of suitable topological spaces that are then quantized using asymptotic morphisms. Several basic concepts of Quantum Algebra are here defined, such as the C*–algebra, which has both an algebraic and a topological structure generated by the properties of quantum operators over a Hilbert space of quantum states. Quantum logics are then linked to Jordan–Banach (JB) and Jordan–Lie algebras, as well as the structure of the quantum 'configuration' space. We also discuss how a C*–Clifford Algebra associated with any given (quantum) Hilbert space is generated beginning with the canonical representation of bounded linear operators on the Fock space (Plymen and Robinson, 1994, ref. [**?**]). This construction seems to be limited however to the free field representations. Generalizations of the concept of quantum group, in the form of quantum topological groupoids, are then introduced together with their associated Quantum Groupoid C*–algebra (GCA), their continuous form (Lie–algebroids) and the (compact) quantum metric spaces.

3.3.1. *Quantum Group (Hopf) Algebras.* We commence here by establishing the concept of Hopf algebras which are the fundamental building blocks of quantum group algebras or 'quantum groups';. For further details we refer to Chaician and Demichev (1996),and also Magid (1995).

Firstly, a unital associative algebra consists of a linear space A together with two linear maps

$$\begin{aligned} m &: A \otimes A \longrightarrow A \ , \ (\textit{multiplication}) \\ \eta &: \mathbb{C} \longrightarrow A \ , \ (\textit{unity}) \end{aligned}$$

satisfying the conditions

$$\begin{aligned} m(m \otimes \mathbf{1}) &= m(\mathbf{1} \otimes m) \\ m(\mathbf{1} \otimes \eta) &= m \\ (\eta \otimes \mathbf{1}) &= \mathrm{id} \ . \end{aligned}$$

This first condition can be seen in terms of a commuting diagram :

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes \mathrm{id}} & A \otimes A \\ {\scriptstyle \mathrm{id} \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{m} & A \end{array}$$

Next suppose we consider 'reversing the arrows', and take an algebra A equipped with a linear homorphisms $\Delta : A \longrightarrow A \otimes A$, satisfying, for $a, b \in A$:

$$\Delta(ab) = \Delta(a)\Delta(b)$$

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta \ .$$

We call Δ a *comultiplication*, which is said to be *coassociative* in so far that the diagram

$$\begin{array}{ccc} A \otimes A \otimes A & \xleftarrow{\Delta\otimes \mathrm{id}} & A \otimes A \\ {\scriptstyle \mathrm{id}\otimes\Delta}\uparrow & & \uparrow{\scriptstyle \Delta} \\ A \otimes A & \xleftarrow{\Delta} & A \end{array}$$

commutes. There is also a counterpart to η, the *counity* map

$\varepsilon : A \longrightarrow \mathbb{C}$ satisfying

$(\mathrm{id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$.

A *bialgebra* $(A, m, \Delta, \eta, \varepsilon)$ is a linear space A with maps $m, \Delta, \eta, \varepsilon$ satisfying the above properties.

Now to recover anything resembling a group structure, we must append such a bialgebra with an antihomomorphism $S : A \longrightarrow A$, satisfying $S(ab) = S(b)S(a)$, for $a, b \in A$. This map is defined implicitly via the property : $m(S \otimes \mathrm{id}) \circ \Delta = m(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \varepsilon$.

We call S the *antipode map* .

A *Hopf algebra* is a bialgebra $(A, m, \eta, \Delta, \varepsilon)$ equipped with an antipode map S .

Commutative and non–commutative Hopf algebras form the backbone of quantum group algebraic representations and they are also essential to the generalizations of the key concept of *symmetry*. Indeed, in many respects a 'quantum group' is identifiable with a Hopf algebra where an algebra duality provides one with a concept of symmetry between observables and states. When such algebras are associated to *matrix groups* there is considerable scope for *representations on both finite and infinite dimensional Hilbert spaces*. Analogous to how quantum mechanics relates to the classical limit according to the correspondence principle discussed above, 'quantum groups' in quantum physics can be seen to correspond to Lie groups in the classical context. Furthermore, the corresponding Lie algebras of certain Lie groups– when expressed in terms of quantum commutators– can provide algebraic solutions to specific quantum problems in physics or chemistry. On the other hand, the mainstream applications of Hopf algebras are directed towards such fields as quantum statistical mechanics, conformal field theory, and the theory of knots and braids. Very similar to the differential geometry of non–commutative Lie groups, 'quantum groups' have their own intrinsic notions of connection and curvature.

Example 3.3.1: The SU(2) 'Quantum' Group

Let us consider the structure of the ubiquitous 'quantum' SU(2) group (Woronowicz 1987, Chaician and Demichev 1996). Here A is taken to be a C*–algebra generated by elements α and β subject to the relations

$$\alpha\alpha^* + \mu^2\beta\beta^* = 1 \ , \ \alpha^*\alpha + \beta^*\beta = 1 \ ,$$
$$\beta\beta^* = \beta^*\beta \ , \ \alpha\beta = \mu\beta\alpha \ , \ \alpha\beta^* = \mu\beta^*\alpha \ ,$$
$$\alpha^*\beta = \mu^{-1}\beta\alpha^* \ , \ \alpha^*\beta^* = \mu^{-1}\beta^*\alpha^* \ ,$$

where $\mu \in [-1, 1]\backslash\{0\}$. In terms of the matrix

$$u = \begin{bmatrix} \alpha & -\mu\beta^* \\ \beta & \alpha^* \end{bmatrix}$$

the coproduct Δ is then given via $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$.

3.3.2. *Jordan–Banach and JBL algebras.* Our development here follows Alfsen and Schultz (2003), Landsman (1998). Firstly, an *algebra* consists of a vector space E over a ground field (typically $\mathbb{R}$ or $\mathbb{C}$) equipped with a bilinear and distributive multiplication $\circ$. Note that E is not necessarily commutative or associative.

A *Jordan algebra* (over $\mathbb{R}$), is an algebra over $\mathbb{R}$ for which

$$S \circ T = T \circ S \ ,$$
$$S \circ (T \circ S^2) = (S \circ T) \circ S^2$$

for all elements S, T of the algebra.

It is worthwhile remarking now that in the algebraic theory of Jordan algebras, an important role is played by the
Jordan triple product $\{STW\}$ as defined by:

$$\{STW\} = (S \circ T) \circ W + (T \circ W) \circ S - (S \circ W) \circ T \ ,$$

which is linear in each factor and for which $\{STW\} = \{WTS\}$. Certain examples entail setting $\{STW\} = \frac{1}{2}\{STW + WTS\}$.

A *Jordan Lie algebra* is a real vector space $\mathfrak{A}_{\mathbb{R}}$ together with a *Jordan product* $\circ$ and *Poisson bracket*

$\{\ ,\ \}$, satisfying :

1. for all $S, T \in \mathfrak{A}_{\mathbb{R}}$, $\begin{array}{c} S \circ T = T \circ S \\ \{S, T\} = -\{T, S\} \end{array}$
2. the *Leibniz rule* holds

   $\{S, T \circ W\} = \{S, T\} \circ W + T \circ \{S, W\}$ for all $S, T, W \in \mathfrak{A}_{\mathbb{R}}$, along with
3. the *Jacobi identity* :

   $\{S, \{T, W\}\} = \{\{S, T\}, W\} + \{T, \{S, W\}\}$
4. for some $\hbar^2 \in \mathbb{R}$, there is the *associator identity* :

   $(S \circ T) \circ W - S \circ (T \circ W) = \frac{1}{4}\hbar^2\{\{S, W\}, T\}$.

3.3.3. *Poisson algebra.* By a *Poisson algebra* we mean a Jordan algebra in which $\circ$ is associative. The usual algebraic types of morphisms automorphism, isomorphism, etc.) apply to Jordan Lie (Poisson) algebras (see Landsman, 2003).

Consider the classical configuration space $Q = \mathbb{R}^3$ of a moving particle whose phase space is the cotangent bundle $T^*\mathbb{R}^3 \cong \mathbb{R}^6$, and for which the space of (classical) observables is taken to be the real vector space of smooth functions

$\mathfrak{A}^0_{\mathbb{R}} = C^\infty(T^*R^3, \mathbb{R})$. The usual pointwise multiplication of functions fg defines a bilinear map on $\mathfrak{A}^0_{\mathbb{R}}$, which is seen to be commutative and associative. Further, the Poisson bracket on functions

$\{f, g\} := \frac{\partial f}{\partial p^i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p^i}$,

which can be easily seen to satisfy the Liebniz rule above. The axioms above then set the stage of passage to quantum mechanical systems which the parameter $\hbar^2$ suggests.

3.3.4. *C^*–algebras (C^*–A), JLB and JBW Algebras.* An *involution* on a complex algebra $\mathfrak{A}$ is a real–linear map $T \mapsto T^*$ such that for all

$S, T \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$, we have $T^{**} = T$, $(ST)^* = T^*S^*$, $(\lambda T)^* = \bar{\lambda}T^*$.

A **–algebra* is said to be a complex associative algebra together with an involution $*$.

A *C^*–algebra* is a simultaneously a *–algebra and a Banach space $\mathfrak{A}$, satisfying for all $S, T \in \mathfrak{A}$:

$\|S \circ T\| \leqslant \|S\| \; \|T\|$,
$\|T^*T\|^2 = \|T\|^2$.

We can easily see that $\|A^*\| = \|A\|$. By the above axioms a C*–algebra is a special case of a Banach algebra where the latter requires the above norm property but not the involution (*) property. Given Banach spaces E, F the space $\mathcal{L}(E, F)$ of (bounded) linear operators from E to F forms a Banach space, where for $E = F$, the space $\mathcal{L}(E) = \mathcal{L}(E, E)$ is a Banach algebra with respect to the norm

$$\|T\| := \sup\{\|Tu\| : u \in E \ , \ \|u\| = 1\} \ .$$

In quantum field theory one may start with a Hilbert space H, and consider the Banach algebra of bounded linear operators $\mathcal{L}(H)$ which given to be closed under the usual algebraic operations and taking adjoints, forms a $*$–algebra of bounded operators, where the adjoint operation functions as the involution, and for $T \in \mathcal{L}(H)$ we have :

$\|T\| := \sup\{(Tu, Tu) : u \in H \ , \ (u, u) = 1\}$, and $\|Tu\|^2 = (Tu, Tu) = (u, T^*Tu) \leqslant \|T^*T\| \; \|u\|^2$.

By a morphism between C*–algebras $\mathfrak{A}, \mathfrak{B}$ we mean a linear map $\phi : \mathfrak{A} \longrightarrow \mathfrak{B}$, such that for all $S, T \in \mathfrak{A}$, the following hold :

$$\phi(ST) = \phi(S)\phi(T) \ , \ \phi(T^*) = \phi(T)^* \ ,$$

where a bijective morphism is said to be an isomorphism (in which case it is then an isometry). A fundamental relation is that any norm-closed $*$–algebra $\mathcal{A}$ in $\mathcal{L}(H)$ is a C*–algebra, and conversely, any C*–algebra is isomorphic to a norm–closed $*$–algebra in $\mathcal{L}(H)$ for some Hilbert space H .

For a C*–algebra $\mathfrak{A}$, we say that $T \in \mathfrak{A}$ is *self–adjoint* if $T = T^*$. Accordingly, the self–adjoint part $\mathfrak{A}^{sa}$ of $\mathfrak{A}$ is a real vector space since we can decompose $T \in \mathfrak{A}^{sa}$ as :

$$T = T' + T'' := \tfrac{1}{2}(T + T^*) + \iota(\tfrac{-\iota}{2})(T - T^*) \ .$$

A *commutative* C*–algebra is one for which the associative multiplication is commutative. Given a commutative C*–algebra $\mathfrak{A}$, we have $\mathfrak{A} \cong C(Y)$, the algebra of continuous functions on a compact Hausdorff space Y .

A *Jordan–Banach algebra* (a JB–algebra for short) is both a real Jordan algebra and a Banach space, where for all $S, T \in \mathfrak{A}_{\mathbb{R}}$, we have

$\|S \circ T\| \leqslant \|S\| \; \|T\|$,
$\|T\|^2 \leqslant \|S^2 + T^2\|$.

A *JLB–algebra* is a JB–algebra $\mathfrak{A}_{\mathbb{R}}$ together with a Poisson bracket for which it becomes a Jordan–Lie algebra for some $\hbar^2 \geqslant 0$. Such JLB–algebras often constitute the real part of several widely studied complex associative algebras.

For the purpose of quantization, there are fundamental relations between $\mathfrak{A}^{sa}$, JLB and Poisson algebras. In fact, if $\mathfrak{A}$ is a C*–algebra and $\hbar \in \mathbb{R}/0$, then $\mathfrak{A}^{sa}$ is a JLB–algebra when it takes its norm from $\mathfrak{A}$ and is equipped with the operations :

$$S \circ T := \frac{1}{2}(ST + TS) \ , \ \{S,T\}_{\hbar} \ := \frac{\iota}{\hbar}[S,T] \ .$$

Conversely, given a JLB–algebra $\mathfrak{A}_{\mathbb{R}}$ with $\hbar^2 \geqslant 0$, its complexification $\mathfrak{A}$ is a C*–algebra under the operations :

$$\begin{aligned} ST &:= S \circ T - \frac{\iota}{2}\hbar\{S,T\} \ , \\ (S + \iota T)^* &:= S - \iota T \ . \end{aligned}$$

For further details see Landsman (2003) (Thm. 1.1.9).

A JB–algebra which is monotone complete and admits a separating set of normal sets is called a JBW–algebra. These appeared in the work of von Neumann who developed a (orthomodular) lattice theory of projections on $\mathcal{L}(H)$ on which to study quantum logic (see later). BW–algebras have the following property: whereas $\mathfrak{A}^{sa}$ is a J(L)B–algebra, the self adjoint part of a von Neumann algebra is a JBW–algebra.

A *JC–algebra* is a norm closed real linear subspace of $\mathcal{L}(H)^{sa}$ which is closed under the bilinear product $S \circ T = \frac{1}{2}(ST + TS)$ (non–commutative and nonassociative). Since any norm closed Jordan subalgebra of $\mathcal{L}(H)^{sa}$ is a JB–algebra, it is natural to specify the exact relationship between JB and JC–algebras, at least in finite dimensions. In order to do this, one introduces the 'exceptional' algebra $H_3(\mathbb{O})$, the algebra of 3×3 Hermitian matrices with values in the octonians $\mathbb{O}$. Then a finite dimensional JB–algebra is a JC–algebra if and only if it does not contain $H_3(\mathbb{O})$ as a (direct) summand [3].

3.3.5. *Reversibility.* Given the relationship between the self–adjoint part of an associative *–algebra and Jordan algebras, we mention here the criteria for a JC–algebra to be the self–adjoint part of the real *–algebra it generates.

A Jordan subalgebra $\mathfrak{A}$ of an associative *–algebra is said to be *reversible* if for all $n > 0$, $a_1, a_2, \ldots, a_n \in \mathfrak{A} \implies a_1 a_2 \cdots a_n + a_n \cdots a_2 a_1 \in \mathfrak{A}$.

This condition can be seen to hold for $n = 2$, as it does for $n = 3$ by virtue of the Jordan triple product in a special Jordan algebra :

$$a_1 a_2 a_3 + a_3 a_2 a_1 = 2\{a_1 a_2 a_3\} \ .$$

However, the condition fails for $n \geqslant 4$. Further criteria for reversibility are given by the following. If $\mathfrak{A} \subseteq \mathcal{L}(H)^{sa}$ is a (concrete) JC–algebra, let $R_0(\mathfrak{A})$ be the real subalgebra generated by $\mathfrak{A}$ in $\mathcal{L}(H)$, and let $R(\mathfrak{A})$ be the norm closure of $R_O(\mathfrak{A})$ and $\overline{R(\mathfrak{A})}$ the σ–weak closure. Then following Alfsen and Schultz ref. [?, ?, AS] one has

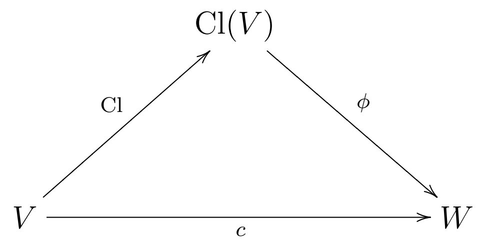
1. $\mathfrak{A}$ is reversible if and only if
   $\mathfrak{A} = R_0(\mathfrak{A})^{sa}$.
2. If $\mathfrak{A}$ is a reversible JC–subalgebra of
   $\mathcal{L}(H)^{sa}$, then $\mathfrak{A} = R(\mathfrak{A})^{sa}$.
3. If in addition to 2. above, $\mathfrak{A}$ is σ–weakly closed, then $\mathfrak{A} = \overline{R(\mathfrak{A})}^{sa}$ (in the σ–weak closure).

A JC–algebra $\mathfrak{A}$ is said to be *universally reversible* if for every faithful representation $\pi : \longrightarrow \mathcal{L}(H)^{sa}$ we have $\pi(\mathfrak{A})$ is reversible. Consequently, it is shown in ref. [3] that the self–adjoint part of every C*–algebra is universally reversible (Prop.4.34).

3.3.6. *C*-Category of Intertwiners.* Returning to C*–algebras and their representations p* one may wish to 'compare', or transform, different representations of C*–algebras. A morphism between two such representations of C*-algebras is called an *intertwiner.* The category of C*–algebra representations as objects and with intertwiners as morphisms is called the *C*–category.* Furthermore, one can define a tensor product of such C*-algebra representations and introduce additional structure to a C*–category, thus forming a *tensor C*–category* (Roberts, 2004).

3.3.7. *The Non–Commutative Quantum Observable Algebra –A Clifford Algebra.* In view of this last observation, let us make a short digression and recall the notion of a *Clifford algebra.* Consider a pair (V, Q), where V denotes a real vector space and Q is a quadratic form on V . *The Clifford algebra associated to* V denoted $\mathrm{Cl}(V) = \mathrm{Cl}(V, Q)$, is the algebra over $\mathbb{R}$ generated by V, where for all $v, w \in V$, the relations $v \cdot w + w \cdot v = -2Q(v, w)$, are satisfied; in particular,
$v^2 = -2Q(v, v)$.

If W is an algebra and $c : V \longrightarrow W$ is a linear map satisfying $c(w)c(v) + c(v)c(w) = -2Q(v, w)$, then there exists a unique algebra homomorphism $\phi : \mathrm{Cl}(V) \longrightarrow W$ such that the diagram



commutes. It is in this sense that Cl(V) is considered to be 'universal'.

For a given Hilbert space H, there is an associated *C^*–Clifford algebra* Cl[H] which admits a canonical representation on $\mathcal{L}(\mathbb{F}(H))$ the bounded linear operators on the Fock space $\mathbb{F}(H)$ of H as in Plymen and Robinson (1994), and hence we a have a natural sequence of maps $H \longrightarrow \mathrm{Cl}[H] \longrightarrow \mathcal{L}(\mathbb{F}(H))$.

3.4. **General Quantization Procedures: The First Quantization.** The initial quantization procedures invoked either the correspondence principle by substitution of a 'quantum Poisson bracket' containing a factor i/h, (where **h** is the Planck action constant) for a classical Poisson bracket, or the application of the Heinsenberg Uncertainty Principle in the form of a commutator bracket between non-commuting, quantum observable operators such as position and 'linear' momentum.We have already discussed above alsom the severe limitations and drawbacks of such an over-simplified approach. Therefore, more involved algebraic, as well as analytic, quantization procedures were developed; the analytic ones being numerically computable are of substantial interest. One other governing 'principle' of quantization involves 'deforming', in a certain way, an algebra of functions on a phase space to an algebra of operator kernels.

3.4.1. *Wigner–Weyl–Moyal Quantization Procedures.* The more general techniques revolve around using such operator kernels in representing *asymptotic morphisms.* A fundamental example is an *asymptotic morphism* $C_0(T^*\mathbb{R}^n) \longrightarrow \mathcal{K}(L^2(\mathbb{R}^n))$ as expressed by the *Moyal deformation* :

$[T_\hbar(a)f](x) := \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} a(\frac{x+y}{2}, \xi) \exp[\frac{\iota}{\hbar}] f(y)\, dy\, d\xi$, where $a \in C_0(T^*\mathbb{R}^n)$ and the operators $T_\hbar(a)$ are of trace class. In Connes (1994), it is called the *Heisenberg deformation.*

An elegant way of generalizing this construction entails introducing *the tangent groupoid* $\mathcal{T}X$ of a suitable space X and using asymptotic morphisms. Putting aside a number of technical details which can be found in Connes (1994) or Landsman (1998), the tangent groupoid $\mathcal{T}X$ is defined as the normal groupoid of a pair Lie groupoid $X \times X \rightrightarrows X$ obtained by 'blowing up' the diagonal $diag(X)$ in X (we will recall for the reader's benefit the concept of a 'groupoid' at a later stage). More specifically, if X is a (smooth) manifold let $G' = X \times X \times (0,1]$ and $G'' = TX$, from which it can be seen $diag(G') = X \times (0,1]$ and $diag(G'') = X$. Then in terms of disjoint unions we have

$$\begin{aligned} \mathcal{T}X &= G' \bigvee G'' \\ diag(\mathcal{T}X) &= diag(G') \bigvee diag(G'') \ . \end{aligned}$$

In this way $\mathcal{T}X$ shapes up both as a smooth groupoid, as well as a manifold with boundary.

Quantization relative to $\mathcal{T}X$ is outlined by Várilly (1997) to which we refer for details. The procedure entails characterizing a function on $\mathcal{T}X$ in terms of a pair of functions on G' and G'' respectively, the first of which will be a kernel and the second will be the inverse Fourier transform of a function defined on T^*X . It will be instructive to consider the case

$X = \mathbb{R}^n$ as a suitable example. So we take a function $a(x,\xi)$ on $T^*\mathbb{R}^n$ whose inverse Fourier transform

$\mathcal{F}^{-1}(a(u,v)) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \exp[\iota\xi v]a(u,\xi)\ d\xi$,thus yields a function on $T\mathbb{R}^n$. Consider next the terms

$$x := \exp_u[\tfrac{1}{2}\hbar v] = u + \tfrac{1}{2}\hbar v\ ,\ y := \exp_u[-\tfrac{1}{2}\hbar v] = u - \tfrac{1}{2}\hbar v\ ,$$

which on solving leads to $u = \frac{1}{2}(x+y)$ and $v = \frac{1}{\hbar}(x-y)$. Then the following family of operator kernels

$$k_a(x,y,\hbar) := \hbar^{-n}\mathcal{F}^{-1}a(u,v) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} a(\tfrac{x+y}{2},\xi)\exp[\tfrac{\iota}{\hbar}(x-y)\xi]\ a(u,\xi)\ d\xi\ ,$$

realize the Moyal quantization.

This mechanism can be generalized to quantize any function on T^*X when X is a Riemannian manifold, and produces an asymptotic morphism $C_c^\infty(T^*X) \longrightarrow \mathcal{K}(L^2(X))$. Furthermore, there is the corresponding K—theory map $K^0(T^*X) \longrightarrow \mathbb{Z}$, which is the analytic index map of Atiyah–Singer (see Berline at al., 1991, Connes, 1994). As an example, suppose X is an even dimensional spin manifold together with a 'prequantum' line bundle $L \longrightarrow X$. Then we have a twisted Dirac operator D_L and a 'virtual' Hilbert space given by the index of D_L : $\mathrm{Ind}(D_L) = \ker D_L^+ - \ker D_L^-$.

3.4.2. *Asymptotic Morphisms.* We describe here the important notion of an *asymptotic morphsim* following Connes (1994). Suppose we have two C*–algebras (see below) $\mathfrak{A}$ and $\mathfrak{B}$, together with a continuous field $(\mathfrak{A}(t),\Gamma)$ of C*–algebras over $[0,1]$ whose fiber at 0 is $\mathfrak{A}(0) = \mathfrak{A}$,and whose restriction to $(0,1]$ is the constant field with fiber $\mathfrak{A}(t) = \mathfrak{B}$, for $t > 0$. This may be called a *strong deformation* from $\mathfrak{A}$ to $\mathfrak{B}$.

For any $a \in \mathfrak{A} = \mathfrak{A}(0)$, it can be shown that there exists a continuous section $\alpha \in \Gamma$ of the above field satisfying $\alpha(0) = a$. Choosing such an $\alpha = \alpha_a$ for each $a \in \mathfrak{A}$, we set $\varphi_t(a) = \alpha_a(\frac{1}{t}) \in \mathfrak{B}$, for all $t \in [1,\infty)$.

Given the continuity of norm $\|\alpha(t)\|$ for any continuous section $\alpha \in \Gamma$, consider the following conditions :

(1) For any $a \in \mathfrak{A}$, the map $t \to \varphi_t(a)$ is norm continuous.

(2) For any $a, b \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$, we have
$$\lim_{t\to\infty}(\varphi_t(a) + \lambda\varphi_t(b) - \varphi_t(a+\lambda b)) = 0$$
$$\lim_{t\to\infty}(\varphi_t(ab) - \varphi_t(a)\varphi_t(b)) = 0$$
$$\lim_{t\to\infty}(\varphi_t(a^*) - \varphi_t(a)^*) = 0\ .$$

Then an *asymptotic morphism from* $\mathfrak{A}$ *to* $\mathfrak{B}$ is given by a family of maps $\{\varphi_t\}, t \in [1, \infty)$, from $\mathfrak{A}$ to $\mathfrak{B}$ satisfying conditions (1) and (2) above.

3.5. **Quantum Problem Solving by Algebraic Methods: The Finite Lie 'Algebra' of Quantum Commutators and the corresponding, unique (*continuous*) Lie Groups. Applications to Chemical, Quantum–Molecular Structure and Quantum Physics.** As we have already discussed in Section 1, one often deals with continuity and continuous transformations in natural systems, be they physical, chemical or self-organizing. Such continuous 'symmetries' often have a special type of underlying continuous group, called a *Lie group*. The *Lie group* is generally defined as a (continuous, i.e.,usually infinite) group defined compatibly over a C^∞ manifold, $\mathbf{M}$. Such a *globally smooth* structure is surprisingly simple in two ways: it always admits an *Abelian fundamental group*, and seemingly also related to this global property, it admits an associated, *unique*– as well as *finite*– *'Lie algebra'* that completely specifies *locally* the properties of the Lie group everywhere. Therefore, Lie 'algebras' can greatly simplify quantum computations and the initial problem of defining the form and symmetry of the quantum Hamiltonian subject to boundary and initial conditions in the quantum system under consideration. However, unlike most regular abstract algebras, a Lie 'algebra' is *not associative*, and it is in fact a *vector space* (ref.[110]). It is also perhaps this feature that makes the Lie algebras somewhat compatible, or 'consistent', with quantum logics that are also thought to have *non–asociative, non–distributive and non–commutative lattice* structures. Nevertheless, the need for 'quantizing' Lie algebras in the sense of a certain *non-commutative* 'deformation' apparently remains for a quantum system , especially if one starts with a 'classical' *Poisson* algebra (ref. [128]). This requirement remains apparently even for the generalized version of a Lie algebra, called a *Lie algebroid* (see its definition and related remarks in Section 5.9).

The presence of Lie groups in many classical physics problems, in view of its essential *continuity* property and its *Abelian* fundamental group, is not surprising. What is surprising, however, at first sight, is the appearance of *Lie groups and Lie 'algebras'* in the context of commutators of observable operators even in quantum systems *with no classical analogue observables such as the spin*, as– for example– the SU(2) and its corresponding, unique su(2)–algebra (*see also Example 3.3.1* above). As a result of quantization, one would expect to deal with an algebra such as the Hopf (quantum group) which is *associative*. On the other hand, the application of the correspondence principle to the simple, classical harmonic oscillator system leads to a quantized harmonic oscillator and remarkably simple *commutator* algebraic expressions, which corresponds precisely to the definition of a Lie 'algebra'. Furthermore, this (*Lie*) algebraic procedure of assembling the quantum Hamiltonian from simple observable operator commutators is readily extended to *coupled, quantum harmonic oscillators*, as shown in great detail in ref. [100].

3.5.1. *The Lie Algebra of a Quantum Harmonic Oscillator.* One wishes to solve the time-independent Schrödinger equations of motion in order to determine the stationary states of the quantum harmonic oscillator which has a quantum Hamiltonian of the form:

$$\mathbf{H} = (\tfrac{1}{2m}) \cdot P^2 + \tfrac{k}{2} \cdot X^2,$$

where X and P are, respectively, the coordinate and conjugate momentum operators. X and P satisfy the Heisenberg commutation/'uncertainty' relations:

$[X, P] = i\hbar I$,

where the identity operator I is employed to simplify notation.

A simpler, equivalent form of the above Hamiltonian is obtained by defining physically dimensionless coordinate and momentum:

$\mathbf{x} = (\frac{X}{\alpha}), \mathbf{p} = (\frac{\alpha P}{\hbar})$ and $\alpha = \sqrt{\frac{\hbar}{\sqrt{mk}}}$. With these new dimensionless operators , **x** and **p**, the quantum Hamiltonian takes the form:

$\mathbf{H} = (\frac{\hbar\omega}{2}) \cdot (\mathbf{p}^2 + \mathbf{x}^2)$, which in units of $\hbar \cdot \omega$ is simply: $\mathbf{H'} = (\frac{1}{2}) \cdot (\mathbf{p}^2 + \mathbf{x}^2)$.

The commutator of **x** with its conjugate operator **p** is simply

$[\mathbf{x}, \mathbf{p}] = i$.

Next one defines the superoperators $S_{Hx} = [H, x] = -i \cdot p$, and $S_{Hp} =[H, p] =$ i $\cdot x$,

that will lead to new operators that act as generators of a Lie algebra for this quantum harmonic oscillator. The eigenvectors Z of these superoperators are obtained by solving the equation

$S_H \cdot Z = \zeta Z$, where ζ are the eigenvalues, and Z can be written as $(c_1 \cdot x + c_2 \cdot p)$. The solutions are

$\zeta = \pm 1$ and $c_2 = \mp i \cdot c_1$. Therefore, the two eigenvectors of S_H can be written as :

$a^\dagger = c_1 * (x - ip)$ and $a = c_1(x + ip)$, respectively for $\zeta = 1$ and $\zeta = -1$. For c $_1 = \sqrt{2}$ one obtains normalized operators H, a and $a\dagger$ that generate a 4–dimensional Lie algebra with commutators:

$[H, a] = -a$, $[H, a^\dagger] = a^\dagger$ and $[a, a^\dagger] = I$.

a is called the *annihilation* operator and $a\dagger$ is called the *creation* operator. This Lie algebra is solvable and generates after repeated application of $a\dagger$ all the eigenvectors of the quantum harmonic oscillator:

$\Phi_n = (\frac{(a\dagger)^n}{\sqrt{(n!)}}) \cdot \Phi_0$.

The corresponding, possible eigenvalues for the energy, derived then as solutions of the Schrödinger equations for the quantum harmonic oscillator are: $E_n = \hbar \cdot \omega(n + \frac{1}{2})$, where $n = 0, 1, ..., N$.

The position and momentum eigenvector coordinates can be then also computed by iteration from *(finite)* matrix representations of the *(finite)* Lie algebra, using, for example, a simple computer programme to calculate linear expressions of the annihilation and creation operators. For example, one can show analytically that:

$$[a, x^k] = (\frac{k}{\sqrt{2}}) \cdot (x_{k-1}).$$

One can also show by introducing a *coordinate* representation that the eigenvectors of the harmonic oscillator can be expressed as *Hermite polynomials* in terms of the coordinates. In the coordinate representation the quantum *Hamiltonian* and *bosonic* operators have, respectively, the simple expressions: $H = (\frac{1}{2}) \cdot [-\frac{d^2}{dx^2}) + (x^2)]$,

$$a = (\frac{1}{\sqrt{2}}) \cdot (x + \frac{d}{dx}) \text{ and } a\dagger = (\frac{1}{\sqrt{2}}) \cdot (x - \frac{d}{dx}).$$

The ground state eigenfunction normalized to unity is obtained from solving the simple first-order differential equation $a\Phi_0(x) = 0$ and which leads to the expression:

$$\Phi_0(x) = \pi^{(\frac{-1}{4})} \cdot exp^{(-\frac{x^2}{2})}.$$

By repeated application of the creation operator written as

$$a\dagger = (-\frac{1}{\sqrt{2}}) \cdot (exp(\frac{x^2}{2})) \cdot (\frac{d}{dx^2}) \cdot exp(-\frac{x^2}{2}),$$

one obtains the n-th level eigenfunction:

$$\Phi_n(x) = (\frac{1}{\sqrt{(\sqrt{\pi})2^n n!)}}) \cdot (\mathbf{He}_n(x)) \;,$$

where $\mathbf{He}_n(x)$ is *the Hermite polynomial* of order n.

With the special generating function of the Hermite polynomials

$$F(t, x) = (\pi^{-\frac{1}{4}}) \cdot (exp((-\frac{x^2}{2}) + tx - (\frac{t^2}{4})))$$

one obtains explicit analytical relations bewtteen the eigenfunctions of the quantum harmonic oscillator and the above special generating function:

$$F(t, x) = \Sigma_{n=0}(\frac{t^n}{\sqrt{(2^n \cdot n!)}}) \cdot \Phi_n(x).$$

Such applications of the Lie algebra, and the related algebra of the *bosonic* operators as defined above are quite numerous in theoretical physics, and especially for various quantum field carriers in QFT that are all *bosons.* (Please note also additional examples of special Lie algebras for gravitational and other fields in Sections 4 throuh 6, such as gravitons and Goldstone quanta that are all *bosons of different spin* and 'Penrose' *homogeneity*).

In the interesting case of a *two-mode* bosonic quantum system formed by the tensor (direct) product of *one-mode* bosonic states: $\mid m, n >:=\mid m > \otimes \mid n >$, one can generate a 3-dimensional Lie algebra in terms of *Casimir* operators. *Finite*– dimensional Lie algebras are far more tractable, or easier to compute, than those with an infinite basis set. For example, such a Lie algebra as the 3-dim one considered above for the two-mode, bosonic states is quite useful for numerical computations of vibrational (IR, Raman, etc.) spectra of two-mode, *diatomic* molecules, as well as the computation of scattering states. Other perturbative calculations for more complex quantum systems, as well as calculations of exact solutions by means of Lie algebras for quantum systems are, respectively, described in detail in Chs. 6 and 9 of ref. [100].

3.6. **Extensions of Quantum Mechanics to Infinite Systems and Electrodynamics.** In Quantum Algebraic Topology one of the most promising approachs consists in linking *local to global* space properties as discussed above and also in the subsequent sections. Mathematically rigorous developments are also involved in one of the areas of QAT often called Local Quantum Physics or Algebraic Quantum Field Theory (AQFT), to be discussed in the sequel. Its results will be later compared in the Lattice QFT (LQFT) section with those rapidly emerging from digital computations of QFT on lattices.

3.6.1. *Local Quantum Physics in AQFT..* Quantum theories are naturally divided into Quantum Mechanics (QM)– concerned with *finite* quantum systems, that is quantum systems that have a finite number of degrees of freedom, and Quantum Field Theories (QFT)/Algebraic Quantum Statistical Mechanics (AQSM) – representing *infinite* quantum systems, i.e. quantum fields and particle systems in the thermodynamic limit that have an infinite number of degrees of freedom. The latter were recognized as being the distinct domain of QFT by Dirac (1927). An *infinite* quantum system is defined in algebraic quantum statistical mechanics by specifying an *abstract algebra of observables.* The algebraic approach has proven most productive in algebraic quantum statistical mechanics, and AQSM has been extensively and very successfully utilized to characterize macroscopic quantum effects that are involved in many important physical phenomena, such as : *crystallization, structural phase transition, ferromagnetism*, *superfluidity* and *superconductivity.*

A further significant breakthrough occurred in QFT by focusing on the local quantum physics (R. Haag and D. Kastler, 1964). The focus on axiomatics and local quantum physics is the main concern of Algebraic Quantum Field Theory (AQFT), with the original approach introduced by R. Haag and D. Kastler (1964). Haag (1955) pointed out that in quantum field theory the representations of *free* fields are *unitarily inequivalent* to those of *interacting* fields, and this is a significant obstruction to the development of a consistent and complete QFT that covers both cases. On the other hand, Segal (1959) pointed out that one of the advantages of the algebraic approach to QFT is that it offers the possibility of obtaining interacting fields from free fields by an automorphism on the algebra, which need not be *unitarily implementable*. A similar, but unpublished, approach was considered by von Neumann earlier, in 1937, by *representing the field dynamics as the automorphism on the (von Neumann) algebra* that involves the observable operator ring closed with respect to the operator *weak* topology. Segal, however, proposed specifically to utilize for this purpose a ring closed with respect to the *uniform topology*. On the other hand, it is only the special (or 'exceptional') Segal algebra which is isomorphic to the $C*$–subalgebra, $C*_{adj}$ that contains only the adjoint observable operators, which can only have real eigenvalues. Following Gel'fand and Naimark, Segal (1947a) introduced the "definitive" *GNS* procedure for constructing *concrete Hilbert space representations* of an abstract C*-algebra. The Hilbert space involved need not be *separable* (i.e., need not have a countable basis), as it is the case of the von Neumann formalism for finite systems. Subsequently, it was proposed that only the measurable field–theoretic variables should be represented in terms of a finite number of canonical operators in an abstract C*–algebra. Recent developments (such as, for example, (Roberts, 2004), [156]) seem to differ significantly on this issue. AQFT itself has now split amongst several groups pursuing significantly different approaches, each with their respective advantages and weaknesses. The basic reasons for this process will become even more evident in the next Section 4.

In recent AQFT approaches, an algebra of observables is associated with *bounded* regions of space–time (either Minkowski or curved) which are required to satisfy six axioms for the local structure of space–time, the quantum fields and their relativistic transformations, such as: *the additivity* of regions, *the domain axiom for quantum fields*, (i.e., quantum fields correspond to operator-valued distributions), *the local commutativity* (i.e., the independence of measurements in spatially-separated regions), *the asymptotic completeness*, (i.e., unitarity of the scattering matrix), *the positively-defined energies and real masses* (i.e., the spectral axiom), *the covariance* and the *the 'uniqueness'* of the 'invariant' vacuum state. One of the main concepts of AQFT that emerged from such basic considerations is that of a *local net of algebras* which are a set of local observable–operator algebras on space–time that satisfy the six axioms. Observable fields are derived function operators that depend upon the structure of the local net, whereas the field intensity values are defined, for example, as 'smoothed' distributions by 'test functions' over such local net regions. The 'local' algebras associated with bounded space-time regions, or boundaries–such as wedges, are of special interest in AQFT approaches aimed at linking such algebras to relativity theory. Furthermore, the QFT net was recently defined in terms of *partially ordered sets* and *quantum causal sets* (see also subsection 3.6 for definitions and further details) as covering regions of a (curved) quantum space-time with non–commutative geometry. This construction was said in ref. [156] to lead to a *2-category* structure. Several algebraic and cohomology properties of such QFT nets were derived and physical representations were developed especially for *hyperbolic* space-time geometry (loc.cit.)

3.6.2. *Axiomatic Quantum Field Theory (AXQFT).*. In the earlier work on AXQFT, Wightman at Princeton University proceeded to develop an abstract, axiomatic formulation of quantum field theory (Wightman,1956) which utilized heavily Schwartz's theory of distributions. The approach was further refined by Bogolioubov and co-workers. This theory based on distribtutions was later re-formulated in the theory of topological vector spaces by Gel'fand and co-workers.

In AXQFT the conceptual framework is reversed by comparison with AQFT: the fundamental concept is the *quantum field*, or fields, defined in terms of observable operator distributions that are subject to the restrictions of the six physical axioms set mentioned above. Somewhat 'conveniently' for analytical and algebraic computations the space-time structure plays the role of a 'background' and the local quantum net loses the central importance that it has in AQFT.

3.6.3. *Quantum Automata.* A third domain of applicability for Algebraic Quantum theories (AQT) is still concerned with finite, but *'organized'*, quantum systems, and has now become distinct from standard QM. The basic concepts in this still maturing field were formally introduced in 1971 as the 'Quantum Automaton' (Section 4., pp. 349-350 of Baianu, 1971a), [13], and as 'Quantum Computers and Algebraic Machine, Quantum Computations' (Baianu, 1971b), [14]. Thus, *quantum automata* were defined in 1971 (in ref. 1) as generalized, probabilistic automata with quantum state spaces. Their next-state functions operate through transitions between quantum states defined by the quantum equations of motions in the Schrödinger representation, with both initial and boundary conditions in space-time, as already discussed above in subsections 3.1 and 3.2. The original concept of Quantum Automaton [13] was formulated in the time–dependent Schrödinger representation through a one–to–one mapping between the automaton *state semigroup* and the Hilbert space of its observables, with *all* the quantum states being specified as being *non-degenerate.* This concept was then employed to explore the *quantum oscillatory* behaviors of complex genetic networks in several, different self-organizing systems [13]. A new theorem is proven here which states that the *category of quantum automata and automata– homomorphisms has both limits and colimits.* Therefore, both categories of quantum automata and classical automata (sequential machines) are *bicomplete.* A second new theorem establishes that the standard automata category is a subcategory of the quantum automata category. A new *category of quantum computers* is also defined in terms of *reversible* quantum automata with quantum state spaces represented by topological groupoids that admit a local characterization through unique 'quantum' *Lie algebroids.* On the other hand, the category of n–Łukasiewicz algebras has a subcategory of *centered* n– Łukasiewicz algebras (ref.2) which can be employed to design and construct subcategories of quantum automata based on n–Łukasiewicz diagrams of existing VLSI. Furthermore, as shown in ref.(2) the category of centered n–Łukasiewicz algebras and the category of Boolean algebras are naturally equivalent.

At the time when the original concepts of *Quantum Automaton* and *Quantum Computation* (through categorical, symbolic programming) were introduced, technological means were not yet available for either practical experimentation or numerical simulations of such organized quantum systems with a finite number of degrees of freedom. Their possible role in anticipatory control systems and biological evolution was further studied through *Algebraic Geometry* concepts, such as *projective variety*, as well as through the application of *singular Homology Group* theory to *Feynman diagrams* and the *dynamic multi-stability problems* in such microphysical systems. Further details are available in ref.[13], and at the following

URLs: http://cogprints.org/3674/ and
http://doc.cern.ch//archive/electronic/other/ext/ext-2004-072.p@ df.

The corresponding AQT field of studies has much accelerated over the last decade and it is likely to experience a hightened degree of interest over the current and next decades as there are many related, potential applications based on *nanoscale* devices such as *Quantum Nano-Automata* and *Quantum Nano-Detectors* (Baianu, 2004), [12]). QAT developments in this new field will also utilize higher-dimensional algebra representations for *optimizing quantum computation algorithms*, as well as QFT *computations on a finite lattice* (QFTL).

In the near future, several *practical* developments of Quantum Automata, Nano–Automata and *'Quantum Computers'* may greatly benefit from further developments of AQFT, and *especially QAT*, concepts that are expected to be utilized in combination with AQSM (which has already made huge progress in quantum solid-state physics and its practical applications).

3.6.4. *Quantum Space–Time Topology in AXQFT and AQFT. Nuclear Fréchet Spaces and the Gel'fand Triple.* The question of quantum space–time topology can be addressed in either AQFT or AXQFT by constructing a Gel'fand Triple (Φ, $\boldsymbol{H}$, Φ^x), where $\boldsymbol{H}$ is a Hilbert space; the major roles are played here by a pair of dual spaces, Φ, and Φ^x, that are constructed from sequences of 'local' Hilbert spaces, as the unique colimit of such a sequence of increasingly finer topologies in the case of the non-Hilbert space Φ, and is the unique limit of increasingly coarser topologies of the non–Hilbert space Φ^x construction. Firstly, a field operator is perhaps 'derived' by analogy with a classical field function by quantizing the classical field function in the canonical manner (e.g., pp. 1–17 of Mandl, 1959). Then, the quantum field operators, $\varphi[f\Box]$, or 'operator–valued distributions' are obtained through smoothing (by the use of suitably selected test-functions that are 'well-behaved'). In the beginning, the Schwartz distributions were the preferred test-functions selected for this purpose. Their definition is as follows:

$$\varphi[f\Box] = \int d\Box^4 x f\Box(x)\varphi(x) \ ,$$

and the field operator is 'smoothed out' over its space-time domain. A basic postulate of AXQFT is that the quantum field operator may be represented as an unbounded operator on a separable Hilbert space $\boldsymbol{H}$. Secondly, the Gel'fand triple is introduced by way of the domain axiom. According to Bogoliubov et al. (p. 34, 1975), [ ]: "... *it is precisely the consideration of the space triple* $\Omega \subseteq \mathbf{H} \subseteq \Omega^*$ *which gives a natural basis for both the construction of a general theory of linear operators and the correct statement of certain problems of quantum field theory.*" In this quote, Ω is the nuclear Frechet space Φ and Ω^* is its dual space, Φ^x.

3.6.5. ***Lattice Quantum Field Theories (LQFT)****: The finite computation approach.* In an independently developing part of QFT, there are finitary – and most often perturbative – approaches to QFT which are achieving remarkable successes through numerical computations and simulations on a finite lattice that complement AQFT. LQFT succeeded recently in solving numerically problems that had alluded AQFT in the past for a long time. Amongst such recent developments are successful perturbative *Quantum Chromodynamics* (QCD) and quantum computations for localized charges on a lattice. Goldstone boson computations, and especially, heavy quark mass predictions are in reasonable agreement with experimental data in spite of the complexity of the strong interactions involving 'color', 'charm' and 'flavor' represented by the SU(3) group of symmetry. For a significant review of QFTL advances and its physical applications the reader is referred to Smit (2002) and references cited therein.

Huge amounts of non-perturbative QCD, high–precision numerical computations are rapidly accumulating for quark, gluon, meson masses, topological 'charges' and topological susceptibilities as a result of several very large collaborations such as, for example, the UKQCD collaboration [**?**]ref.[], 2004) down to lattice spacings of $10^{-17}m$. Obviously such spacings are a very long way from the Planck scale ($10^{-33}m$) where quantum gravity effects are expected to become noticeable. It is interesting, however, that the LQFT computations are quite sensitive to the combined selection of the Quantum System *Topology* (QST) [**?**]ref [ ] in arXiv Dec.7, 2004) and (of course) the quantum operator algebra. Recent developments also include the interesting construction of TQFT and QG theories on a 4–dimensional space-time lattice [(ref [], arXiv 2004; see also the next subsections 7.6 and 7.7.

3.6.6. *Topological and Homotopy Quantum Field Theories (TQFT and HQFT).* TFT/TQFT and HQFT are concerned mostly with *"topological" invariants* in *'lower'* dimensional spaces (i.e., $n < 4$) and partition functions or 'state sums'. HQFT can be defined as a *'TQFT with background'*, but it also utilizes Homotopy concepts and other tools from Algebraic Topology to investigate– in characteristic TFT style – the invariants of 'lower' dimensional ($n \leqslant 3$) manifolds and their associated vector spaces. HQFT has considerably accelerated progress with identifying QSS invariants through 'standard' algebraic topology procedures even though its extensions to higher dimensions have not yet appeared. Its more interesting, potential applications might be in the future to *Spin Networks and Quantum 'Spin Foams'*.

We are proposing here a new conjecture for *extensions* of TQFT to time-dependent QSS (ETQFTs) that also include Quantum 'Foams' of Spin Networks as lower-dimensional ($n = 2$), specific examples.

Conjecture 3.6.7. *The Quantum Fundamental Groupoid* , $\Pi_1(D_{QS})$ *, of any time-dependent ETQFT State Space,* D_{QS}*, can be computed* via *the Generalized van Kampen Theorem (see Section 8.3) as the colimit of the sequence of fundamental groupoids* $\{\pi_1^i(CW_i)\}$ *of the sequence of CW–complex subspaces,* $\{CW_i\}$*, forming the CW-approximation (colimit) sequence of the time-dependent ETQFT. In categorical form, this is concisely stated as:*

$$\Pi_1(D_{QS}) \approx colim_{i=1,\ldots n}\ \{\Pi_1^i(CW_i)\}.$$

Note: In the simpler cases of one- and two- dimensional CW complexes (simplices), such as, respectively, the Quantum Spin Networks (QSN) and the time-dependent QSNs (or Spin 'Foams'), this general conjecture can be proven directly through a step-by-step graph decomposition procedure for QSNs.

Proposition 3.6.8

The Quantum Groupoid, $\Pi_1(M_{QS})$*, of any time-dependent ETQFT State Space* Model *of quantum crossed complexes,* M_{QS}*, can be computed* via *the Generalized van Kampen Theorem (see Section 8.3) as the colimit of the sequence of fundamental groupoids* $\{\pi_1^i(M_i(CW_i))\}$ *of the sequence of crossed complex models* $\{M_i\}$ *of CW– complex subspaces,* $\{CW_i\}$*, forming the CW-approximation (colimit) sequence of the time-dependent ETQFT. In categorical form, this is concisely stated as:*

$$\Pi_1(M_{QS}) \cong colim_{i=1,\dots n}\ \{\Pi_1^i(M_i(CW_i))\}.$$

Proof

4. Generalizations of Fundamental Quantum Concepts required by Modern Quantum Theories: From Superconductivity to Quantum Gravity

4.1. **Concrete Representations of C*–Algebras: The Gel'fand Triples.** Gel'fand and Naimark (1943) followed the previous work by Murray and von Neumann on rings of operators and focused their attention on *abstract normed *–rings.* They showed that any C*–algebra can be given a concrete representation in a Hilbert space (which does not need to be a separable topological space) by proving the existence of an isomorphic mapping of a C*–algebra elements into the set of bounded operators of the Hilbert space. Subsequently, Segal (1947) completed the work of Gel'fand and Naimark by specifying the *definitive procedure for constructing concrete (Hilbert space) representations of an abstract C*-algebra*, which is called the *GNS construction* (after Gel'fand, Naimark, and Segal).

Furthermore, *the Gel'fand–Naimark theorem* states that any *abelian* C*–algebra is isomorphic to the *–algebra $C_0(X)$ of all continuous complex valued functions vanishing at infinity on a locally compact Hausdorff space X. The spectral theorem shows that an *abelian* von Neumann algebra is isomorphic to $L^1(X; -)$ for some finite measure space $(X; -)$. Thus, the collection of commutative C*–algebras is large and complicated, whereas the list of commutative types of von Neumann algebras is quite short: $L^1([0, 1]; dx^1$, $L^1(S)$ for some countable set S, and direct sums of the two.

<< In ordinary quantum mechanics the von Neumann algebras one encounters are of type I. But as soon as one talks about quantum statistical mechanics, factors of types II and III are required. Putting together infinitely many particles requires however taking limits first. This was first approached by von Neumann in [26x] and is now best understood in terms of various von Neumann algebra completions of C*–algebras. Quantum Field theory- 'whatever it may do'- certainly retains von Neumann's Hilbert space foundations, and we can expect to see factors in all their splendour as the mathematical receptacle for the quantum fields. The Algebraic Quantum Field Theory (AQFT) of Haag, Kastler and others (see [11x] and Section 3.y, above) is an attempt to approach quantum field theory by seeing what constraints are imposed on the underlying operator algebras by general physical principles such as relativistic invariance and positivity of the energy. A von Neumann algebra of "localised observables" is postulated for each bounded region of space–time. Causality implies that these von Neumann algebras commute with each other if no physical signal can travel between the regions in which they are localized. The degree of a field extension is the dimension of the big field as a vector space over the small one. But in von Neumann algebras there is also a notion of dimension - of a module over a II_1 factor (the "coupling constant" of Murray and von Neumann). Indeed this is the first seductive aspect of operator algebras, as this dimension is a *continuously varying*, real number. One can now attempt to simply copy the *Galois theory*: given a subfactor $[N_M]$, define the degree of the extension to be the dimension of M as a left N-module. For historical reasons it is called the index of N in M, written $[M : N]$. The first surprise is the answer to the question: "What are the possible values of $[M : N]$?" To answer this we need to find lots of subfactors. *Galois theory* suggests looking at fixed points for group actions on M. A Galois-like theory for such subfactors was worked out in the 1950's [18x], but that theory, no matter how one tries to fiddle with it, supplies only integer values for the index $[M : N]$. On the other hand the index is a von Neumann dimension, so we expect it to vary *continuously.* The intriguing answer to the question is that there must be both a *discrete* <u>*and*</u> *a continuous* part to the set of all possible values. If $[M : N] \leqslant 4$, then the index is necessarily one of the numbers $4cos^2\pi/n$ for some integer $n \geqslant 3$, whereas all numbers $\geqslant 4$ can occur. If we were to think of these subfactors in *Galois theory* terms, they would correspond to finite groups of real order! (ref.x.12) >>

4.2. **Galois Extension Conjecture.** *There exists a crossed ϕ –module consisting of a discrete quantum group and a (*continuous*) Lie groupoid generated by von Neumann subfactors such that their extended Galois theory will provide both discrete and continuous values for the index family* $[M : N]$.

4.3. **Quantum Statistical Mechanics of Phase Transitions in Macroscopically Coherent, Quantum Systems: Spontaneous Symmetry Breaking Mechanisms.** The following is just a short list of very important examples of physical phenomena that involve a *Fundamental Spontaneous Symmetry Breaking process.* Several such mechanisms that are based on spontaneous symmetry breaking were either found or validated experimentally [170].

• *Vacua, Metastable State Transitions through Quantum Tunneling: The Role played by Local, Spontaneous Symmetry Breaking;*

• *Approximate* $SU(2)\times U(1)$*, Spontaneous Symmetry Breaking in The Unified Electroweak Theory: The prediction of* $W\mp$ *and* Z^o *massive carriers of the weak interactions;*

• *SO(4) to SU(3) Spontaneous Symmetry Breaking: Pions predicted as 'Goldstone' bosons;*

• *Spontaneous Supersymmetry Breaking and Goldstone Bosons in Unified Field Theories and Quantum Gravity;*

• *Spontaneous Symmetry Breaking in Quantum Phase Transitions in Macroscopically Coherent, Quantum Systems:* (QPTDG)

◦ *Quantum Liquid Helium-3 /quantized fluids;*

◦ *Superconductivity : Generalized, and Dynamically–Enhanced, Symmetry;*

◦ *Ferromagnetism;*

◦ *Colossal Magnetoresistance.*

In terms of simplicity of mathematical representation and equations it makes sense to begin deriving results for the 'simplest' possible physical 'components' of the universe: its *vacua.*

4.3.1. *Vacua Metastable State Transitions Mediated by Quantum Tunneling: The Role played by both Global and Local, Spontaneous Symmetry Breaking.* According to Weinberg (2001; p.135x in vol.2 of ref. [170]), modern QFT predicts the existence of a set of *vacua metastable states* of different symmetry that result from fluctuations of the vacuum ground-state, which are allowed according to the Heisenberg Uncertainty Principle (see Section 3.2). Therefore, we are able to derive the following result for such metastable states that would also hold for relativistic QST representations.

Proposition 4.3.2

Assume the presence of vacua metastable states with local symmetries that are different from the 'global' symmetry of the (quantum) ground–state vacuum; also assume quantum tunneling through 'barriers' between such states, as well as the 'global' symmetry breaking made possible by such quantum tunneling. Then, there exists a representation of the vacuum ground–state in terms of a unique colimit *of the QST representations (sequence) for all of the allowed metastable vacua coexisting with the vacuum ground–state.*

Proof (outline). The proof consists in the construction of the QST representation of the vacuum ground-state as a colimit of a sequence of filtered spaces with broken-symmetry that represent mixtures of metastable state vacua regions with local, ground-state vacuum regions. On the basis of the *Fundamental Approximation Theorem* (see Section 7.3), such filtered spaces admit unique CW-complex approximations. This unique colimit is computable (up to an isomorphism)–at least in principle– in terms of the *GvKT colimit constructions* (which are here recalled in Section 8.3, *Theorems 8.3.1* through *8.3.3*, respectively, from refs. [**?**],[32] and [33]). There should also exist a CW-complex approximation of this colimit in terms of the CW-complexes that approximate the filtered spaces in the above specified sequence.

Corollary 4.3.3

The 'global' symmetry breaking of the vacuum ground-state through vacuum fluctuations to the assumed metastable vacua states that were specified above in Proposition 4.3.2 generates Goldstone bosons.

Proof (outline)

Remark 4.3.4

The Goldstone bosons specified in Corollary 4.3.3 might be the hypothetical *'Higgs' bosons* required by superstring theory and postulated also by effective field theories utilizing *non-Abelian gauge* transformations (see also the next Section 5.1), if it were not for the fact that *Higgs bosons* are predicted to be quite massive, and therefore they would not be able to propagate at the speed of light either between, or through, such regions in vacua.

4.4. **Quantum Symmetry Groupoids and Pseodogroups. Quantum 6j–Groupoids of Type II_1 von Neumann Subfactor Paragroups. Christoffel symbols and ETQFT Symmetry Tetrahedron.** NewInfo1

4.5. **Quantum Field Configuration J-Groupoids of Type III von Neumann Subfactors. Locally Compact Quantum Groups and Groupoids.** Newinfo2

4.6. **Interacting Quantum Spin Groups and Quasi-Particles: Explicit Hamiltonians and Numerical Computations Compared with Experimental Results in Solid Electrolytes and Ferromagnetic Glasses.** New Info3; refs: [20], [23], [24]

4.6.1. *Spin Wave Excitations in Solids.* refs: [21], [22]

4.6.2. *Quantum Phase Transition Double Groupoids (QPTDG).* newinfo2

5. Novel Representations of Global and Local Symmetry Breaking in Quantum Systems. Quantum Algebraic Topology of Space–Time Representations Consistent with the Symmetries of Quantum Fields: *Quantum Crossed Modules, Quantum 'Convolution' Groupoids and Quantum Cross Complexes over a Groupoid; Non-Abelian Gauge Theories in AQFT and Quantum Gravity.*

We are proposing to construct next novel representations for quantum systems and quantum field configuration spaces that are consistent with known symmetries. Then, we will consider representations of *'patterns of symmetry breaking'* [170] and their quantum effects both at the local and global levels, either static or dynamic.

5.1. **Quantum Algebraic Topology of Space–Time Representations Consistent with the Symmetries of Quantum Fields: *Local–to–Global (LG) Construction Principles based on AQFT and Quantum 'Axiomatics'.*** Several different strategies can be employed, and selection criteria based on physical considerations, such as quantum field axiomatics, will then be utilized to decide on the adoption of the most appropriate representation strategy. These will be discussed next.

5.1.1. *Quantum Space–Time Representation Strategies based on Algebraic Topology.* Recently, there are a number of alternative approaches designed for 'building up', or constructing, mathematically the global structure of QST from local regions, as in both Algebraic and Topological Quantum Field theories, but with the emphasis placed on either Quantum Algebra (QA) or the 'Quantum' Topology of space–time. The third alternative approach to this problem, and/or construction, is that provided by *Quantum Algebraic Topology (QAT)* and involves considering *jointly* the *algebraic and topological* structures of QST, as well as defining and determining the fundamental *algebraic invariants* of possible QST topologies that might be relevant to corresponding *Quantum Gravity* theories. Although there is only a physically *unique* QST, there is already a rapid proliferation of proposed *mathematical representations* of the physical space–time, ranging from partially ordered sets (i.e., with *discrete* topology) to *continuous* topological space representations such as various manifolds (with dimensions of 4, (e.g., $Riemann, R^4$), 10, 11, 26 or n-dimensions), 'group manifolds', 'monoidal' categories, small "intertwiner" categories, 2-categories, 'tensor' 2-categories, and a 'quantum' topos. The systematic classification and rigorous characterization of such potential candidates for the mathematical representation of QST can also be considered as a significant task in Quantum Algebraic Topology which is defined in this first paper of a series. Furthermore, a completely satisfactory resolution of the problem of QST structural representation will undoubtedly involve the consistent linking of Quantum Logics (QL) with *Quantum Algebraic Topology*, thus relating back the theoretical constructions of QST to quantum measurements and experimental data in terms of systematic QL analyses of quantum events and their consequences for both QA (Alfsen and Schultz, 2003), ref. [3], and QAT. Linking consistently QL with QAT for representing the structure of QST is an approach that will be pursued in the second paper of this series (Baianu, Glazebrook, Georgescu and Brown, 2004). Algebraic developments related to quantum theories have a long and successful history. The more challenging aspects of such developments are recently based on

Algebraic Topology, and also in algebraic treatments of 'Quantum Geometry'.

The consideration of possible candidates for representing the complete structure of our physical space–time thus runs into the basic problem of classifying such space–time candidates into equivalent classes determined by homeomorphisms of topological spaces. As the explicit mathematical construction of homemorphisms can be a very daunting problem for topological spaces in general, the computation of *algebraic invariants* of such spaces is the chosen, basic methodology of Algebraic Topology (AT). Thus, if one can assign the algebraic structure of a group to a topological space, then one can compare two homeomorphic, or equivalent, topological spaces and find that their corresponding groups are *isomorphic*. However, the converse does not necessarily hold: even though two arbitrary topological spaces may have assigned *isomorphic* groups, the two spaces are not necessarily *homeomorphic*. Therefore, one needs to consider first the simpler problem of finding a coarser equivalence of topological spaces in terms of the *homotopy equivalence* and associated homotopy groups by assembling *equivalence classes* of continuous path deformations in such topological spaces. Whereas many *homotopy groups* may be readily computed for n-spheres (S^n), certain polyhedra-like spaces ('simplicial complexes') and their generalized form–the *CW–complexes* and 'Eilenberg- MacLane spaces'– their computation for arbitrary spaces with corresponding, 'dual' higher dimensional algebras is not yet solved. Therefore, other refined algebraic approaches to topological space *classification* were developed. One such approach to topological space classification was developed in terms of *map transformations* and exact sequences that involve both *singular homology and cohomology* constructions allowing the systematic computation of certain required homology groups, or *groupoids*, especially for CW–complexes. *CW–complexes* can be constructed either as equivalent 'cellular spaces' by attaching cells to spaces in a systematic, precisely-defined construction, or else they can be defined as a special type of Hausdorff space subject to several restrictions imposed by their equivalent cellular construction.

An alternative approach involves generalizing fundamental theorems of algebraic topology from specialized, 'globally well-behaved' topological spaces, to arbitrary ones. In this category are both the *generalized van Kampen theorem (GvKT)* and the generalized Hurewicz theorem of AT. Several fundamental theorems of Algebraic Topology, such as the Hurewicz (1955), the J.H.C. Whitehead (1965) and the van Kampen (1933) theorems were first proven for 'simpler' spaces and subsequently extended or generalized to arbitrary topological spaces, through *non-abelian higher dimensional algabra*. (AT fundamental theorems will be stated without proof in Sections 7 and 8). Such theorems greatly aid the calculation of homology, cohomology and homotopy groups of topological spaces. In the case of the Hurewicz theorem, for example, this was generalized to arbitrary topological spaces (Spanier,1966), and establishes that certain homology groups are isomorphic to 'corresponding' homotopy groups of an arbitrary topological space. R. Brown and coworkers (1999, 2004 a,b,c) went further and generalized the van Kampen theorem, at first to homotopy groupoids (Brown,1967), and then, to higher dimensional algebras involving, for example, homotopy double groupoids and 2-categories (Brown, 2004a). The more sensitive *algebraic invariant* of topological spaces seems to be, however, captured only by *cohomology* theory through an algebraic *ring* structure that is not accessible either in homology theory, or in the existing homotopy theory. Thus, two arbitrary topological spaces that have isomorphic homology groups may not have isomorphic cohomological ring structures, and may also not be homeomorphic, even if they are of the same homotopy type. The corollary of this statement may lead to an interesting cohomology-based classification in a category of certain 'Coh' topological spaces that have isomorphic ring structures and are also homeomorphic. Furthermore, several *non–Abelian* results in algebraic topology could only be derived from the generalized van Kampen theorem (cf. Brown, 2004a), so that one may find links of such results to the expected *'non–commutative* geometrical' structure of quantized space–time (Connes, 1994; Varilly, 1997). In this context, the important algebraic-topological concept of a *Fundamental Homotopy Groupoid (FHG) is applied to a Quantum Topological Space (QTS)* as a "partial classifier" of the *invariant* topological properties of quantum spaces of *any* dimension; quantum topological spaces are then linked together in a *crossed complex over a quantum groupoid* (Section 5.4), thus suggesting the construction of *global* topological structures from *local* ones with well-defined *quantum homotopy groupoids*. The latter theme is then further pursued through defining *locally topological* groupoids that can be globally characterized by applying *the Globalization Theorem* [**?**, **?**, x1] which involves the *unique* construction of *the Holonomy Groupoid*. In a real quantum system, a unique quantum holonomy groupoid may represent parallel transport processes and the 'phase-memorizing' properties of such remarkable quantum systems. This theme can be similarly pursued in the continuous case through *locally Lie* groupoids and their corresponding Globalization theorem. The converse approach may involve the use of fundamental theorems of Algebraic Topology such as the generalized van Kampen theorem for characterizing the topological invariants of a higher- dimensional, or 'composite', topological space in terms of the (known) invariants of its 'simpler' subspaces (as in the case of Whitehead's theorem and the original version of the van Kampen theorem).

Another very interesting aspect of such algebraic constructions leading from local to global structures of topological spaces is the representation of a topological space as the categorical *colimit* of a sequence of 'simpler' spaces, such as CW–complexes, at least as an approximation. This also occurs in the generalized van Kampen theorem in terms of colimits of homotopy double groupoids. As an illustration, a specific example will be given in Subsection 5.6.5 for local subgroupoids that are defined as a *sheaf*, thus leading towards the concept of a *Generalized Topos with a Quantum Logic, subobject classifier* (Baianu, Glazebrook, Georgescu and Brown, 2004) which links Quantum Multi–Valued Logics with generalized QAT structures in categories generated by sheaves, such as the Grothendieck categories. The relevance of such colimit constructions to the QAT representation of fundamental quantum space–time structure in our inflationary universe will be shown in this and Section 7. Therefore, instead of utilizing flat, or almost-flat, pieces of space–time as the local, *'linearized'* structure that approximates our inflationary universe only for small masses with *weak* gravitational fields (as in the 'standard' supergravity theory that will be concisely reviewed in Section 6), one should also be able to employ categorical *colimits* to construct representations of quantized space–time that incorporate huge masses and correspondingly *intense* gravitational fields. Such generalized space–time representations –based on QAT constructions– will also be endowed with the prerequisite covariance, metric and broken supersymmetry properties. It is conjectured at this point that such a physical representation of the emerging, *nonlinear* supergravity theory for *intense* gravitational (and other coupled) fields– which is obtained by including the appropriate QAT structure of space–time (both local and global)–will be at least *consistent* with the accepted results of the Standard Model in the limit of the currently attainable energies with the existing particle accelerators (i.e. $E < 0.2TeV$ in the laboratory reference frame).

Before introducing topological space constructions based on the strategies discussed above we shall briefly consider the discrete approach of 'causal sets' to 'quantizing space-time' even though its justification in terms of the underlying quantum logics does not seem to be forthcoming.

5.2. **Quantizing Space–Time: Causal Sets.** In Sorkin [162] 'finitary topological spaces' (or sequences of these) were introduced to approximate or to reproduce in the limit, a topological space such as a manifold. The motivation concerns the patent inadequacies of the traditional differentiable manifold structure of space–time based on several reasons. The main premise is that the smooth structure at small time scales breaksn down to one that is more discrete– and 'quantum'–in form; there is an ideal character of the event as observed classically and this occurs within the presence of singularities. The continuum of events and their infinitesimal separation do not yield to the usual experimental analysis. While not neglecting the large scale classical model, one may propose the structure of 'ideal observations' as manifest in a limit, in some sense, of discrete measurements, where such a limit accomodates the classical event. Then the latter is represented as a *'point'* which is not influenced by quantum interference; nevertheless, the idea is to admit *coherent* quantum superposition of events. Thus, at the quantum level, the events can *decohere* to the classical point in the limit, somewhat in accordance with the correspondence principle. Expositions and applications of the finitary approach were reported in [150] to [152] where further details can be found. We will proceed to cover some of the basic groundwork in the next two subsections.

5.2.1. Posets as 'Finitary Approximations'

Let S be a compact metric space and consider a finite open covering $\mathcal{U}$ of S . An equivalence relation "$\sim_{\mathcal{U}}$" is defined on S by the following :

if $x, y \in S$, then $x \sim_{\mathcal{U}} y$, if and only if $\forall U \in \mathcal{U}\ ,\ x \in U \Leftrightarrow y \in U$.

The quotient space $S_{\mathcal{U}} = S/{\sim_{\mathcal{U}}}$ is a finite topological space which is endowed with the quotient topology via $S \longrightarrow S_{\mathcal{U}}$. The space $S_{\mathcal{U}}$ can be shown to be a T_0 space in which each point $x \in S_{\mathcal{U}}$ belongs to a unique minimal open set U_x . Accordingly a partial order on $S_{\mathcal{U}}$ is obtained by :

$$x \preceq y \ \Leftrightarrow\ U_x \subseteq U_y \ \Leftrightarrow x \in U_y \ .$$

Sorkin [162] considers the resulting posets $P(S)$ as 'finitary approximations' to S by reducing elements and intersections of $\mathcal{U}$ to a 'point' system, thus comprising the 'causal sets'. The dictum is that posets provide a useful structure of space–times in which the '$\sim$' is the small scale correspondent of the relation defining the past and future distinctions in the space–time continuum. One convenient way of doing this is via a Hasse diagram whose rules we recall.

1. If $x \prec y$, then say that y is 'higher' than x .

2. If $x \prec y$, and $\nexists z$ such that $x \prec z \prec y$, then x and y are connected by an edge (or say "y covers x").

When the poset $P(S)$ contains $2N$ points we write this as $P_{2N}(S)$. The $P_N(X)$ in the projective limit $\lim \leftarrow P_N X$ recovers a space homeomorphic to X [162].

Example 5.2.1

Consider the simple example where S is the circle S^1 as covered by four open sets $\{O_1, O_2, O_3, O_4\}$ where $O_1, O_3 \subseteq O_2 \cap O_4$, $O_1 \cap O_3 = \emptyset$. To see how

$P_4(S^1)$ is obtained we make the assignments

$$O_1 \mapsto x_1 \ , \ O_2 \setminus [O_2 \cap O_4] \mapsto x_2$$
$$O_3 \mapsto x_3 \ , \ O_4 \setminus [O_2 \cap O_4] \mapsto x_4$$

Then consider the map $S^1 \longrightarrow P_4(S^1)$ given by $\{x_1\}$

{x3} $\{x_1, x_2, x_3\}$ $\{x_1, x_4, x_3\}$, which leads to a

poset structure on $P_4(S^1)$ given by the partial ordering

$x_1 \prec x_2$, $x_1 \prec x_4$, $x_3 \prec x_2$, $x_3 \prec x_4$

More generally, a cell decomposition of a manifold (simplicial or otherwise) decomposition can be associated with an open covering which is assumed finite, and as such can be viewed as a poset. For instance, from a simplex K of X, we obtain a poset as follows.

If $c^{\ell}_{(n)}$ denotes an n–cell, we obtain a partial ordering by setting $c^k_{(m+1)} \preceq c^j_{(m)}$, if $c^j_{(m)}$ is a face of $c^k_{(m+1)}$. The resulting poset P is a topological space and on regarding each cell as consisting only of its interior points, then the open sets of P form an open covering of X .

Let us see this specifically in terms of the *nerve of an open covering* $N\mathcal{U}$. Consider an open covering $\mathcal{U} = \{U_0, \ldots, U_k\}$ of X . This open covering, as the above simple example illustrates, can be seen to be a vertex set forming a k–simplex $N\mathcal{U}$ of the poset if and only if $\{U_0, \ldots, U_k\} \in N\mathcal{U} \iff U_0 \cap U_1 \cdots \cap U_k \neq \emptyset$. This specifies $N\mathcal{U}$, and the fact that it has a simplicial representation follows from a simplicial isomorphism $K \cong N\mathcal{U}$ for a k–simplex K underlying X [163]. Accordingly, $N\mathcal{U}$ is a poset : points of $N\mathcal{U}$ are the simplices and arrows abide by the rule $p \iff p$ is a face of q (as above).

5.2.2. The Rota algebra

Let us consider now the fact that associated to a any poset there is a *Rota incidence algebra* [155] which is non–commutative and associative. If we consider $\mathcal{P} = (S, \longrightarrow)$ where S is a set of elements '$\longrightarrow$' is a reflexive, antisymmetric, transitive binary relation (i.e., the partial order), then the Rota algebra $\Omega(\mathcal{P}p, q)$ is specified by

$$\Omega\mathcal{P} = \{p \longrightarrow q : p, q \in S\}$$
$$\Omega(\mathcal{P}p, q) = span_{\mathbb{C}}\{p \longrightarrow q\}$$

.

$$(\mathrm{p} \longrightarrow q) \cdot (r \longrightarrow s) = \begin{cases} p \longrightarrow s \ , \ \text{if } q = r \\ 0 \ , \ \text{otherwise.} \end{cases}$$

The algebra Ω associated with the finitary T_0–poset $P(S)$ can be realized as a topological space which is essentially quantum : Ω is non–commutative, the partial ordering of the arrows defining the T_0 topology can superpose coherently with each other. If one follows the dictum that spacetimes at small time-scales should be seen in the context of 'quantum' sets, then points extracted from Ω might be labeled as 'quantum' once identified with the kernels of (equivalence classes) of irreducible finite dimensional Hilbert space representations of non–commutative incidence algebras whose kernels turn out to be primitive ideals in the latter structures. Thus, the posets defined above, and with this interpretation, constitute an example of *'quantum' causal sets.*

5.3. **Quantizing Groupoids: Compact Quantum Groupoids (CQG), CQG–Algebras, Quantum Metrics and Quantum Principal Bundles.** A natural starting point for this subsection is the introduction of the mathematical concept of *topological groupoid* which is essential for defining either *generalized* or *broken symmetries* (that were discussed above in Section 4.3 for important physical phenomena), as well as for constructing extensions of both quantum state spaces and QST representations.

5.3.1. *Topological Groupoids.* Recall that a groupoid $\mathbb{G}$ is, loosely speaking, a small category with inverses over its set of objects $X = \mathrm{Ob}(\mathbb{G})$. One often writes $\mathbb{G}_x^y$ for the set of morphisms in $\mathbb{G}$ from x to y .

Definition 5.1. A (topological) *groupoid* consists of a set $\mathbb{G}$, a distinguished subset $\mathbb{G}^{(0)} = \mathrm{Ob}(\mathbb{G}) \subseteq \mathbb{G}$, called *the set of objects* of $\mathbb{G}$, together with maps $r, s \ : \ \mathbb{G} \underset{s}{\overset{r}{\rightrightarrows}} \mathbb{G}^{(0)}$, called the *range* and *source maps* respectively, together with a law of composition $\circ \ : \ \mathbb{G}^{(2)} := \mathbb{G} \times_{\mathbb{G}^{(0)}} \mathbb{G} = \{ \ (\gamma_1, \gamma_2) \in \mathbb{G} \times \mathbb{G} \ : \ s(\gamma_1) = r(\gamma_2) \ \} \ \longrightarrow \ \mathbb{G}$,

such that the following hold :

(1) $s(\gamma_1 \circ \gamma_2) = r(\gamma_2)$, $r(\gamma_1 \circ \gamma_2) = r(\gamma_1)$, for all $(\gamma_1, \gamma_2) \in \mathbb{G}^{(2)}$.

(2) $s(x) = r(x) = x$, for all $x \in \mathbb{G}^{(0)}$.

(3) $\gamma \circ s(\gamma) = \gamma$, $r(\gamma) \circ \gamma = \gamma$, for all $\gamma \in \mathbb{G}$.

(4) $(\gamma_1 \circ \gamma_2) \circ \gamma_3 = \gamma_1 \circ (\gamma_2 \circ \gamma_3)$.

(5) Each γ has a two–sided inverse γ^{-1} with $\gamma\gamma^{-1} = r(\gamma)$, $\gamma^{-1}\gamma = s(\gamma)$.

It is usual to call $\mathbb{G}^{(0)} = \mathrm{Ob}(\mathbb{G})$ *the set of objects* of $\mathbb{G}$. For $u \in \mathrm{Ob}(\mathbb{G})$, the set of arrows $u \longrightarrow u$ forms a group $\mathbb{G}_u$, called the *isotropy group of* $\mathbb{G}$ *at* u.

5.3.2. *Compact Quantum Groupoids (CQG).* 5.3.2

Compact quantum groupoids were introduced in Landsman (1999) as a simultaneous generalization of a compact groupoid and a quantum group. To commence, let $\mathfrak{A}$ and $\mathfrak{B}$ be C*–algebras equipped with a *–homomorphism $\eta_s : \mathfrak{B} \longrightarrow \mathfrak{A}$, and a *–antihomomorphism $\eta_t : \mathfrak{B} \longrightarrow \mathfrak{A}$ whose images in $\mathfrak{A}$ commute. A non–commutative Haar measure is defined as a completely positive map $P : \mathfrak{A} \longrightarrow \mathfrak{B}$ which satisfies $P(A\eta_s(B)) = P(A)B$. Alternatively, the composition $\mathcal{E} = \eta_s \circ P : \mathfrak{A} \longrightarrow \eta_s(B) \subseteq \mathfrak{A}$ is a faithful conditional expectation.

The next step requires a little familiarity with the theory of Hilbert modules, as in e.g. Lance (1995). We define a left $\mathfrak{B}$–action λ and a right $\mathfrak{B}$–action ϱ on $\mathfrak{A}$ by $\lambda(B)A = A\eta_t(B)$ and $\varrho(B)A = A\eta_s(B)$. For the sake of localization of the intended Hilbert module, we implant a $\mathfrak{B}$–valued inner product on $\mathfrak{A}$ given by $\langle A, C\rangle_{\mathfrak{B}} = P(A^*C)$. Since P is faithful, we fit a new norm on $\mathfrak{A}$ given by $\|A\|^2 = \|P(A^*A)\|_{\mathfrak{B}}$. The completion of $\mathfrak{A}$ in this new norm is denoted by $\mathfrak{A}^-$ leading then to a Hilbert module over $\mathfrak{B}$.

The tensor product $\mathfrak{A}^- \otimes_{\mathfrak{B}} \mathfrak{A}^-$ can be shown to be a Hilbert bimodule over $\mathfrak{B}$, which for $i = 1, 2$, leads to *–homorphisms $\varphi^i : \mathfrak{A} \longrightarrow \mathcal{L}_{\mathfrak{B}}(\mathfrak{A}^- \otimes \mathfrak{A}^-)$. Next is to define the (unital) C*–algebra $\mathfrak{A} \otimes_{\mathfrak{B}} \mathfrak{A}$ as the C*–algebra contained in $\mathcal{L}_{\mathfrak{B}}(\mathfrak{A}^- \otimes \mathfrak{A}^-)$ that is generated by $\varphi^1(\mathfrak{A})$ and $\varphi^2(\mathfrak{A})$. The last stage of the recipe for defining a compact quantum groupoid entails considering a certain coproduct operation $\Delta : \mathfrak{A} \longrightarrow \mathfrak{A} \otimes_{\mathfrak{B}} \mathfrak{A}$, together with a coinverse $Q : \mathfrak{A} \longrightarrow \mathfrak{A}$ that it is both an algebra and bimodule antihomomorphism. Finally, the following axiomatic relationships are observed :

$$
\begin{aligned}
(\mathrm{id} \otimes_{\mathfrak{B}} \Delta) \circ \Delta &= (\Delta \otimes_{\mathfrak{B}} \mathrm{id}) \circ \Delta \\
(\mathrm{id} \otimes_{\mathfrak{B}} P) \circ \Delta &= P \\
\tau \circ (\Delta \otimes_{\mathfrak{B}} Q) \circ \Delta &= \Delta \circ Q
\end{aligned}
$$

where τ is a flip map : $\tau(a \otimes b) = (b \otimes a)$.

Suppose now $\mathbb{G}$ is a *Lie groupoid*. Then the isotropy group G_x is a Lie group, and for a (left or right) Haar measure μ_x on G_x, we can consider the Hilbert spaces $\mathcal{H}_x = L^2(\mathbb{G}_x, \mu_x)$ as exemplifying the above sense of a representation. Putting aside some technical details which can be found in Connes (1994), Landsman (1998), the overall idea is to define an operator of Hilbert spaces $\pi_x(f) : L^2(\mathbb{G}_x, \mu_x) \longrightarrow L^2(\mathbb{G}_x, \mu_x)$, given by $(\pi_x(f)\xi)(\gamma) = \int f(\gamma_1)\xi(\gamma_1^{-1}\gamma)\, d\mu_x$, for all $\gamma \in \mathbb{G}_x$, and $\xi \in \mathcal{H}_x$.

For each $x \in X = \mathrm{Ob}\ \mathbb{G}$, π_x defines an involutive representation $\pi_x : C_c(\mathbb{G}) \longrightarrow \mathcal{H}_x$. We can define a norm on $C_c(\mathbb{G})$ given by $\|f\| = \sup_{x \in X} \|\pi_x(f)\|$, whereby the completion of

$C_c(\mathbb{G})$ in this norm, defines *the (reduced) C*–algebra* $C_r^*(\mathbb{G})$ *of* $\mathbb{G}$. It is the most commonly used C*–algebra for Lie groupoids (groups) in non–commutative geometry.

5.3.3. *Quantum Groupoid C*–Algebra (QGCA).* Let $\mathbb{G}$ be a (topological) groupoid. We denote by $C_c(\mathbb{G})$ the space of smooth complex–valued functions with compact support on $\mathbb{G}$. In particular, for all $f, g \in C_c(\mathbb{G})$, the function defined via convolution

$$(f \ * \ g)(\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2) \ ,$$

is again an element of $C_c(\mathbb{G})$, where the convolution product defines the composition law on $C_c(\mathbb{G})$. We can turn $C_c(\mathbb{G})$ into a *–algebra once we have defined the involution $*$, and this is done by specifying $f^*(\gamma) = \overline{f(\gamma^{-1})}$.

Following Landsman (1998), a *representation* of a groupoid $\mathbb{G}$, consists of a family (or field) of Hilbert spaces $\{\mathcal{H}_x\}_{x \in X}$ indexed by $X = \text{Ob}\ \mathbb{G}$, along with a collection of maps $\{U(\gamma)\}_{\gamma \in \mathbb{G}}$, satisfying

1. $U(\gamma) : \mathcal{H}_{s(\gamma)} \longrightarrow \mathcal{H}_{r(\gamma)}$, is unitary;
2. $U(\gamma_1\gamma_2) = U(\gamma_1)U(\gamma_2)$, whenever $(\gamma_1, \gamma_2) \in \mathbb{G}^{(2)}$;
3. $U(\gamma^{-1}) = U(\gamma)^*$, for all $\gamma \in \mathbb{G}$ [41].

5.3.4. *Quantum Metric Spaces.* Let G be a countable discrete group and let $C_c(G)$ denote the convolution *–algebra of complex–valued functions with finite support on G . We follow a procedure as described in [154] similar in part to that of the last subsection. The unitary representation by left translations on the Hilbert space $\ell^2(G)$, leads to a representation $\pi : C_c(G) \longrightarrow \ell^2(G)$. Again using the norm completeness we obtain the C*–algebra $C_r^*(G)$ with dense inclusion $C_c(G) \subseteq C_r^*(G)$. Let a length function ℓ be defined on G; for$x, y \in G$ such a function satisfies

$$\ell(xy) \leqslant \ell(x) + \ell(y) \ , \ \ell(x^{-1}) = \ell(x) \ ,$$

with $\ell(x) \geqslant 0$ and $\ell(x) = 0$ if and only if $x = e$, the identity in G . The point here is that ℓ defines a Lipschitz seminorm on $C_r^*(G)$ leading to define a metric on the latter. This is essentially the definition in [154] of a (compact) quantum metric space.

Next let M_ℓ denote the (generally unbounded) operator on $\ell^2(G)$ defined by pointwise multiplication by the length function ℓ, where given a function $f \in C_c(G)$, the commutators $[M_\ell, \pi_f]$ are bounded. Let $\alpha_z(\ell)$ denote left–translation of ℓ by z (although translation bounded it is not necessarily a length function, but we will finesse this point). Thus the above operator transforms as $M_\ell \longrightarrow M_{\alpha_z(\ell)}$. With the convention $(\alpha_z \ell)(x) = \ell(z^{-1}x)$, the relationship

$$\pi_y M_h = M_{\alpha_y(h)} \pi_y \ ,$$

holds for any function h on G and any $y \in G$. As pointed out in [154], this construction easily generalizes when h is taken to be an (operator) algebra–valued function on G .

5.3.5. *Quantum Principal Bundles (QPB).* 5.3.4.

Relevant to our conceptual developments in this paper is the notion of a *quantum principal bundle* following Durdevich (1996, 1997). Firstly, one considers a Hopf $*$–algebra α representing a compact quantum group G, and a quantum space P represented by a $*$–algebra β on which G acts on the right, with a co–action given by a $*$–homomorphism $F : \beta \longrightarrow \beta \otimes \alpha$. Taking $\phi : \alpha \longrightarrow \alpha \otimes \alpha$ to be the coproduct map in α, it is required that the diagram below commutes :

$$\begin{array}{ccc} \beta & \xrightarrow{F} & \beta \otimes \alpha \\ {\scriptstyle F}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}\otimes\phi} \\ \beta \otimes \alpha & \xrightarrow{F\otimes \mathrm{id}} & \beta \otimes \alpha \otimes \alpha \end{array}$$

It is necessary to specify what is 'freeness' of the action in this case: for all $a \in \alpha$, there exists elements $q_k, b_k \in \beta$ such that

$$\sum_k q_k F(b_k) = \mathbf{1} \otimes a \ .$$

Thus with the sense of the action of G on P, the base manifold M is defined as the orbit space of the action, so the set–up can be realized in usual notation for a principal G–bundle, viz $G \hookrightarrow P \longrightarrow M$. We refer to Durdevich (1996, 1997) for the full treatment of differential calculus on quantum principal bundles, whereas here we will extract some of the essential ingredients.

Let Γ denote a bicovariant $*$–calculus over G and $\Gamma^{\vee}$ a suitable differential $*$–algebra over Γ . Elements of Γ are regarded as first–order differential forms over G and those of $\Gamma^{\vee}$ are regarded as the higher order differential forms. In this way one obtains a differential graded $*$–algebra (DGA) denoted $\Omega(P)$, such that

(1) $\Omega(P)^0 = \beta$ (that is, the space of functions on the bundle).

(2) $\Omega(P)$ is generated by β . The spaces $\Omega^n(P)$ are spanned by elements of the form

$$\omega = b_0 \ d(b_1) \cdots d(b_n)$$

where $b_i \in \beta$ and $d : \Omega(P) \longrightarrow \Omega(P)$.

(3) The action map induces a DGA homomorphism $F : \Omega(P) \longrightarrow \Omega(P) \otimes \Gamma^{\vee}$.

The space $\Omega(P)$ can be regarded as the *universal differential envelope of* β, on which the appropriate differential calculus is fixed once the context has been determined.

The traditional theory of principal bundles specifies a 'connection' as a choice of a horizontal distribution relative to the fibration $P \longrightarrow M$ (see e.g. Kobayashi and Nomizu 1963), and conversely. In the quantum situation this is achieved by firstly considering a graded $*$–algebra of (quantum) horizontal forms

$hor(P) = F^{-1}[\Omega(P) \otimes \alpha]$,

Next, let Γ_{inv} denote the space of left–invariant elements of the first order calculus Γ . A *connection* on P is then given by every first–order term linear map $\omega : \Gamma_{inv} \longrightarrow \Omega(P)$ satisfying

$F[\omega(\theta)] = \sum_k \omega(\theta_k) \otimes c_k + \mathbf{1} \otimes \theta$,

where $\sum_k \theta_k \otimes c_k = (\theta)$ for which : $\Gamma_{inv} \longrightarrow \Gamma_{inv} \otimes \alpha$ is the corresponding quantum adjoint action. Now referring back to our earlier remark, relative to $P \longrightarrow M$, the connection ω determines a horizontal–vertical splitting via

$\mu_\omega : \Omega(P) \leftrightarrow hor(P) \otimes \Gamma^{\vee}_{inv}$, where $\Gamma^{\vee}_{inv}$ denotes the algebra of left–invariant forms on the group, and μ_ω is left–linear over (quantum) horizontal forms. Next, we have a horizontal projection $h_\omega : \Omega(P) \longrightarrow hor(P)$ (annihilating vertical forms) which leads to defining the covariant derivative $D_\omega = h_\omega d : \Omega(P) \longrightarrow hor(P)$, with the curvature tensor defined as $R_\omega = D_\omega \omega$.

5.4. **Assembling Groupoids: Crossed C-Modules and Crossed Complexes over a Groupoid. The Fundamental Groupoid of a Crossed Complex.** We say that a group G is a *crossed C–module* if there exists a group homomorphism $\theta : G \longrightarrow C$ and an action of C on G denoted $(c, g) \mapsto cg$, such that :

(CM1) for all $g, g' \in G$, we have ${}^{\theta(g)}g' = gg'g^{-1}$;

(CM2) for all $g \in G, c \in C$, we have $\theta(g^c) = c\theta(g)c^{-1}$.

Note that $\theta(G)$ is normal in C .

A generalization of crossed modules is that of *crossed complexes* (see [31]).

A *crossed complex* C (over a groupoid) is a sequence of morphisms of groupoids over C_0 :

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & C_n & \xrightarrow{\delta_n} & C_{n-1} & \longrightarrow \cdots \longrightarrow & C_2 & \xrightarrow{\delta_2} & C_1 \\
 & \downarrow t & & \downarrow t & & \downarrow t & & t \downarrow\downarrow s \\
 & C_0 & & C_0 & & C_0 & & C_0 .
\end{array}
$$

Here $\{C_n\}_{n \geqslant 2}$ is a family of groups with base point map t, and s, t are the source and targets for the groupoid C_1. We assume an operation of the groupoid C_1 on each family of groups C_n for $n \geqslant 2$ such that :

(1) each δ_n is a morphism over the identity on C_0;

(2) $C_2 \longrightarrow C_1$ is a crossed module over C_1;

(3) C_n is a C_1-module for $n \geqslant 3$;

(4) $\delta : C_n \longrightarrow C_{n-1}$ is an operator morphism for $n \geqslant 3$;

(5) $\delta\delta : C_n \longrightarrow C_{n-2}$ is trivial for $n \geqslant 3$;

(6) δC_2 acts trivially on C_n for $n \geqslant 3$.

5.4.1. *The Fundamental Groupoid of a Crossed Complex over Groupoids.* Let C be a crossed complex. Its *fundamental groupoid* $\pi_1 C$ is the quotient of the groupoid C_1 by the normal, totally disconnected subgroupoid δC_2. The rules for a crossed complex give C_n, for $n \geqslant 3$, the induced structure of $\pi_1 C$–module.

5.4.2. *The Fundamental Groupoid of a Topological Space.* newinfo1

5.5. **The Quantum Fundamental Groupoid (QFG) in Quantum Space–Time Representations.** As it will be explained in further detail in Section 5.7, one may wish to consider the structure of quantum space–time in terms of *crossed complexes of quantum groupoids* that represent the local topology of a *quantum field net* in curved space–time. In this representation, the *quantum fundamental groupoid* of curved quantum space–time can be computed, at least in principle, as the groupoid quotient $\pi_1 C$ described above for the 'quantum crossed complex over quantum local groupoids' representing the local algebras of the local, observable quantum fields over the (AQFT) net. The *Quantum Fundamental Groupoid (QFG)*, $\pi_1 C$, is the quotient of the groupoid C_1 by the normal, totally disconnected subgroupoid δC_2 and will reflect the basic *covariance* properties of quantum space–time as it is an algebraic invariant of the space–time topology. As explained next, QFG leads to a new functor between two important categories. First, a *morphism* $f : C \longrightarrow D$ of crossed complexes is defined as a family of groupoid morphisms $f_n : C_n \longrightarrow D_n (n \geqslant 0)$ which preserves all the structure. Secondly, one defines the category Crs of crossed complexes and their morphisms. Then, the fundamental groupoid gives a functor $\pi_1 : Crs \to Gpds$. This functor is left–adjoint to the functor $i : Gpds \to Crs$ where for a groupoid G the crossed complex iG agrees with G in dimensions 0 and 1, and is otherwise trivial. The right–adjoint functor $\underline{i}$ is obviously both *full and faithful*, and also preserves *limits.* In the case of the QFG representation for the quantum space–time, the right–adjoint functor i serves the purpose of naturally embedding, or 'assembling both smoothly and faithfully', the local quantum net 'pieces'– that is, the *locally topological groupoids* (for their definition see the following subsection 5.6.1.1) representing the *local quantum net* regions– into the global structure of a relativistic space–time 'continuum'. On the other hand, the left–adjoint functor QFG preserves *colimits*, which can be further exploited by applying fundamental algebraic topology theorems (see, for example, the CW–complex Approximation and the generalized van Kampen theorems, respectively, in Section 7 and 8) to compute the global homotopy groupoids from local ones. A more detailed example of such a construction is provided in the following Section 5.6 for the unique Holonomy Groupoid that utilizes the Globalization Theorem.

5.6. **The Holonomy Groupoid–A Unique Local to Global Construction based on Locally Topological Groupoids.**

5.6.1. *5.6.1 Locally Topological Groupoids and Local Subgroupoids. Definition of a Wide Subgroupoid.* As before, let $\mathbb{G}$ denote a groupoid and $\mathrm{Ob}(\mathbb{G})$ the set of objects of $\mathbb{G}$, together with the range and source maps $\alpha, \beta : \mathbb{G} \longrightarrow \mathrm{Ob}(\mathbb{G})$. The product hg of two elements of $\mathbb{G}$ is defined if and only if $\alpha h = \beta g$ and so the product map $\gamma : (h, g) \mapsto hg$ is defined on the pullback $\mathbb{G}_\alpha \times_\beta \mathbb{G}$ of α and β . The *difference map* $\delta : \mathbb{G}_\alpha \times_\beta \mathbb{G} \longrightarrow \mathbb{G}$, is given by $\delta(g, h) = gh^{-1}$, and is defined on the double pullback of $\mathbb{G}$ by α . We assume that $X = \mathrm{Ob}(\mathbb{G})$ has the structure of a topological space.

5.6.1.1. Locally Topological Groupoid

A *locally topological groupoid* is a pair $(\mathbb{G}, W)$ consisting of a groupoid $\mathbb{G}$ and a topological space W, such that :

($\mathbb{G}_1$) $\mathrm{Ob}(\mathbb{G}) \subseteq W \subseteq \mathbb{G}$.

($\mathbb{G}_2$) $W = W^{-1}$.

($\mathbb{G}_3$) The set $W_\delta = \{W \times_\alpha W\} \cap \delta^{-1}(W)$ is open in $W \times_\alpha W$ and the restriction to W_δ of the difference map $\delta : \mathbb{G} \times_\alpha \mathbb{G} \longrightarrow \mathbb{G}$ given by $(g, h) \mapsto gh^{-1}$, is continuous.

($\mathbb{G}_4$) The restriction to W of α, β are continuous and (α, β, W) admits enough continuous admissible local sections.

($\mathbb{G}_5$) W generates $\mathbb{G}$ as a groupoid.

5.6.1.2. Local Subgroupoid

A *local subgroupoid* of $\mathbb{G}$ on the topological space X is a continuous global section of the sheaf $p_{\mathbb{G}} : \mathcal{L}_{\mathbb{G}} \longrightarrow X$ associated to the presheaf $L_{\mathbb{G}}$.

An *atlas* $\mathcal{U}_H = \{(U_i, H_i) : i \in I\}$ for a local subgroupoid S of $\mathbb{G}$ consists of an open cover $\mathcal{U} = \{U_i : i \in I\}$ of X, and for each $i \in I$, a wide subgroupoid H_i of $\mathbb{G}|U_i$ such that the following compatibility condition holds, namely, for all $i, j \in I$ and $x \in U_i \cap U_j$, there is an open set W such that $x \in W \subseteq U_i \cap U_j$ and $H_i|W = H_j|W$. Then the local subgroupoid of the atlas is defined as $S(x) = [U_i H_i]_x$.

Suppose $\mathcal{U}'_{H'} = \{(U'_j, H'_j) : j \in J\}$ is another such atlas. Then the compatibility of this atlas with that of above follows if, for all $i \in I,\ j \in J$ and $x \in U_i \cap U'_j$, there is an open set W such that $x \in W \subseteq U_i \cap U'_j$ and $H_i|W = H'_j|W$. Two such compatible atlases define the same local subgroupoid.

5.6.1.3. Local and Global subgroupoids for a Wide Subgroupoid

If H is a *wide subgroupoid* of $\mathbb{G}$ on X, then we define $\mathrm{loc}(H)$ to be the *local subgroupoid* given by

$$\mathrm{loc}(H)(x) = [X, H]_x \ .$$

Given a local subgroupoid S of $\mathbb{G}$, we define $\mathrm{glob}(S)$ to be the wide subgroupoid of $\mathbb{G}$ which is the intersection of all wide subgroupoids H of $\mathbb{G}$ such that $S \leqslant \mathrm{loc}(H)$ where the partial order $\leqslant$ is defined as in [41].

Let $\mathcal{U}_S = \{(U_i, H_i) : i \in I\}$ be an atlas for the local subgroupoid S . We define $\mathrm{glob}(\mathcal{U}_S)$ to be the *Global subgroupoid* of $\mathbb{G}$ generated by all the $H_i,\ i \in I$. An atlas $\mathcal{U}_S$ for S is said to be globally adapted if $\mathrm{glob}(S) = \mathrm{glob}(\mathcal{U}_S)$.

5.6.2. *Admissible Local Section of a Groupoid.* An *admissible local section* of $\mathbb{G}$ is a function $s : U \longrightarrow \mathbb{G}$ from an open subset U of X such that s satisfies :

(1) $\alpha s x = x$ for all $x \in U$.

(2) $\beta s(U)$ is open in X, and

(3) βs maps U homeomorphically to $\beta s(U)$.

The open set U is called the domain of s, denoted $\mathrm{Dom}(s)$. For admissible local sections s, t, the product ts is defined as

$$(ts)x = (t\beta s x)(sx) \ ,$$

where the product on the right is the usual product in $\mathbb{G}$. In this way, $\mathrm{Dom}(ts)$ is an open subset of $\mathrm{Dom}(s)$, and the product yields another admissible local section. If s is an admissible local section, then denote by s^{-1} the admissible local section with domain $(\beta s)\mathrm{Dom}(s)$ and given by $\beta s x \mapsto (sx)^{-1}$. This shows that the set $\Gamma(\mathbb{G})$ of admissible local sections is an inverse semi-group.

5.6.3. *The Globalization Theorem and the Holonomy Groupoid.* The following statement containing the Globalization Theorem may also be applied to locally Lie Groupoids and to the construction of a Lie Holonomy Groupoid.

5.6.3.1. The Globalization Theorem [7]

Let $(\mathbb{G}, W)$ be a locally topological groupoid. Then there is a topological groupoid $\mathbb{H}$, a morphism $\phi : \mathbb{H} \longrightarrow \mathbb{G}$ of groupoids and an embedding $\iota : W \longrightarrow \mathbb{H}$ onto an open neighborhood of $\mathrm{Ob}(\mathbb{H})$*, such that :*

(1) $\phi = \mathrm{id}_W$, *ϕ is the identity on objects (units), $\phi^{-1}(W)$ is open in $\mathbb{H}$ and the restriction $\phi_W : \phi^{-1}(W) \longrightarrow W$ of ϕ, is continuous.*

(2) *If A is a topological groupoid and $\zeta : A \longrightarrow \mathbb{G}$ is a morphism of groupoids such that :*

   (a) *ζ is the identity on objects.*

(b) *The restriction* $\zeta_W : \zeta^{-1}(W) \longrightarrow W$ *is continuous and* $\zeta^{-1}(W)$ *is open in* A *and generates* A .

(c) *The triple* (α_A, β_A, A) *has enough continuous admissible local sections. Then there exists a unique morphism* $\zeta' : A \longrightarrow \mathbb{H}$ *of topological groupoids such that* $\phi \circ \zeta' = \zeta$ *and* $\zeta'(a) = \iota \circ \zeta(a)$ *for* $a \in \zeta^{-1}(W)$.

The above theorem characterizes the existence of the *holonomy groupoid* $\mathbb{H} = \mathrm{Hol}(\mathbb{G}, W)$ the uniqueness of which is given by (2). In summary, we have the diagram of maps :

$$\begin{array}{ccc}
\mathrm{Hol}(\mathbb{G}, W) = \mathbb{H} & \xrightarrow{\ \phi\ } & \mathbb{G} \\
{\scriptstyle =}\big\uparrow & & \big\uparrow \\
\mathrm{Hol}(\mathbb{G}, W) = \mathbb{H} & \xleftarrow{\ \iota\ } & W \supseteq \mathrm{Ob}(\mathbb{G})
\end{array}$$

Let $J(\mathbb{G})$ be defined as the sheaf of germs of admissible local sections of $\mathbb{G}$. The product structure induces a groupoid structure on $J(\mathbb{G}) \rightrightarrows X$. Let $\psi : J(\mathbb{G}) \longrightarrow \mathbb{G}$ and set $J_0 = J^r(W) \cap \ker \psi$, where $J^r(W)$ denotes the sheaf of germs of local admissible sections with values in W . Letting $J^r(W, \mathbb{G})$ denote the subgroupoid generated by $J^r(W)$,and denoting the sheaf of germs of elements of $\Gamma^r(W)$, one has $\mathbb{H} = \mathrm{Hol}(\mathbb{G}, W) = J^r(\mathbb{G}, W)/J_0$.

5.6.4. *Locally Lie Groupoids and Locally Lie Subgroupoids.* We begin by defining a *locally Lie groupoid.*

5.6.4.1. Locally Lie Groupoids

A *locally Lie groupoid* is a pair $(\mathbb{G}, W)$ consisting of a groupoid $\mathbb{G}$ and a smooth manifold W, such that :

($\mathbb{G}_1$) $\mathrm{Ob}(\mathbb{G}) \subseteq W \subseteq \mathbb{G}$.

($\mathbb{G}_2$) $W = W^{-1}$.

($\mathbb{G}_3$) The set $W_\delta = \{W \times_\alpha W\} \cap \delta^{-1}(W)$ is open in $W \times_\alpha W$ and the restriction to W_δ of the difference map $\delta : \mathbb{G} \times_\alpha \mathbb{G} \longrightarrow \mathbb{G}$ given by $(g, h) \mapsto gh^{-1}$, is smooth.

($\mathbb{G}_4$) The restrictions to W of α, β are smooth and (α, β, W) admits enough smooth admissible local sections.

($\mathbb{G}_5$) W generates $\mathbb{G}$ as a groupoid.

5.6.4.2. Local Lie subgroupoids

A Lie local subgroupoi*d* S of a Lie groupoid $\mathbb{G}$, is a local subgroupoid S given by an atlas $\mathcal{U}_S = \{(U_i, H_i) : i \in I\}$, such that for $i \in I$, each H_i is a Lie subgroupoid of $\mathbb{G}$.

5.6.5. *A Locally Topological Subgroupoid defined as a Sheaf.* Let X be a topological space and $\alpha, \beta : \mathbb{G} \rightrightarrows X = \mathrm{Ob}(\mathbb{G})$, a groupoid. For an open set $U \subseteq X$, let $\mathbb{G}|U$ be the full subgroupoid of $\mathbb{G}$ on U . Let $L_{\mathbb{G}}(U)$ be the set of all wide subgroupoids of $\mathbb{G}|U$. For $V \subseteq U$, there is a restriction map $L_{UV} : L_{\mathbb{G}}(U) \longrightarrow L_{\mathbb{G}}(V)$ sending $H \mapsto H|V$. So $L_{\mathbb{G}}$ has the structure of a presheaf on X .

Consider the sheaf $p_{\mathbb{G}} : \mathcal{L}_{\mathbb{G}} \longrightarrow X$ formed from the presheaf $L_{\mathbb{G}}$. For $x \in X$, the stalk $p_{\mathbb{G}}^{-1}(x)$ of $\mathcal{L}_{\mathbb{G}}$ has elements the germs of classes of equivalence relations $[U, H_U]_x$, for U open in X and $x \in U$. Here, $H|U$ is a wide subgroupoid of $\mathbb{G}|U$ and the equivalence relation $\sim_x$ yielding the germ at x, is such that $H_U \sim_x K_V$, where K_V is a wide subgroupoid of $G|V$ if and only if there exists a neighbourhood W of x such that $W \subseteq U \cap V$ and $H_U|W = K_V|W$.

The topology of $\mathcal{L}_{\mathbb{G}}$ is the usual sheaf topology with a sub–base of sets $\{[U, H_U]_x : x \in X\}$, for all open subsets U of X and wide subgroupoids H of $\mathbb{G}|U$.

5.7. **Emergence of Extended 'Supersymmetry' by Assembling Quantum Topological Groupoids (QTGs): *Linking Quantum Metric Spaces to QTGs in QST Representations.*** Let us consider a Quantum Crossed Complex (QCC), Q_{XC}, consisting of a sequence of QTGs, $\{Q_{T_i}\}$ over a Quantum Compact Groupoid (QCG), Q_{CG}, as defined above, respectively, in Sections 5.2, 5.6.1 (or in 5.6.2 as well as 5.6.5) and 4.2.1. In particular, one can construct the QCC over a *Quantum Metric Space (QMC)* treated as a Quantum Groupoid and thus obtain a 'standard' *quantum groupoid representation* of QST based on the QSS generated by the QMC [154]. Note that the latter was *not*, however, employed in ref [154] to actually construct a quantized space–time compatible with the compact quantum metric specified by an extended quantum groupoid symmetry, as we shall now proceed to accomplish. We utilize the *quantum 'algebraic- topological-metric'* concept described in ref. [154], to construct here an extended (n-dimensional) version of a GR Riemannian QST representation which will be based on the Quantum State Space generated by the QMC; its symmetry properties will be, however, determined by the QCC construction and its defining sequence of QTGs, $\{Q_{T_i}\}$. Its *metric* is not only *compatible* with the extended, n-dimensional GR Riemannian space–time but also is thus *quantized* (at least as far as the *first quantization* goes)! Furthermore, one can now obtain the correct expression for the *Quantum Fundamental Groupoid* of this novel QST representation as described above in more detail in Sections 5.3 and 5.4. The *Quantum Fundamental Groupoid (QFG)*, $\pi_1(Q_{XC})$, is the quotient of the groupoid Q_{T_1} by the normal, totally disconnected subgroupoid δQ_{T_2} and will reflect the basic *covariance* properties of the n-dimensional quantum space–time as it is an *algebraic invariant* of the space–time topology. The definitions and general preservation properties of the adjoint functors associated with the Quantum Fundamental Groupoid computation carry over from Section 5.4 to our assembly of the quantum crossed complex, Q_{XC}. Our construction is therefore a powerful *higher-dimensional extension* of all the existing AQFT, TQFT and HQFT constructions combined. The category of QCCs and the adjoint functors involved are endowed with a higher-dimensional, or meta-categorical, *'supersymmetry'* that allows one to develop generalized versions of either AQFTs and HQFTs, either 'with' or *'without a background'*. Furthermore, our novel constructions– unlike TFTs– do contain also a *valid quantum metric* for the extended higher-dimensional GR. Therefore, one can employ our QMC-based representations of QST to generate entire classes of *higher-dimensional Quantum Gravity theories* that are *consistent* with the extended, n-dimensional GR space–time and all of the GR theories based on a *Riemannian space–time of any finite dimension.* The selection of the 'right' relativistic QG theory, or of the 'right' *equivalence* class of such theories, will have to be made, however, based on physical constraints and restrictions rather than 'purely mathematical' considerations. Our construction therefore validates **Conjecture 1** that was made in Section 1 *(Introduction).* This is indeed the case at least for the class of relativistic QG theories that are built on a *quantum metric space* and employ the compatible, quantum groupoid symmetries assembled into the quantum crossed complex, Q_{XC}. Note, however, that neither our new construction nor *Conjecture 1* that we proposed in *Section 1* do *either prove or disprove* the internal, logical consistency of any GR theory in existence at the present time. Such internal consistency is, indeed, the first open question that we short-listed at the beginning of our discussion in the *"Introduction"* section.

One expects that there may be other, *symmetrically analogous*–but *dynamically different*–constructions, as illustrated next. The latter could also be developed to readily include M-theories, n-branes, twistors and also 'supersymmetric' superstring theories. In the following section (5.7) we shall take this construction one step further by considering the existence issue of a 'complete', or 'global' quantum measurement for any higher-dimensional QSS (with $n > 2$) in terms of a Quantum Atlas.

5.7 Building the Quantum Atlas. The Existence Question of a Global Section in a Generalized QSS

5.7.1 The Atlas Definition

An atlas $\mathcal{U}_S = \{(U_i, H_i) : i \in I\}$ for a Lie local subgroupoid S of $\mathbb{G}$, is said to be *regular* if the groupoid (α_i, β_i, H_i) is locally sectionable for all $i \in I$. We say that S is *regular* if it has a regular atlas.

5.7.2 Definition of a Strictly Regular Atlas

An atlas $\mathcal{U}_S = \{(U_i, H_i) : i \in I\}$ for a *Lie local subgroupoid* S of $\mathbb{G}$, is said to be strictly regular if :

- $\mathcal{U}_S$ is globally adapted to S .
- $\mathcal{U}_S$ is regular.
- $W(\mathcal{U}_S)$ defined with its topology as a subset of $\mathbb{G}$,has the structure of a smooth submanifold containing each H_i, $i \in I$, as an open submanifold of $W(\mathcal{U}_S)$ and such that $W(\mathcal{U}_S)(\delta)$ is open in $W(\mathcal{U}_S) \times_\alpha W(\mathcal{U}_S)$.

5.8. **'Global' Non-Abelian Gauge Transformations. Quantum Crossed Modules (QCM), Quantum Convolution Groupoids (QCG) and Their Quantum Cross Complexes (QCC)..**

5.9. **Lie Algebroids.** One can think of a *Lie algebroid* as generalizing the idea of a *tangent bundle* where the tangent space at a point is effectively the equivalence class of curves meeting at that point (thus suggesting a groupoid approach), as well as serving as a site on which to study infinitesimal geometry. Specifically, let M be a manifold and let $\mathfrak{X}(M)$ denote the set of vector fields on M . A *Lie algebroid over* M consists of a vector bundle $E \longrightarrow M$, equipped with a Lie bracket $[\ ,\ ]$ on the space of sections $\Gamma(E)$, and a bundle map $\Upsilon : E \longrightarrow TM$, called the *anchor*. Further, there is an induced map $\Upsilon : \Gamma(E) \longrightarrow \mathfrak{X}(M)$, which is required to be a map of Lie algebras, such that given sections $\alpha, \beta \in \Gamma(E)$ and a differentiable function f, the following Leibniz rule is satisfied :

$$[\alpha, f\beta] = f[\alpha, \beta] + (\Upsilon(\alpha))\beta \ .$$

The problem of 'integrability' of Lie algebroids can be traced all the way back to Lie's work on those classes of algebras and groups which were to bear his name. When M is just a point, we recover the definition of a Lie algebra and Lie's third theorem says that this Lie algebra can be integrated to a *'unique'* Lie group (see also Section 3.5, above). More general results follow when the anchor map Υ is constrained to satisfy certain conditions. So one can see that the development of this subject has its roots in the possible extension of Lie's three structure theorems to Lie algebroids (groupoids); we refer to Crainic and Fernandes (2003), Mackenzie (1987), (ref. [131]) for historical coverage.

Now suppose we have a *Lie groupoid* $\mathbb{G}$

$$r, s \ : \ \mathbb{G} \underset{s}{\overset{r}{\rightrightarrows}} M = \mathbb{G}^{(0)}$$

; then, there exists an *associated Lie algebroid* $\alpha = \alpha(\mathbb{G})$, which in the guise of a *vector bundle*, as it is the *restriction to* M of the bundle of tangent vectors along the fibers of s (ie. *the s–vertical vector fields*). Also, the space of sections $\Gamma(\alpha)$ can be indentified with the space of s–vertical, right–invariant vector fields $\mathfrak{X}^s_{inv}(\mathbb{G})$ which can be seen to be closed under $[\ ,\ ]$, and the latter induces a bracket operation on $\Gamma(A)$ thus turning α into a Lie algebroid. Subsequently, *a Lie algebroid α is integrable if there exists a Lie groupoid $\mathbb{G}$ inducing α* . So what are the computable obstructions to the (local) integrability of Lie algebroids? These and other questions have been answered quite recently in Crainic and Fernandes (2003). Furthermore, when attempting to apply the concept of a Lie algebroid to Quantum Field Theory, and especially to Quantum Electrodynamics (QED) one finds this concept to be insufficient to completely represent, for example, gauge theories (Baez, 2002). Therefore, the generalized concept of Lie 2-groups, Lie 2-algebras and 2-bundles were considered in the context of Yang-Mills actions and categorified Yang-Mills equations (loc.cit.). In the context of AQFT (Roberts, 2004), the structure of a 2-category emerged from considerations about posets as a basis for curved quantum space–time (see also Sections 6 and 7).

5.10. **Applications of 2–Lie Groups and Groupoids in Non-Abelian Gauge Theories, AQFT and Quantum Gravity.** New Info3

5.11. **Lattice Quantum Gravity (LQG), Gauge Theories and New Approaches to Many-Body Problems.** *Pfeiffer 2-Lie groups, 2-categories and gauge theories*

6. Supergravity Theories. The Metric Superfield and Supersymmetry Algebras.

In this subsection we shall briefly review the 'standard' supergravity theories in a quasi-linear form for *weak* gravitational fields and with a *QST metric consistent with GR.*

6.1. **Supergravity Theories.** Supergravity, in essence, is an extended supersymmetric theory of both matter and gravitation (Weinberg, 2000). A first approach to supersymmetry relies on a curved 'superspace' (Wess and Bagger, 2000) and is analogous to supersymmetric gauge theories (see, for example, Sections 27.1 to 27.3 of Weinberg, 2000). Unfortunately, a complete non-linear supergravity theory would be "forbiddingly complicated" and , furthermore, the constraints that need be made on the graviton superfield appear somewhat subjective (cf. Weinberg, 2000). On the other hand, the second approach to supergravity is much more transparent than the first, albeit theoretically less elegant. The physical components of the gravitational superfield can be identified in this approach based on flat-space superfield methods (Chs. 26 and 27 of Weinberg, 2000). By employing the weak-field approximation one obtains several of the most important consequences of supergravity theory, including masses for the hypothetical gravitino and gaugino 'particles' whose existence is expected from supergravity theories. Furthermore, by adding on the higher order terms in G (the gravitational 'constant') to the supersymmetric transformation, the general coordinate transformations form a closed algebra and the Lagrangian that describes the interactions of the physical fields is invariant under such transformations. Quantization of such a flat-space superfield would obviously involve its 'deformation' as discussed in Section 3 above, and as a result its corresponding *supersymmetry algebra* (see Section 6.3) would become *non–commutative.*

6.2. **The Metric Superfield.** Because in supergravity both spinor and tensor fields are being considered, the gravitational fields are represented in terms of *tetrads*, $e^a_\mu(x)$, rather than in terms of the general relativistic metric $g_{\mu\nu}(x)$. The connections between these two distinct representations are as follows:

$$g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\gamma(x),$$

with the general coordinates being labeled by μ, ν, etc., whereas local coordinates that are being defined in a locally inertial coordinate system are labeled with superscripts a, b, etc.; η_{ab} is the diagonal matrix with elements +1, +1, +1 and -1. The tetrads are invariant to two distinct types of symmetry transformations–the local Lorentz transformations:

$$e^a_\mu(x) \longmapsto \Lambda^a_b(x) e^b_\mu(x),$$

(where Λ^a_b is an arbitrary real matrix), and the general coordinate transformations:

$x^\mu \longmapsto x'^\mu(x)$.

In a weak gravitational field the tetrad may be represented as:

$e^a_\mu(x) = \delta^a_\mu(x) + 2\kappa\Phi^a_\mu(x)$,

where $\Phi^a_\mu(x)$ is small compared with $\delta^a_\mu(x)$ for all x values, and $\kappa = \sqrt{8\pi G}$, where G is Newton's gravitational constant. As it will be discussed next, the supersymmetry algebra (SA) implies that the graviton has a fermionic superpartner, the hypothetical *gravitino*, with helicities ± 3/2. Such a self-charge-conjugate massless particle as the gravitiono with helicities ± 3/2 can only have *low-energy* interactions if it is represented by a Majorana field $\psi_\mu(x)$ which is invariant under the gauge transformations:

$\psi_\mu(x) \longmapsto \psi_\mu(x) + \delta_\mu\psi(x)$,

with $\psi(x)$ being an arbitrary Majorana field as defined by Grisaru and Pendleton in 1977. The tetrad field $\Phi_{\mu\nu}(x)$ and the graviton field $\psi_\mu(x)$ are then incorporated into a *vector superfield* $H_\mu(x,\theta)$ defined as the *metric superfield*. The relationships between $\Phi_{\mu\nu}(x)$ and $\psi_\mu(x)$, on the one hand, and the components of the metric superfield $H_\mu(x,\theta)$, on the other hand, can be derived from the transformations of the whole metric superfield:

$H_\mu(x,\theta) \longmapsto H_\mu(x,\theta) + \Delta_\mu(x,\theta)$

by making the simplifying- and physically realistic- assumption of a weak gravitational field. Further details can be found, for example, in Ch.31 of vol.3. of Weinberg (2000). The interactions of the whole superfield $H_\mu(x)$ with matter would be then described by considering how a weak gravitational field, $h_{\mu\nu}$ interacts with an energy-momentum tensor $T^{\mu\nu}$ represented as a linear combination of components of a real vector superfield Θ^μ. Such interaction terms would, therefore, have the form:

$I_{matter} = 2\kappa \int dx^4 [H_\mu \Theta^\mu]_D$,

where the integration space is a four-dimensional ('Minkowski-like') space-time with the metric defined by the superfield $H_\mu(x,\theta)$. Θ^μ, as defined above, is physically a *supercurrent* and satisfies the conservation conditions:

$\gamma^\mu \boldsymbol{D}\Theta_\mu = \boldsymbol{D}X$,

where $\boldsymbol{D}$ is the four-component super-derivative and X is a real chiral scalar superfield. This leads immediately to the calculation of the interactions of matter with a weak gravitational field as:

$I_{matter} = \kappa \int d^4x T^{\mu\nu}(x) h_{\mu\nu}(x)$.

It is quite interesting that the gravitational actions for the superfield that are invariant under the generalized gauge transformations $H_\mu \longmapsto H_\mu + \Delta_\mu$ leads to solutions of the Einstein field equations for a homogeneous, non-zero vacuum energy density ϱ_V that are either a deSitter space for $\varrho_V > 0$, or an anti-deSitter space for $\varrho_V < 0$. Such spaces can be represented then as surfaces :

$$x_5^2 \pm \eta_{\mu\nu} x^\mu x^\nu = R^2$$

in a quasi-Euclidean five-dimensional space with the "distance" (line element) specified as:

$$ds^2 = \eta_{\mu\nu} x^\mu x^\nu \pm dx_5^2,$$

with '+' for deSitter spaces and '-' for anti-deSitter space, respectively.

The space-time symmetry groups, or groupoids –as the case may be– are different from the 'classical' Poincare symmetry group of translations and Lorentz transformations. Such space-time symmetry groups, in the simplest case, are therefore the O(4,1) group for the deSitter space and the O(3,2) group for the anti- deSitter space. A detailed calculation indicates that the transition from ordinary flat space to a bubble of anti-deSitter space is not favored energetically and, therefore, the ordinary (deSitter) flat space is stable (cf. Coleman and deLuccia, 1980), even though quantum fluctuations might occur to an anti-deSitter bubble within the limits permitted by the Heisenberg uncertainty principle.

6.3. **The Supersymmetry and Graded Lie Algebras.** The expression of supersymmetry in a similar manner to the generation of Lie algebras will be discussed here together with the introduction of *Graded Lie* Algebras.

6.3.1. *Graded Lie Algebras and Graded Parameters.* It was shown in subsections 3.4 and 5.9 how continuous symmetry transformations can be represented in terms of a Lie algebra of linearly independent symmetry generators t_j that satisfy commutation relations:

$$[t_j, t_k] = i\Sigma_l C_{jk} t_l.$$

Supersymmetry is similarly expressed in terms of the symmetry generators t_j of a *graded* Lie algebra (**ref**) which satisfy relations of the general form:

$$t_j t_k - (-1)^{\eta_j \eta_k} t_k t_j = i\Sigma_l C^l_{jk} t_l.$$

The generators for which $\eta_j = 1$ are called **fermionic** whereas those for which $\eta_j = 0$ are called *bosonic*. The coefficients C^l are called *structure constants* and must satisfy the following conditions: $C^l_{jk} = -(-1)^{\eta_j \eta_k} C^l_{jk}$.

If the generators t_j are quantum Hermitian operators, then the structure constants satisfy the reality conditions: $C^{l*}_{jk} = -C^{l}_{jk}$.

The standard computational approach in QM utilizes the S-matrix approach, and therefore,one needs consider here the general graded Lie algebra of supersymmetry generators that commute with the S-matrix. If one denotes the fermionic generators by Q, then $U^{-1}(\Lambda)QU(\Lambda)$ will also be of the same type when $U(\Lambda)$ is the quantum operator corresponding to arbitrary, homogeneous Lorentz transformations Λ^{μ}_{ν}. Such a group of generators provide therefore a representation of the homogeneous Lorentz group of transformations, $\underline{\boldsymbol{L}}$. The irreducible representation of the homogeneous Lorentz group of transformations provides therefore a classification of such individual generators.

6.3.2. *The Supersymmetry Algebras.* A set of quantum operators $Q^{AB}{}_{j\ k}$ form an (**A**,**B**) representation of the group $\underline{\boldsymbol{L}}$ defined above which satisfy the commutation relations:

$[\mathbf{A}, Q^{AB}{}_{j\ k}] = -[\Sigma^{\prime}_{j} J^{A\prime}_{j\ j} Q^{AB\prime}{}_{j\ k}]$
and

$[\mathbf{B}, Q^{AB}{}_{j\ k}] = -[\Sigma^{\prime}_{j} J^{A\prime}_{k\ k} Q^{AB\prime}{}_{j\ k}]$,

with the generators **A** and **B** defined by
$\mathbf{A} \equiv (1/2)(\mathbf{J} \pm i\mathbf{K})$ and $\mathbf{B} \equiv (1/2)(\mathbf{J} - i\mathbf{K})$,

with **J** and **K** being, respectively, the Hermitian generators of rotations and 'boosts'.

In the case of the two-component Weyl-spinors Q_{jr} the Haag-Lopuszanski-Sohnius (HLS) theorem applies, and thus one has that the fermions form a *supersymmetry algebra* defined by the anti-commutation relations

$[Q_{jr}, Q^{*}_{ks}] = 2\delta_{rs}\sigma^{\mu}_{jk}P_{\mu}$,
$[Q_{jr}, Q_{ks}] = e_{jk}Z_{rs}$,

where P_{μ} is the 4-momentum operator, $Z_{rs} = -Z_{sr}$ are the bosonic symmetry generators, and σ_{μ} and **e** are 2x2 matrices:

$$\sigma_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\sigma_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $\mathbf{e} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Furthermore, the fermionic generators commute with both energy and momentum operators:

$$[P_\mu, Q_{jr}] = [P_\mu, Q^*_{jr}] = 0.$$

The bosonic symmetry generators Z_{ks} and Z^*_{ks} represent the set of *central charges* of the *supersymmetric algebra*:

$$[Z_{rs}, Z^*_{tn}] = [Z^*_{rs}, Q_{jt}] = [Z^*_{rs}, Q^*_{jt}] = [Z^*_{rs}, Z^*_{tn}] = 0.$$

7. Fundamental Algebraic Topology Theorems with Applications to *Local-to-Global* Constructions of Quantum Space-Time and Quantum State Space Representations

7.1. **The Fundamental Theorem of Hurewicz and The Whitehead Theorem: *Related Propositions and Their Applications*.** Until recently, *homology* and *cohomology* groups were, more readily computed than homotopy groups for topological spaces of somewhat arbitrary complexity. Cohomology does provide, however, a *more sensitive algebraic invariant* of topological spaces than homology by virtue of being able to introduce a *ring* structure through the definition of a product which is not possible for homology. Thus, *cohomology* can distinguish between topological spaces that have *isomorphic homology* groups.

Furthermore, the fundamental Hurewicz theorem and its generalization to arbitrary topological spaces establishes a direct link between *homology* and *homotopy* that could be exploited for a wide category of topological spaces to link certain isomorphic homotopy groups with homology ones, the latter being the easier to compute for a wide range of topological spaces. The precise formulation of the generalized Hurewicz fundamental theorem is presented next.

7.1.1. *The Generalized Hurewicz Fundamental Theorem.* The Hurewicz theorem was generalized from connected CW–complexes to arbitrary topological spaces (Spanier, 1966).

Theorem 7.1.1.1

If $\pi_r(K,L) = 0$ *for* $1 \leqslant r \leqslant n$, $(n \geqslant 2)$, *then* $h_\pi : \pi_n^*(K,L) \simeq H_n(K,L)$, *where* π_n *are homotopy groups,* H_n *are homology groups, K and L are arbitrary topological spaces, and '*$\simeq$*' denotes an isomorphism.*

7.1.2. *Some Basic Algebraic Topology Concepts : CW-Complexes. N-connected CW (Complex) Models and Graphs. Weak Homotopy Equivalence.* A few key, AT concepts are here necessary in order to be able to present three fundamental AT theorems and then to apply such theorems to Quantum Spin Networks and Quantum Spin 'Foam' representations in terms of *CW-complexes* and *n-connected CW (complex) Models.*

7.1.2.1. Definition of Weak Homotopy Equivalence

If $f_* : \pi_o(X) \longrightarrow \pi_o(Y)$ is 1-1 and if $f_* : \pi_r(X, x_o) \simeq \pi_r()Y, f(x_o))$ for all $r \geqslant 1$, and for any x_o in X, then f_* is called a *weak homotopy equivalence*, where $' \simeq'$ denotes an isomorphism.

7.1.2.2: Definition of a CW–complex

A *CW–complex* X_c is a topological space which is the union of an expanding sequence of subspaces X^n such that, inductively, X^0 is a discrete set of points called vertices and X^{n+1} is the pushout obtained from X^n by attaching disks D^{n+1} along "attaching maps" $j : S^n \to X^n$. Each resulting map D^{n+1} is called a "cell". The subspace X^n is called the "n-skeleton" of X.

Example: A graph is a one-dimensional CW–complex. Spin networks are one-dimensional CW–complexes, whereas 'spin foams' are two-dimensional CW–complexes representing two local spin networks with quantum transitions between them, sometimes represented also as functors [**?**]Baez, 1998; ref.[]).

7.1.2.3. Definition of the n-connected CW (Complex) Model

7.1.3. *The J.H.C. Whitehead Theorem.* [137]

(Short version) *If* $f : K \longrightarrow L$ *is a weak homotopy equivalence of CW–complexes K and L, then f is a homotopy equivalence.*

Theorem 7.1.3.1

If $f : K \longrightarrow L$ is a map of CW–complexes that induces $f_* : \pi_r(K) \simeq \pi_r(L)$ for all $r \geqslant 0$ then f is a homotopy equivalence.

7.2. **The Cohomology Group Theorem.** [137]

The cohomology group $H^{\sim n}(K;G)$ is equivalent to the homotopy group $[K, K(G,n)]$ when K(G,n) is a CW–complex such that: $\pi_r(K(G,n)) \simeq G$, if r= n, or 0 otherwise.

7.3. **The Approximation Theorem for an Arbitrary Space: *The Colimit of a Sequence of CW–complexes*.** [137]

There is a functor $\Gamma : \mathbf{hU} \longrightarrow \mathbf{hU}$ where $\mathbf{hU}$ is the homotopy category for unbased spaces (p. 14 of May, 1999;[?]), and a natural transformation $\gamma : \Gamma \longrightarrow Id$ that asssigns a CW–complex ΓX and a weak equivalence $\gamma_e : \Gamma X \longrightarrow X$ to an arbitrary space X, such that the following diagram commutes:

$$\begin{array}{ccc} \Gamma X & \xrightarrow{\Gamma f} & \Gamma Y \\ \downarrow{\scriptstyle \gamma(X)} & & \downarrow{\scriptstyle \gamma(Y)} \\ X & \xrightarrow{f} & Y \end{array}$$

and

$$\Gamma f : \Gamma X \to \Gamma Y$$

is unique up to homotopy equivalence.

The CW–complex specified in the Approximation Theorem is constructed as the colimit ΓX of a sequence of cellular inclusions of CW–complexes $X_1, ..., X_n$, so that one obtains $X \equiv colim[X_i]$. As a consequence of J.H.C. Whitehead's Theorem, one also has that:

$\gamma* : [\Gamma X, \Gamma Y] \longrightarrow [\Gamma X, Y]$ is an isomorphism.

Furthermore, the homotopy groups of the CW–complex ΓX are the colimits of the homotopy groups of X_n and $\gamma_{n+1} : \pi_q(X_{n+1}) \longmapsto \pi_q(X)$ is a group epimorphism.

7.4. **The Quantum Algebraic Topology of CW–complex Representations: New QAT Theorems for Quantum State Spaces of Spin Networks and Quantum Spin 'Foams' based on CW n-Connected Models and AT fundamental theorems.** .

7.4.1. *Spin Network Dynamics and Quantum Spin 'Foam' Definitions.* We shall consider first a Lemma to facilitate the proof of subsequent theorems concerning Spin Networks and Quantum Spin 'Foams'.

7.4.2. *Lemma 7.4.2. Let Z be a CW–complex that has the (three-dimensional) Quantum Spin 'Foam' (QSF) as a subspace. Furthermore, let $f : Z \to QSS$ be a map so that $f \mid QSF = 1_{QSF}$, with QSS being an arbitrary, local quantum state space (which is not necessarily finite). There exists an n-connected CW model (Z,QSF) for the pair (QSS,QSF) such that :*

$f_* : \pi_i(Z) \to \pi_i(QST)$ is an isomorphism for $i > n$ and it is a monomorphism for $i = n$. The n-connected CW model is unique up to homotopy equivalence.

(The CW–complex, Z, considered here is a homotopic 'hybrid' between QSF and QSS).

Proof

Theorem 7.4.3.

For every pair (QSS, QSF) of topological spaces defined as in Lemma 7.4.2, with QSF nonempty, there exist n-connected CW models $f : (Z, QSF) \to (QSS, QSF)$ for all $n \geqslant 0$ and such models can be selected to have the property that the CW–complex Z is obtained from QSF by attaching cells of dimension $n > 2$, and therefore (Z,QSF) is n-connected. Following Lemma 7.4.2 one also has that the map

$f_* : \pi_i(Z) \to \pi_i(QSS)$ is an isomorphism for $i > n$, and it is a monomorphism for $i = n$.

Proof

Theorem 7.4.4.

Let ${QF_j}_{j=1,\dots,n}$ be a complete sequence of commuting quantum spin 'foams' (QSFs) in an arbitrary Quantum State Space (QSS), and let (QF_j, QSS_j) be the corresponding sequence of pair subspaces of QST. If Z_j is a sequence of CW–complexes such that for any j, $QF_j \subseteq Z_j$, then there exists a sequence of n-connected models (QF_j, Z_j) of (QF_j, QSS_j) and a sequence of induced isomorphisms $f^j_* : \pi_i(Z_j) \to \pi_i(QSS_j)$ for $i > n$, and a sequence of induced monomorphisms for $i = n$.

Note that there exist *weak* homotopy equivalences between each Z_j and QSS_j spaces in such a sequence. Therefore, there exists a CW-approximation of QSS defined by the sequence ${Z_j}_{j=1,\ldots,n}$ of CW–complexes with dimension $n \geqslant 2$. This CW-approximation is unique up to (*regular*) homotopy equivalence.

Corollary 7.4.5.

The n-connected models (QF_j, Z_j) *of* (QF_j, QSS_j) *form the* Model Category of Quantum Spin 'Foams' (QF_j) *whose morphisms are maps* $h_{jk} : Z_j \to Z_k$ *such that* $h_{jk} \mid QF_j = g : (QSS_j, QF_j) \to (QSS_k, QF_k)$*, and such that the following diagram is commutative:*

$$\begin{array}{ccc} Z_j & \xrightarrow{f_j} & QSS_j \\ {\scriptstyle h_{jk}}\downarrow & & \downarrow{\scriptstyle g} \\ Z_k & \xrightarrow{f_k} & QSS_k \end{array}$$

Furthermore, the maps h_{jk} *are unique up to the homotopy rel* QF_j *(and* QF_k*).*

Proof

7.5. **The Quantum Algebraic Topology of Loop Quantum Gravity (LQG) and Closed Superstrings in QST.** newinfo5

7.6. **Novel Homotopic Representations of GR-QST: The Quantum Gravitonic Homotopy (QGH).** New Info QGH

As discussed next, the generalized van Kampen theorem offers a method for the local-to-global (isomorphic) construction of a topological space X in terms of the homotopy double groupoids in the ϱ-sequence of the open covering of X. This allows one to build more complex topological spaces from simpler ones with known homotopy (double) groupoids.

8. The Generalized van Kampen Theorem and Its Possible Applications to Physical Representations in Quantum Algebraic Topology

[The first three sections 8.1 to 8.3. are the former "sections 6.1 to 6.3" in **QAT v18**].

8.1. **The van Kampen Theorem, Its Generalizations and Applications.** There are several generalizations of the original van Kampen theorem, such as its extension to crossed complexes, its extension in categorical form in terms of colimits, and its generalization to higher dimensions, i.e., its extension to 2-groupoids and 2-categories [33][]

In this way one obtains comparatively quickly not only classical results such as the Brouwer degree and the relative Hurewicz theorem, but also *non–commutative* results on second relative homotopy groups, as well as *higher dimensional* results involving the action of and also presentations of the *fundamental group.* For example, *the fundamental crossed complex* ΠX_* of the skeletal filtration of a CW–complex X is a useful generalisation of the usual cellular chains of the universal cover of X. It also gives a replacement for singular chains by taking X to be the geometric realisation of a singular complex of a space.

8.2. **Applications of the Van Kampen Theorem to Crossed Complexes and Representations of Quantum Space-Time in terms of Quantum Crossed Complexes over a Quantum Groupoid.** Crossed complexes have several *advantages* in Algebraic Topology such as:

• They are good for *modelling CW-complexes.* Free crossed resolutions enable calculations with *small CW-models* of $K(G, 1)$s *and their maps* (Whitehead, Wall, Baues).

•Also, they have an interesting relation with the Moore complex of simplicial groups and of *simplicial groupoids.*

• They *generalise groupoids and crossed modules to all dimensions.* Moreover, the natural context for the second relative homotopy groups is crossed modules of groupoids, rather than groups.

• They are convenient for *calculation*, and the functor Π is classical, involving *relative homotopy groups.*

• They provide a kind of *'linear model'* for homotopy types which includes all 2-types. Thus, although they are not the most general model by any means (they do not contain quadratic information such as Whitehead products), this simplicity makes them easier to handle and to relate to classical tools. The new methods and results obtained for crossed complexes can be used as a model for more complicated situations. For example, this is how a general n-adic Hurewicz Theorem was found [67].

• Crossed complexes have a *good homotopy theory*, with a *cylinder object, and homotopy colimits.* (A *homotopy classification* result generalises a classical theorem of Eilenberg-Mac Lane).

• They are close to chain complexes with a group(oid) of operators, and related to some classical homological algebra (e.g. *chains of syzygies*). In fact if SX is the simplicial singular complex of a space, with its skeletal filtration, then the crossed complex $\Pi(SX)$ can be considered as a slightly *non commutative version of the singular chains of a space.*

Note that all these important advantages also apply to our construction of QST representations in terms of a Quantum Crossed Complex over a Quantum Metric Space that we developed in Section 7.6.

Also note that a replacement for the excision theorem in homology is obtained by using cubical methods to prove *a colimit theorem for the fundamental crossed complex functor on filtered spaces*. This *colimit theorem* is a higher dimensional version of a classical example of a *non–commutative local-to-global theorem*, which itself was the initial motivation for the work by R. Brown on generalizations of the Van Kampen Theorem. This Seifert-Van Kampen Theorem (SVKT) determines completely the fundamental group $\pi_1(X, x)$ of a space X with base point which is the union of open sets U, V whose intersection is path connected and contains the base point x; the *'local information'* is on the morphisms of fundamental groups induced by the inclusions $U \cap V \to U, U \cap V \to V$. The importance of this result reflects the importance of the *fundamental group* in algebraic topology, *algebraic geometry*, *complex analysis*, and many other, mathematical subjects. Indeed, the origin of the fundamental group was in Poincaré's work on *monodromy for complex variable theory.* Essential to this use of crossed complexes and the colimit theorem, is a construction of *higher homotopy groupoids*, with properties described by *an algebra of cubes.* Such a construction is particularly important for conjecturing and proving *local-to-global* theorems since homotopical methods play a key role in many areas. There are applications to *local-to-global problems in homotopy theory* which are more powerful than purely classical tools, while shedding light on those tools.

Furthermore, with the advent of Quantum Group Algebras, Quantum Groupoids, Quantum Algebra, and especially Quantum Algebraic Topology, such fundamental theorems in Algebraic Topology will acquire an enhanced importance through their applications to current problems in Theoretical Physics such as those described in the previous sections of this paper, and especially in Sections 4 to 7. Moreover, there are several applications of the generalizations of the van Kampen theorem, and especially those related to the concept of *fundamental groupoid* that was discussed in Section 5. Thus, the Van Kampen Theorem was generalized by formulating it for the *fundamental groupoid* $\pi_1(X, X_0)$ *on a set* X_0 *of base points*, therefore enabling computations in the *non-connected* case, including those in Van Kampen's original paper [166]. This use of groupoids in dimension $\underline{1}$ suggested the possibility of utilising groupoids in *higher homotopy theory*, and especially the question of the existence of *higher homotopy groupoids.* It will be useful to consider briefly the statement and special features of this generalised Van Kampen Theorem for the fundamental groupoid. First, if X_0 is a set , and X is a space, then $\pi_1(X, X_0)$ denotes the fundamental groupoid on the set $X \cap X_0$ of base points. This allows the set X_0 to be chosen in a way which is appropriate to the geometry. Consider the simple example of the circle S^1 written as the union of two semicircles $E_+ \cup E_-$, then the intersection $\{-1, 1\}$ of the semicircles is not connected, so it is not clear where to take the base point. Instead one takes $X_0 = \{-1, 1\}$, and so has two base points. This flexibility is very important in computations, and this simple example of S^1 was a motivating example for this development, as described in further detail in ref.[31].

8.3. **The Generalized van Kampen Theorem (GvKT).** Consideration of a set of base points leads next to the following theorem for *the fundamental groupoid*:

Theorem 8.3.1. (The Van Kampen Theorem for the Fundamental Groupoid, $\pi_1(X, X_0)$, [46])

Let the space X be the union of open sets U, V with intersection W, and let X_0 be a subset of X meeting each path component of U, V, W. Then
(C) (connectivity) *X_0 meets each path component of X and*
(I) (isomorphism) *the diagram of groupoid morphisms induced by inclusions*

$$\begin{array}{ccc} \pi_1(W, X_0) & \xrightarrow{i} & \pi_1(U, X_0) \\ {\scriptstyle j}\downarrow & & \downarrow \\ \pi_1(V, X_0) & \longrightarrow & \pi_1(X, X_0) \end{array} \tag{8.1}$$

is a pushout of groupoids.

Note that this theorem is a generalization of an analogous Van Kampen theorem for the fundamental group [31]. From this theorem, one can compute a particular fundamental group $\pi_1(X, x_0)$ using combinatorial information on the graph of intersections of path components of U, V, W, but for this it is useful to develop the algebra of groupoids. Notice two special features of this result:

(i) The computation of the *invariant* one wants to obtain, *the fundamental group*, is obtained from the computation of a larger structure, and so part of the work is to give *methods for computing the smaller structure from the larger one.* This usually involves non canonical choices, such as that of a maximal tree in a connected graph. The work on applying groupoids to groups gives many examples of such methods [111, 112, 50].

(ii) The fact that the computation can be done is surprising in two ways: (a) The fundamental group is computed *precisely*, even though the information for it uses input in two dimensions, namely 0 and 1. This is contrary to the experience in homological algebra and algebraic topology, where the interaction of several dimensions involves exact sequences or spectral sequences, which give information only up to extension, and (b) the result is a *non commutative invariant*, which is usually even more difficult to compute precisely.

The reason for this success seems to be that the fundamental groupoid $\pi_1(X, X_0)$ contains information in *dimensions 0 and 1*, and therefore it can adequately reflect the geometry of the intersections of the path components of U, V, W and the morphisms induced by the inclusions of W in U and V. This fact also suggested the question of whether such methods could be extended successfully to *higher dimensions.*

8.3.1. *The Generalized Van Kampen Theorem (GvKT) for Covering Spaces and Covering Groupoids.* There is yet another approach to the Van Kampen Theorem which goes *via* the theory of *covering spaces*, and the *equivalence* between covering spaces of a reasonable space X and functors $\pi_1(X) \to \mathbf{Set}$ [50]. See also an example [90] that consists in an exposition of the relation of this approach with the Galois theory. Another paper [66] gives a general formulation of conditions for the theorem to hold in the case $X_0 = X$ in terms of the map $U \sqcup V \to X$ being an *'effective global descent morphism'* (the theorem is given in the generality of lextensive categories). The latter work has been developed for topoi [77]. However, analogous interpretations of the topos work for higher dimensional Van Kampen theorems are not known so far.

The justification for changing from groups to groupoids is here threefold:
• the elegance and power of the results obtained with groupoids;
• the increased linking with other uses of groupoids [49], and
• the opening out of new possibilities in higher dimensions, which allowed for new results, calculations in homotopy theory, and also suggested new algebraic constructions.

8.4. **Construction of The Homotopy Double Groupoid.** To proceed further with generalizing the Van Kampen Theorem in terms of groupoids in higher dimensions one needs the concept of a *homotopy double groupoid.* We shall begin by recalling the construction of the Homotopy Double Groupoid.

8.4.1. The Homotopy Double Groupoid, $\boldsymbol{\varrho}^{\square}(X)$

This section is adapted from [33], and the reader should refer to that source for complete details.

The singular cubical set of a topological space.

We shall be concerned with the low dimensional part (up to dimension 3) of the singular cubical set

$$R^{\square}(X) = (R^{\square}_n(X), \partial^-_i, \partial^+_i, \varepsilon_i)$$

of a topological space X. We recall the definition (cf. [35]).

For $n \geqslant 0$ let

$$R^{\square}_n(X) = \mathbf{Top}(I^n, X)$$

denote the set of *singular n–cubes* in X, i.e. continuous maps $I^n \longrightarrow X$, where $I = [0, 1]$ is the unit interval of real numbers.

We shall identify $R^{\square}_0(X)$ with the set of points of X. For $n = 1, 2, 3$ a singular n–cube will be called a *path*, resp. *square*, resp. *cube*, in X.

The *face maps*

$$\partial^-_i, \partial^+_i : R^{\square}_n(X) \longrightarrow R^{\square}_{n-1}(X) \quad (i = 1, \dots, n)$$

are given by inserting 0 resp. 1 at the i^{th} coordinate whereas the *degeneracy maps*

$$\varepsilon_i : R^{\square}_{n-1}(x) \longrightarrow R^{\square}_n(X) \quad (i = 1, \dots, n)$$

are given by omitting the i^{th} coordinate. The face and degeneracy maps satisfy the usual cubical relations (cf. [35], § 1.1; [**?**] , § 5.1). A path $a \in R^{\square}_1(X)$ has *initial point* $a(0)$ and *endpoint* $a(1)$. We will use the notation $a : a(0) \simeq a(1)$. If a, b are paths such that $a(1) = b(0)$, then we denote by $a + b : a(0) \simeq b(1)$ their *concatenation*, i.e.

$$(a + b)(s) = \begin{cases} a(2s), & 0 \leqslant s \leqslant \frac{1}{2} \\ b(2s - 1), & \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

If x is a point of X, then $\varepsilon_1(x) \in R^{\square}_1(X)$, denoted e_x, is the *constant path* at x, i.e.

$$e_x(s) = x \text{ for all } s \in I.$$

If $a : x \simeq y$ is a path in X, we denote by $-a : y \simeq x$ the *path reverse* to a, i.e. $(-a)(s) = a(1 - s)$ for $s \in I$. In the set of squares $R^{\square}_2(X)$ we have two partial compositions $+_1$ (*vertical composition*) and $+_2$ (*horizontal composition*) given by concatenation in the first resp. second variable.

Similarly, in the set of cubes $R^{\square}_3(X)$ we have three partial compositions $+_1, +_2, +_3$.

The standard properties of vertical and horizontal composition of squares are listed in [35], §1. In particular we have the following *interchange law*. Let $u, u', w, w' \in R^{\square}_2(X)$ be squares, then

$$(u +_2 w) +_1 (u' +_2 w') = (u +_1 u') +_2 (w +_1 w'),$$

whenever both sides are defined. More generally, we have an interchange law for rectangular decomposition of squares. In more detail, for positive integers m, n let $\varphi_{m,n} : I^2 \longrightarrow [0, m] \times [0, n]$ be the homeomorphism $(s, t) \longmapsto (ms, nt)$. An $m \times n$ *subdivision* of a square $u : I^2 \longrightarrow X$ is a factorization $u = u'\ \varphi_{m,n}$; its *parts* are the squares $u_{ij} : I^2 \longrightarrow X$ defined by

$$u_{ij}(s, t) = u'(s + i - 1, t + j - 1).$$

We then say that u is the *composite* of the array of squares (u_{ij}), and we use matrix notation $u = [u_{ij}]$. Note that as in §1, $u +_1 u'$, $u +_2 w$ and the two sides of the interchange law can be written respectively as

$$\begin{bmatrix} u \\ u' \end{bmatrix}, \qquad [u\ \ w], \qquad \begin{bmatrix} u & w \\ u' & w' \end{bmatrix}.$$

Finally, *connections*

$$\Gamma^-, \Gamma^+ : R^{\square}_1(X) \longrightarrow R^{\square}_2(X)$$

are defined as follows. If $a \in R^{\square}_1(X)$ is a path, $a : x \simeq y$, then let

$$\Gamma^-(a)(s, t) = a(\max(s, t));\ \Gamma^+(a)(s, t) = a(\min(s, t)).$$

The full structure of $R^{\square}(X)$ as a *cubical complex with connections and compositions* has been exhibited in [2].

Thin squares.

In the setting of a geometrically defined double groupoid with connection, as in [35], (resp. [33]), there is an appropriate notion of *geometrically thin* square. It is proved in [35], Theorem 5.2 (resp. [33], Proposition 4), that in the cases given there, geometrically and algebraically thin squares coincide. In our context the explicit definition is as follows:

Definition.
A square $u : I^2 \longrightarrow X$ in a topological space X is *thin* if there is a factorisation of u,

$$u : I^2 \xrightarrow{\Phi_u} J_u \xrightarrow{p_u} X,$$

where J_u is a *tree* and Φ_u is piecewise linear (PWL, see below) on the boundary ∂I^2 of I^2.

Here, by a *tree*, we mean the underlying space $|K|$ of a finite 1-connected 1-dimensional simplicial complex K.

A map $\Phi : |K| \longrightarrow |L|$ where K and L are (finite) simplicial complexes is PWL (*piecewise linear*) if there exist subdivisions of K and L relative to which Φ is simplicial.

(2) Let u be as above, then the homotopy class of u relative to the boundary ∂I^2 of I is called a *double track*. A double track is *thin* if it has a thin representative. ■

8.5. **The Homotopy Double Groupoid of a Hausdorff space.** The data for the homotopy double groupoid, $\boldsymbol{\varrho}^{\square}(X)$, will be denoted by

$$(\boldsymbol{\varrho}^{\square_2(X)}, \boldsymbol{\varrho}_1^{\square(X),\partial_1^-,\partial_1^+,+_1,\varepsilon_1}), \ (\boldsymbol{\varrho}^{\square_2(X),\boldsymbol{\varrho}_1^{\square}(X),\partial_2^-,\partial_2^+,+_2,\varepsilon_2})$$
$$(\boldsymbol{\varrho}^{\square_1(X),X,\partial^-,\partial^+,+,\varepsilon}).$$

Here $\boldsymbol{\varrho}_1(X)$ denotes the *path groupoid* of X of [108]. We recall the definition. The objects of $\boldsymbol{\varrho}_1(X)$ are the points of X. The morphisms of $\boldsymbol{\varrho}_1^{\square}(X)$ are the equivalence classes of paths in X with respect to the following relation $\sim_T$.

Definition of Thin Equivalence

Let $a, a' : x \simeq y$ be paths in X. Then a is *thinly equivalent* to a', denoted $a \sim_T a'$, if there is a thin relative homotopy between a and a'.

We note that $\sim_T$ is an equivalence relation, see [33]. We use $\langle a\rangle : x \simeq y$ to denote the $\sim_T$ class of a path $a : x \simeq y$ and call $\langle a\rangle$ the *semitrack* of a. The groupoid structure of $\boldsymbol{\varrho}_1^{\square}(X)$ is induced by concatenation, $+$, of paths. Here one makes use of the fact that if $a : x \simeq x'$, $a' : x' \simeq x''$, $a'' : x'' \simeq x'''$ are paths then there are canonical thin relative homotopies

$$\begin{aligned}(a + a') + a'' \simeq a + (a' + a'') : x \simeq x''' \ (\mathit{rescale})\\ a + e_{x'} \simeq a : x \simeq x';\ e_x + a \simeq a : x \simeq x' \ (\mathit{dilation})\\ a + (-a) \simeq e_x : x \simeq x \ (\mathit{cancellation}).\end{aligned}$$

The source and target maps of $\boldsymbol{\varrho}_1^{\square}(X)$ are given by

$$\partial_1^- \langle a\rangle = x,\ \ \partial_1^+ \langle a\rangle = y,$$

if $\langle a\rangle : x \simeq y$ is a semitrack. Identities and inverses are given by

$$\varepsilon(x) = \langle e_x\rangle \quad \text{resp.} - \langle a\rangle = \langle -a\rangle.$$

In order to construct $\boldsymbol{\varrho}_2^{\square}(X)$, we define a relation of cubically thin homotopy on the set $R_2^{\square}(X)$ of squares.

Definition of Cubically Thin Homotopy.

Let u, u' be squares in X with common vertices. (1) A *cubically thin homotopy* $U : u \equiv_T^{\square} u'$ between u and u' is a cube $U \in R_3^{\square}(X)$ such that

(i) U is a homotopy between u and u',

$$\text{i.e. } \partial_1^-(U) = u,\ \ \partial_1^+(U) = u',$$

(ii) U is rel. vertices of I^2,

$$\text{i.e. } \partial_2^-\partial_2^-(U),\ \ \partial_2^-\partial_2^+(U),\ \ \partial_2^+\partial_2^-(U),\ \ \partial_2^+\partial_2^+(U) \text{ are constant},$$

(iii) the faces $\partial_i^\alpha(U)$ are thin for $\alpha = \pm 1,\ i = 1, 2$.

(2) The square u is *cubically T-equivalent* to u', denoted $u \equiv_T^{\square} u'$ if there is a cubically thin homotopy between u and u'.

Proposition 8.4.

The relation $\equiv_T^{\square}$ is an equivalence relation on $R_2^{\square}(X)$.

Proof The reader is referred to [33] for a proof. ∎

If $u \in R_2^{\square}(X)$ we write $\{u\}_T^{\square}$, or simply $\{u\}_T$, for the equivalence class of u with respect to $\equiv_T^{\square}$. We denote the set of equivalence classes $R_2^{\square}(X) \equiv_T^{\square}$ by $\boldsymbol{\varrho}_2^{\square}(X)$. This inherits the operations and the geometrically defined connections from $R_2^{\square}(X)$ and so becomes a double groupoid with connections. A proof of the final fine detail of the structure is given in [33].

Definition of a thin representative.

An element of $\boldsymbol{\varrho}_2^{\square}(X)$ is *thin* if it has a thin representative (in the sense of Definition in ref. **??**).

From the remark at the beginning of this subsection we infer:

Lemma 8.5.

Let $f : \boldsymbol{\varrho}^{\square}(X) \to \mathsf{D}$ be a morphism of double groupoids with connection. If $\alpha \in \boldsymbol{\varrho}_2^{\square}(X)$ is thin, then $f(\alpha)$ is thin. ■

The Homotopy Addition Lemma 8.5

Let $u : I^3 \to X$ be a singular cube in a Hausdorff space X. Then by restricting u to the faces of I^3 and taking the corresponding elements in $\boldsymbol{\varrho}_2^{\square}(X)$, we obtain a cube in $\boldsymbol{\varrho}^{\square}(X)$ which is commutative by the homotopy addition lemma for $\boldsymbol{\varrho}^{\square}(X)$ ([33]*, Proposition 5.5). Consequently, if $f : \boldsymbol{\varrho}^{\square}(X) \to \mathsf{D}$ is a morphism of double groupoids with connections, any singular cube in X determines a commutative 3-shell in D.*

Definition 8.1. . *the Homotopy Double Groupoid of a Hausdorff space*

The data for the homotopy double groupoid, $\boldsymbol{\varrho}^{\square}(X)$, will be denoted by :

$$(\boldsymbol{\varrho}^{\square_2(X)}, \boldsymbol{\varrho}_1^{\square(X), \partial_1^-, \partial_1^+, +_1, \varepsilon_1}),$$
$$(\boldsymbol{\varrho}^{\square_2(X), \boldsymbol{\varrho}_1^{\square}(X), \partial_2^-, \partial_2^+, +_2, \varepsilon_2})$$
$$(\boldsymbol{\varrho}^{\square_1(X), X, \partial^-, \partial^+, +, \varepsilon}).$$

Here $(\boldsymbol{\varrho}_1(X))$ denotes the *path groupoid* of X which was defined as follows in ref. [108]. The objects of $(\boldsymbol{\varrho}_1(X))$ are the points of X. The morphisms of $(\boldsymbol{\varrho}^{\square_1(X)})$ are the equivalence classes of paths in X with respect to the following relation $\sim_T$.

.

As already discussed above for crossed complexes, the general setting of the van Kampen theorem is that of a local–to– global problem which can be explained as follows:

> *Given an open covering $\mathcal{U}$ of X and knowledge of each (U) for U in $\mathcal{U}$, give a determination of (X).*

Of course we need also to know the values of on intersections $U \cap V$ and on the inclusions from $U \cap V$ to U and V.

We first note that that the functor on **Top** preserves coproducts $\bigsqcup$, since these are just disjoint union in topological spaces and in double groupoids. It is an advantage of the groupoid approach that the coproduct of such objects is so simple to describe.

Suppose we are given a cover $\mathcal{U}$ of X. Then the homotopy double groupoids in the following ϱ*-sequence of the cover* are well-defined:

$$\bigsqcup_{(U,V)\in\mathcal{U}^2} (U\cap V) \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{U\in\mathcal{U}} (U) \xrightarrow{c} (X). \tag{8.2}$$

The morphisms a, b are determined by the inclusions

$$a_{UV} : U\cap V \to U, b_{UV} : U\cap V \to V$$

for each $(U,V)\in\mathcal{U}^2$ and c is determined by the inclusion $c_U : U \to X$ for each $U\in\mathcal{U}$.

8.6. **The Generalized van Kampen Theorem (GvKT) in terms of Double Groupoids and Its Possible Applications in Quantum Algebraic Topology.** The following is a statement of the Generalized van Kampen Theorem (GvKT) expressed in terms of Double Groupoids with connections as developed and proven in ref. [33].

Theorem 8.6 [*van Kampen theorem*]

If the interiors of the sets of $\mathcal{U}$ cover X, then in the above ϱ-sequence of the cover, c is the coequaliser of a, b in the category of double groupoids with connections.

A special case of this result is when $\mathcal{U}$ has two elements. In this case the coequaliser reduces to a pushout.

Proof The reader is referred to Brown et al.(2004a), [33], for the complete proof of the generalized van Kampen theorem.

Furthermore, a general formulation of the van Kampen theorem is obtained in terms of categorical colimits of groupoid diagrams. This categorical formulation generalizes the van Kampen theorem to a small category or diagram for a ϱ-sequence of covers of the topological space X that have a *unique colimit* (or inverse/projective limit). The proof of this categorical form of the van Kampen theorem consists in the construction of the fundamental homotopy groupoid of X, $\Pi(X)$ (as defined in Section 8.3), in terms of such a colimit.

There are several possible applications of the generalized van Kampen theorem in the development of physical representations of a quantized space-time 'geometry'. For example, a possible application of the generalized van Kampen theorem is the construction of the initial, quantized space-time as the *unique colimit* of *quantum causal sets (posets)* which was precisely described in Subsection 5.5.1 in terms of *the nerve of an open covering* $N\mathcal{U}$ of the topological space X that would be isomorphic to a k-simplex K underlying X [44]. The corresponding, *non–commutative* algebra Ω associated with the finitary T_0-poset $P(S)$ is *the Rota algebra* Ω discussed above in Subsection 5.2.2, and the *quantum topology* T_0 is defined by the partial ordering arrows for regions that can overlap, or superpose, coherently (in the quantum sense) with each other. When the poset $P(S)$ contains $2N$ points we write this as $P_{2N}(S)$. The *unique* (up to an isomorphism) $P(S)$ in the projective limit (*colimit*), $\lim_{\leftarrow} P_N X$, recovers a space homeomorphic to X [162]. Other non–Abelian results derived from the generalized van Kampen theorem are discussed by Brown et al. (2004) and Brown (2005), respectively [33, 43].

8.7. **Novel GvKT Applications and Related Conjectures.** Several new QAT applications to quantum systems *via* GvKT, and also GvKT–related conjectures will be here presented and discussed.

8.7.1. *The Weak Homotopy Double Groupoid (WHDG) of a Compactly–Generated (weak Hausdorff) space,* X_{cg}. Let us re-consider here and generalize the quantum-theoretical *Gel'fand triple* (Φ, H, Φ^*) extension of the Hilbert space (subsection 3.6.4) to a general quantum system with an infinite number of degrees of freedom that includes quantum fields coupled to any type of (low or high density) massive bodies (i.e., consider also intense gravitonic fields such as the Hawking 'white', or 'black' holes, etc., specified under the *Steve Hawking* and *Roger Penrose* 'programmes' in the *Introduction* (Section 1)). The latter demand the construction of a generalized, *non–linear quantum gravity (NQG)* theory, that is, a 'quantized', generally relativistic theory, *albeit* for the omission of covariantly defined arbitrary reference frames inside certain *'inaccessible'*, universe regions. This means explicitely that one conjectures the existence of 'singular' regions, or 'singularities' of QST and QSS where 'events'–if they exist at all– cannot be observed from the outside of the 'event horizon' surrounding such singularities in our observable universe. Such 'inaccessible' regions cannot be related to–and either probed or measured through–any direct, either quantum or classical, observations that would be made from any reference frame located outside such 'singular' regions of extremely intense gravitonic fields).

Our novel approach to *non–linear QG* will also involve an extension of the GNS construction to 'arbitrary', *compactly–generated* topological space representations of Φ and Φ^* of QSS, thus leading to consistent extensions of the C*-algebra concrete representations in a generalized quantum field theory that includes *supersymmetry* (as defined above in Section 6), as well as non–linear supergravity and superfields. The generalization from *compact* topological spaces to *compactly–generated* spaces is physically an important one as it allows us to define *'supersymmetries'* that may exhibit *'patterns of broken symmetry'*, or *spontaneously broken symmetry* in QSS. This novel generalization is potentially important as it permits the development of appropriate theoretical treatments of quantum phase transitions– and other important quantum processes, such as tunneling– in various *macroscopically coherent quantum systems*; such treatments are made possible through the algebraic formulation of the effects of non–linear gravitonic fields on quantized space-time structure (e.g., topological or metric/geometric) as shown in the next Corollary 8.7.1 , Theorem 8.7.2 and subsequent Conjectures (8.7.3.1 to 8.7.3.3).

Let us first define the *weak homotopy double groupoid (WHDG)* of a *compactly–generated space* (weak Hausdorff), X_{cg}. We utilize the construction method developed by R. Brown (ref. [33]) for the homotopy double groupoid of a Hausdorff space (which was recalled above in subsection 8.5), with the important change in this construction that involves a replacement of regular homotopy equivalence with the *weak homotopy equivalence, (Definition 7.1.2.1)* in the definition of the fundamental groupoid, as well as replacement of the Hausdorff space by the compactly–generated space X_{cg}. Therefore, the data for the homotopy double groupoid of X_{cg}, $\boldsymbol{\varrho}^{\square}(X_{cg})$, will now be

$$(\boldsymbol{\varrho}^{\square_2(X_{cg})}, \boldsymbol{\varrho}_1^{\square(X_{cg}),\partial_1^-,\partial_1^+,+_1,\varepsilon_1}), \ (\boldsymbol{\varrho}^{\square_2(X_{cg}),\boldsymbol{\varrho}_1^{\square}(X_{cg}),\partial_2^-,\partial_2^+,+_2,\varepsilon_2})$$
$$(\boldsymbol{\varrho}^{\square_1(X_{cg}),X_{cg},\partial^-,\partial^+,+,\varepsilon}).$$

We have introduced here $\boldsymbol{\varrho}_1(X_{cg})$ which denotes the *path groupoid* of X_{cg}, analogous to the definition in ref. [108] for a Hausdorff space. The objects of $\boldsymbol{\varrho}_1(X_{cg})$ are therefore the points of X_{cg}. The morphisms of $\boldsymbol{\varrho}_1^{\square}(X_{cg})$ are the *weak* homotopy equivalence classes of paths in X_{cg} with respect to the *weak* homotopy equivalence, $\sim_W$, defined on X_{cg}. With the above definitions we can now proceed to present the following Lemma for the Weak Homotopy Equivalence defined for compactly–generated topological spaces, X_{cg}.

Weak Homotopy Addition Lemma 8.7.1. *Consider* $\boldsymbol{u} : I^3 \to X_{cg}$ *to be a singular cube in a compactly-generated space* X_{cg}. *Then by restricting* u *to the faces of* I^3 *and taking the corresponding elements in* $\boldsymbol{\varrho}_2^{\square}(X_{cg})$, *one obtains a cube in* $\boldsymbol{\varrho}^{\square}(X_{cg})$ *which is commutative by an extension of the homotopy addition lemma for* $\boldsymbol{\varrho}^{\square}(X_{cg})$ *(*[33] *, Proposition 5.5) to* weak *homotopy in* X_{cg}. *Consequently, if* $f : \boldsymbol{\varrho}^{\square}(X_{\ cg}) \to \mathsf{D}$ *is a morphism of double groupoids with connections, any singular cube in* X_{cg} *determines a commutative 3-shell in* D.

Proof

Remark 8.7.1.1. With the help of the Weak Homotopy Addition Lemma 8.7.1 and the general, categorical form of GvKT, the weak homotopy double groupoid can be calculated as the categorical *colimit* of a sequence of fundamental groupoids of the spaces defining the filtering sequence for the extended Gel'fand triple $(\Phi_{cg}, \mathbf{H}, \Phi_{cg}^*)$ representation of the generalized QSS of an NQG theory, with Φ_{cg} and Φ_{cg}^* being compactly–generated topological spaces.

8.7.2. *A Novel Application of GvKT: The Quantum Fundamental Groupoid Theorem 8.7.2. The Quantum Fundamental Groupoid (QFG)* of a QSS in a NQG theory, $\pi_1(X_{cg}, X_0)$, defined by the Weak Homotopy of a QSS representation, X_{cg}, is computed in a non–linear quantum gravity theory by the application of the GvKT (**Theorem 8.3.1**, [46]) to an arbitrary, compactly–generated space X_{cg} as follows.

Let the space X_{cg} *be the union of open sets* U, V *with intersection* W, *and let* X_0 *be a subset of* X_{cg} *meeting each path component of* U, V, W. *Then*
(C) (connectivity) X_0 *meets each path component of* X_{cg} *and*
(I) (isomorphism) *the diagram of groupoid morphisms induced by inclusions*

$$\begin{array}{ccc} \pi_1(W, X_0) & \xrightarrow{\ i\ } & \pi_1(U, X_0) \\ {\scriptstyle j}\downarrow & & \downarrow \\ \pi_1(V, X_0) & \longrightarrow & \pi_1(X_{cg}, X_0) \end{array} \tag{8.3}$$

is a ***pushout*** *of groupoids.*

Proof 8.7.2. Note that this theorem results as a direct, specific application of the generalized van Kampen Theorem for the fundamental groupoids (which was proven in ref. [31]) to the QSS representation by X_{cg} in a NQG theory.

Remark 8.7.2.1 From this theorem, one can also compute a particular fundamental group $\pi_1(X_{cg}, x_0)$ using combinatorial information on the graph of intersections of path components of U, V, W, but for this it was previously found useful to develop the algebra of groupoids (*loc.cit.*). As noted above in the discussion of the GvKT (Theorem 8.3) a speciallly important feature of this result is that the effective computation of the *invariant* one wants to obtain, *i.e.*, *the fundamental group*, is obtained through the computation of a larger structure, and so part of the work is to develop *methods for computing the smaller structure from the larger one.* Such methods usually involve non canonical choices, as in the computation of a maximal tree of a connected graph. The recently published work on applying groupoids to groups gives numerous examples of such methods [111, 112, 50].

8.7.3. *New GvKT–Related Conjectures.* Three novel conjectures related to GvKT will be presented next. The first two conjectures are, respectively, potential applications of GvKT to quantum Lie double groupoids (and their corresponding quantum Lie (double) algebroids), and to the Homotopy of Quantum Field Configurations Space (Ch. 6 , Section 2 in ref. [170]). The third conjecture is the statement of a possible extension of GvKT to dimensions higher than 2.

8.7.3.1. A New, GvKT–based, Lie Double Groupoid Conjecture.

Consider a 'quantum' Lie double groupoid (QLDG, not necessarily compact), obtained by appropriate integrations of a graded Lie algebroid derived from the quantum superalgebra consistent with supersymmetry, supergravity and the superfields defined in Section 6. Then, the QLDG corresponding to this 'quantum' graded Lie algebroid can be expressed as a crossed module of two locally compact, symplectic 'quantum' groupoids. The fundamental groupoid of this QLDG can be computed by the application of GvKT in dim 2.

Hints. This conjecture is readily satisfied by an example provided by Mackenzie of a double Lie algebroid which is its own Lie double groupoid (ref.[131]). Furthermore, *any symplectic Lie groupoid can be obtained by integration of a corresponding, unique Lie algebroid* (cf. Theorem 2.1 in ref. [**?**]).

8.7.3.2. The Quantum Field Configuration Space Homotopy Conjecture.

Let us consider the Homotopy of Quantum Field Configurations Space *(*HQFCS*; as presented in Ch. 6 , Section 2 in ref.* [170]*). The Quantum Fundamental Groupoid (QFG) of a quantum crossed complex of quantum field configuration spaces represented as* locally compact *quantum groupoids (LCQG, as specified above in Section 5) can be computed by applying GvKT to such QFCS representations and by utilizing the LCQG construction contained in the premise of* Conjecture 5.3.

8.7.3.3. Three – and n– Dimensional GvKT Extensions Conjecture.

*Recall the GvKT statement for double groupoid structures (*GvK Theorem 8.6*). Moreover, consider the* $\underline{3}$*–category of small double groupoid categories canonically constructed by means of the Yoneda functors* hom_X *and* hom^X *(refs.*[147], [**?**]*), where X is an arbitrary double groupoid. Then, there exist valid GvKT extensions to dimension 3– as well as higher dimensions– of the 2–dim GvKT statement, which will be obtained recursively by iteration of the Yoneda canonical construction.*

Tentative Proof Construction 8.7.3.3

One will utilize a construction similar to that of the double groupoid ϱ in the 2–dim GvKT, but extended to higher dimension groupoids. For example, for $dim = 3$ one employs cube, instead of square, diagrams in order to define $\underline{3}$–groupoids, or higher dimensional symplectic diagrams in order to extend the construction to $\underline{\text{n}}$–dimensions. In the 3–dimensional case, one may also construct crossed modules of double groupoids, $h_X(Y) = hom_X(X, Y) = [X, Y]$, where $[X, Y]$ denotes here the set of double groupoid homomorphisms from X to Y. The *natural transformations*, $\eta_2 : h_X \longrightarrow h'_X$, will then complete the construction in the higher dimension, provided all naturality conditions are appropriately checked through diagram chasing in all higher dimensions.

9. Conclusions and Discussion

Current developments in Quantum Algebraic Topology were here discussed with a view to bridging the gap between Quantum Field Theories and General Relativity, a long standing problem in the foundation of Mathematical and Theoretical Physics which is of considerable conceptual importance. Mathematical generalizations from quantum groups to quantum groupoids, and then further to quantum topological groupoids and double groupoids, as well as higher dimensional algebra are concluded to be logical requirements of the unification between quantum and relativity theories that would be leading towards a deeper understanding of quantum gravity and quantum space-time geometry through QAT. In a subsequent paper (Baianu, Glazebrook, Georgescu and Brown, 2004), we shall further consider quantum algebraic topology from the standpoints of the theory of categories, functors, quantum logics, higher dimensional algebra, as well as the integrated viewpoint of the Generalized 'Topos'–a new concept that links quantum logics with category theory. Other potential applications of quantum algebraic topology to operational quantum nano-automata were also recently suggested (Baianu, 2004). Algebraically simpler representations of quantum space-time than QAT have also been proposed in terms of causal sets and quantized causal sets (see for example, Raptis, 2000a,b; Raptis and Zapatrin, 2000) that might also prove to be useful in emerging quantum gravity theories and that may have a topology compatible with the QAT approach summarized in this paper. Figure 2 summarizes– albeit in a simplified manner– the connections of the mathematical concepts summarized in this paper with their physical representations and applications.

The algebraic structure of lattices, the algebraic-topological structures of *quantum groupoids*–including quantum group algebras, compact groupoids, quantum 2-groupoids and certain categories of sheaves– are suggested as being especially important for further developments of unified quantum field theories. Such concepts could also link quantum field theories with general relativity, thus leading towards generally relativistic quantum gravity.

References

[1] F. Al-Agl, 1989, *Aspects of multiple categories*, Ph.D. thesis, University of Wales, Bangor.

[2] F.A. Al-Agl, R. Brown and R. Steiner, Multiple categories: the equivalence of a globular and a cubical approach, *Advances in Maths.* **170** (2002) 71-118.

[3] E. M. Alfsen and F. W. Schultz : *Geometry of Spaces of Operator Algebras*, Birkäuser, Boston–Basel–Berlin (2003).

[4] J. Anandan : The geometric phase, *Nature* **360** (1992), 307–313.

[5] R. Antolini, 1996, *Cubical structures and homotopy theory*, Ph.D. thesis, Univ. Warwick, Coventry.

[6] N. Ashley, *Simplicial T-Complexes: a non abelian version of a theorem of Dold-Kan*, Dissertationes Math., 165, (1988), 11 – 58.

[7] M. E. – S. A. Aof and R. Brown : The holonomy groupoid of a locally topological groupoid, *Top. Appl.* **47** (1992), 97–113.

[8] R. Attal : Two–dimensional parallel transport: combinatorics and functionality, arXiv:math-ph/0105050

[9] J. Baez. 2004. Quantum quandaries : a category theory perspective, in *'Structural Foundations of Quantum Gravity'*, (ed. S. French et al.) Oxford Univ. Press.

[10] J. Baez. 2002. Categorified Gauge Theory. in Proceedings of the Pacific Northwest Geometry Seminar Cascade Topology Seminar,Spring Meeting–May 11 and 12, 2002. University of Washington, Seattle, WA.

[11] I.C. Baianu, James Glazebrook, G. Georgescu and Ronald Brown. 2005a."Generalized 'Topos' Representations of Quantum Space-Time: Linking Quantum N-Valued Logics with Categories and Higher Dimensional Algebra.", (*manuscript in preparation*)

[12] I.C. Baianu. 2004. Quantum Nano-Automata (QNA): Microphysical Measurements with Microphysical QNA Instruments, CERN Preprint EXT-2004-125.

[13] I.C. Baianu. 1971a. Organismic Supercategories and Qualitative Dynamics of Systems. The Bulletin of Mathematical Biophysics, 33(3):339–354.

[14] I.C. Baianu. 1971b. "Categories, Functors and Quantum Algebraic Computations", Proceed. Fourth Intl. Congress LMPS, September 1–4, 1971, Buch.

[15] Baianu, I.C. 1977. "A Logical Model of Genetic Activities in Łukasiewicz Algebras: The Non-linear Theory." *Bulletin of Mathematical Biology*, 39:249-258 (1977).

[16] Baianu, I.C. 1987. "Computer Models and Automata Theory in Biology and Medicine" (A Review). In: *"Mathematical Models in Medicine."*,vol.7., M. Witten, Ed., Pergamon Press: New York, pp.1513-1577.

[17] Baianu, I.C., J. Glazebrook, G. Georgescu and R.Brown. 2005b. "A Novel Approach to Complex Systems Biology based on Categories, Higher Dimensional Algebra and A Generalized Łukasiewicz Topos. " , *Manuscript in preparation*, 46 pp.

[18] I.C. Baianu and D. Scripcariu. 1973.*On Adjoint Dynamical Systems. The Bulletin of Mathematical Biophysics*, **35**(4):475–486.

[19] I.C. Baianu, K.A. Rubinson and J. Patterson. 1979. Ferromagnetic Resonance and Spin–Wave Excitations in Metallic Glasses., *J. Phys. Chem. Solids*, **40**: 940–951.

[20] I.C. Baianu, J. Patterson and K.A. Rubinson. 1979. Ferromagnetic Resonance Observations of Surface Effects, Magnetic Ordering and Inhomogeneous Anisotropy in a Metallic Glass, *Material Sci. and Engineering*, **40**: 273–284.

[21] I.C. Baianu, N. Boden,Y.K. Levine and D. Lightowlers. 1978. Dipolar Coupling between Groups of Three Spin-1/2 undergoing Hindered Reorientation in Solids., Solid-State Communications. 27: 474-478.

[22] I.C. Baianu, N. Boden, Y.K.Levine and S.M.Ross. 1978. Quasi-quadrupolar NMR Spin-Echoes in Solids Containing Dipolar Coupled Methyl Groups., **J. Phys. (C) Solid State Physics**., **11**: **L**37-41.

[23] I.C. Baianu, N. Boden and D. Lightowlers.1981. NMR Spin-Echo Responses of Dipolar–Coupled Spin-1/2 Triads in Solids., **J. Magnetic Resonance**, **43**:101-111.

[24] I.C. Baianu, K.A. Rubinson and J. Patterson.1979. The Observation of Structural Relaxation in a FeNiPB Glass by X-ray Scattering and Ferromagnetic Resonance., **Physica Status Solidi**(a), **53**: **K**133-135.

[25] A. Bak, R. Brown, G. Minian and T. Porter: Global actions, groupoid atlases and related topics, http://citeseer.ist.psu.edu/bak00global.html

[26] A. P. Balachandran, G. Bimonte, E. Ercolessi and P. Teotonio–Sobrinho, *Finite approximations to quantum physics: quantum points and their bundles*, in 'Constraint theory and quantization methods' (Montepulciano, 1993), 1–17, World Scientific, River Edge, NJ 1994.

[27] H. J. Baues, 1989, *Algebraic Homotopy*, volume 15 of *Cambridge Studies in Advanced Mathematics*, Cambridge Univ. Press.

[28] H. J. Baues and R. Brown, *On relative homotopy groups of the product filtration, the James construction, and a formula of Hopf*, J. Pure Appl. Algebra, 89, (1993), 49–61, ISSN 0022-4049.

[29] H.-J. Baues and A. Tonks, *On the twisted cobar construction*, Math. Proc. Cambridge Philos. Soc., 121, (1997), 229–245.

[30] N. Berline, E. Getzler and M. Vergne : Heat kernels and Dirac operators, Grund. der math. Wissenschaften **298**, Springer Verlag 1991.

[31] R. Brown : Groupoids and crossed objects in algebraic topology, *Homology, Homotopy and Appl.* **1** (1999), 1–78.

[32] R. Brown, Crossed complexes and homotopy groupoids as non commutative tools for higher dimensional local-to-global problems, Proceedings of the Fields Institute Workshop on Categorical Structures for Descent and Galois Theory, Hopf Algebras and Semiabelian Categories, September 23-28, 2002, Contemp. Math. (2004). 30pp

[33] R. Brown, K.A. Hardie, K.H. Kamps and T. Porter, A homotopy double groupoid of a Hausdorff space, *"Theory and Applications of Categories"* **10** (2002) 71-93.

[34] R. Brown and P.J. Higgins, On the connection between the second relative homotopy groups of some related spaces, *Proc. London Math. Soc.* **(3) 36** (1978), 193-212.

[35] R. Brown and P.J. Higgins, On the algebra of cubes, *J. Pure Applied Algebra* **21** (1981), 233–260.

[36] R. Brown and P.J. Higgins, Colimit theorems for relative homotopy groups', *J. Pure Appl. Algebra* **22** (1981) 11-41.

[37] R. Brown and G.H. Mosa, Double categories, thin structures and connections, *Theory and Applications of Categories* **5** (1999), 163-175.

[38] R. Brown, and C.B. Spencer, Double groupoids and crossed modules, *Cahiers Top. Géom.Diff.* **17** (1976) 343-362.

[39] R. Brown.2002. *Categorical Structures for Descent and Galois Theory.* Fields Institute, September 23-28, 2002.

[40] R. Brown and J. F. Glazebrook : Connections, local subgroupoids and a holonomy Lie groupoid of a line bundle gerbe, arXiv:math. DG/0210322 *Univ. Iagel. Acta Math.* (2003) (to appear).

[41] R. Brown and İ. İcen : Lie local subgroupoids and their holonomy and monodromy Lie groupoids, *Top. Appl.* **115** (2001), 125–138.

[42] R. Brown: "Topology: a geometric account of general topology, homotopy types and the fundamental groupoid", Ellis Horwood, Chichester, Prentice Hall, New York, 1988.

[43] R. Brown: "Nonabelian Algebraic Topology, 2005" (*in preparation*).

[44] R. Brown and O. Mucuk : Foliations, locally Lie groupoids and holonomy, *Cah. Top. Géom. Diff. Cat.* **37** (1996), 61–71.

[45] R. Brown, İ. İ*cen* and O. Mucuk : Local subgroupoids II : Examples and properties. *Top. Appl.* **127** (2003), 393–408.

[46] R. Brown, *Groupoids and van Kampen's Theorem*, 3, (1967), 385–340.

[47] R. Brown, *On the second relative homotopy group of an adjunction space: an exposition of a theorem of J. H. C. Whitehead*, J. London Math. Soc. (2), 22, (1980), 146–152, ISSN 0024-6107.

[48] R. Brown, *Coproducts of crossed P-modules: applications to second homotopy groups and to the homology of groups*, Topology, 23, (1984), 337–345.

[49] R. Brown, *From groups to groupoids: a brief survey*, Bull. London Math. Soc., 19, (1987), 113–134, ISSN 0024-6093.

[50] R. Brown, 1988, *Topology*, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester, second edition, a geometric account of general topology, homotopy types and the fundamental groupoid.

[51] R. Brown, 1992, *Computing homotopy types using crossed n-cubes of groups*, in *Adams Memorial Symposium on Algebraic Topology, 1 (Manchester, 1990)*, volume 175 of *London Math. Soc. Lecture Note Ser.*, 187–210, Cambridge Univ. Press, Cambridge.

[52] R. Brown and N. D. Gilbert, *Algebraic models of 3-types and automorphism structures for crossed modules*, Proc. London Math. Soc. (3), 59, (1989), 51–73, ISSN 0024-6115.

[53] R. Brown and J. F. Glazebrook, *Connections, local subgroupoids, and a holonomy Lie groupoid of a line bundle gerbe*, University of Wales, Bangor, Math Preprint, 02.22, (2002), 10 pages.

[54] R. Brown, M. Golasiński, T. Porter and A. Tonks, *Spaces of maps into classifying spaces for equivariant crossed complexes*, Indag. Math. (N.S.), 8, (1997), 157–172, ISSN 0019-3577.

[55] R. Brown, M. Golasiński, T. Porter and A. Tonks, *Spaces of maps into classifying spaces for equivariant crossed complexes. II. The general topological group case*, K-Theory, 23, (2001), 129–155, ISSN 0920-3036.

[56] R. Brown and P. J. Higgins, *On the connection between the second relative homotopy groups of some related spaces*, Proc.London Math. Soc., (3) 36, (1978), 193–212.

[57] R. Brown and P. J. Higgins, *Sur les complexes croisés d'homotopie associés à quelques espaces filtrés*, C. R. Acad. Sci. Paris Sér. A-B, 286, (1978), A91–A93, ISSN 0151-0509.

[58] R. Brown and P. J. Higgins, *Colimit theorems for relative homotopy groups*, J. Pure Appl. Alg, 22, (1981), 11–41.

[59] R. Brown and P. J. Higgins, *The equivalence of crossed complexes and ω-groupoids,*, Cahiers Top. Géom. Diff., 22, (1981), 370 – 386.

[60] R. Brown and P. J. Higgins, *Tensor products and homotopies for ω-groupoids and crossed complexes,*, J. Pure Appl. Alg, 47, (1987), 1–33.

[61] R. Brown and P. J. Higgins, *Crossed complexes and chain complexes with operators*, Math. Proc. Camb. Phil. Soc., 107, (1990), 33–57.

[62] R. Brown and P. J. Higgins, *The classifying space of a crossed complex*, Math. Proc. Cambridge Philos. Soc., 110, (1991), 95–120, ISSN 0305-0041.

[63] R. Brown and P. J. Higgins, 2002, *Cubical abelian groups with connections are equivalent to chain complexes*, Technical Report 02.24.

[64] R. Brown and J. Huebschmann, 1982, *Identities among relations*, in R.Brown and T.L.Thickstun, eds., *Low Dimensional Topology*, London Math. Soc Lecture Notes, Cambridge University Press.

[65] R. Brown and I. Icen, *Towards two dimensional holonomy*, Advances in Mathematics, (to appear).

[66] R. Brown and G. Janelidze.:1997, *Van Kampen theorems for categories of covering morphisms in lextensive categories*, *J. Pure Appl. Algebra*, **119**: 255–263, ISSN 0022-4049.

[67] R. Brown and J.-L. Loday, *Homotopical excision, and Hurewicz theorems, for n-cubes of spaces*, Proc. London Math. Soc., (3)54, (1987), 176 – 192.

[68] R. Brown and J.-L. Loday, *Van Kampen Theorems for diagrams of spaces*, Topology, 26, (1987), 311 – 337.

[69] R. Brown, E. Moore, T. Porter and C. Wensley, *Crossed complexes, and free crossed resolutions for amalgamated sums and HNN-extensions of groups*, Georgian Math. J., 9, (2002), 623–644.

[70] R. Brown and G. H. Mosa, *Double categories, 2-categories, thin structures and connections*, Theory Appl. Categ., 5, (1999), No. 7, 163–175 (electronic), ISSN 1201-561X.

[71] R. Brown and A. Razak Salleh, *Free crossed resolutions of groups and presentations of modules of identities among relations*, LMS J. Comput. Math., 2, (1999), 28–61 (electronic), ISSN 1461-1570.

[72] R. Brown and C. B. Spencer, *Double groupoids and crossed modules*, Cahiers Topologie Géom. Différentielle, 17, (1976), 343–362.

[73] R. Brown and C. B. Spencer, *G-groupoids, crossed modules and the fundamental groupoid of a topological group*, Proc. Kon. Ned. Akad. v. Wet, 79, (1976), 296 – 302.

[74] R. Brown and C. D. Wensley, *On finite induced crossed modules, and the homotopy 2-type of mapping cones*, Theory Appl. Categ., 1, (1995), No. 3, 54–70 (electronic), ISSN 1201-561X.

[75] R. Brown and C. D. Wensley, *Computing crossed modules induced by an inclusion of a normal subgroup, with applications to homotopy 2-types*, Theory Appl. Categ., 2, (1996), No. 1, 3–16 (electronic), ISSN 1201-561X.

[76] R. Brown and C. D. Wensley, *Computations and homotopical applications of induced crossed modules*, J. Symb. Comp., (to appear), ISSN 1201-561X.

[77] M. Bunge and S. Lack, *Van Kampen theorems for toposes*, Advances in Mathematics.

[78] J. Butterfield and C. J. Isham : A topos perspective on the Kochen–Specker theorem I - IV, *Int. J. Theor. Phys*, **37** (1998) No 11., 2669–2733 **38** (1999) No 3., 827–859, **39** (2000) No 6., 1413–1436, **41** (2002) No 4., 613–639.

[79] J. Butterfield and C. J. Isham : Some possible roles for topos theory in quantum theory and quantum gravity, *Foundations of Physics*.

[80] P. Carrasco and A. M. Cegarra, *Group-theoretic Algebraic Models for Homotopy Types*, Jour. Pure Appl. Algebra, 75, (1991), 195–235.

[81] E. A.

[82] E. Ĉech, 1933, *Höherdimensionale Homotopiegruppen*, in *Verhandlungen des Internationalen Mathematiker-Kongresses Zurich 1932*, volume 2, 203, International Congress of Mathematicians (4th : 1932 : Zurich, Switzerland, Walter Saxer, Zurich, reprint Kraus, Nendeln, Liechtenstein, 1967.

[83] M. Chaician and A. Demichev : Introduction to Quantum Groups, World Scientific 1996.

[84] A. Connes, 1994, *Noncommutative geometry*, Academic Press Inc., San Diego, CA.

[85] M. Crainic and R. Fernandes : Integrability of Lie brackets, Ann. of Math. **157** (2003), 575–620.

[86] R. Crowell, *The derived module of a homomorphism*, Advancesin Math., 5, (1971), 210–238.

[87] P. Dedecker, *Sur la cohomologie non abélienne. II*, Canad. J. Math., 15, (1963), 84–93.

[88] Dirac, P. A. M., 1930, The Principles of Quantum Mechanics, Oxford: Clarendon Press. ———-, 1933, The Lagrangian in Quantum Mechanics, Physikalische Zeitschrift der Sowietunion, 3: 64–72. ———-, 1943, Quantum Electrodynamics, Communications of the Dublin Institute for Advanced Studies, A1: 1–36.

[89] Dixmier, J., 1981, "Von Neumann Algebras", Amsterdam: North–Holland Publishing Company. [First published in French in 1957: Les Algebres d'Operateurs dans l'Espace Hilbertien, Paris: Gauthier–Villars.]

[90] A. Douady and R. Douady, 1979, *Algebres et theories Galoisiennes*, volume 2, CEDIC, Paris.

[91] M. Durdevich : Geometry of quantum principal bundles I, Commun. Math. Phys. **175** (3) (1996), 457–521.

[92] M. Durdevich : Geometry of quantum principal bundles II, Rev. Math. Phys. **9** (5) (1997), 531–607.

[93] S.Eilenberg. Relations between Homology and Homotopy Groups. Proc.Natl.Acad.Sci.USA (1966),v:x–xx.

[94] P. J. Ehlers and T. Porter, *Varieties of simplicial groupoids. I. Crossed complexes*, J. Pure Appl. Algebra, 120, (1997), 221–233, ISSN 0022-4049.

[95] P. J. Ehlers and T. Porter, *Erratum to: "Varieties of simplicial groupoids. I. Crossed complexes" [J. Pure Appl. Algebra* **120** *(1997), no. 3, 221–233; MR 98d:55012]*, J. Pure Appl. Algebra, 134, (1999), 207–209, ISSN 0022-4049.

[96] G. Ellis and I. Kholodna, *Three-dimensional presentations for the groups of order at most* 30, LMS J. Comput. Math., 2, (1999), 93–117+2 appendixes (HTML and source code) (electronic), ISSN 1461-1570.

[97] G. J. Ellis, *Homotopy classification the J. H. C. Whitehead way*, Exposition. Math., 6, (1988), 97–110, ISSN 0723-0869.

[98] Z. F. Ezawa, G. Tsitsishvilli and K. Hasebe : non–commutative geometry, extended W_∞ algebra and Grassmannian solitons in multicomponent Hall systems, arXiv:hep–th/0209198.

[99] J. M. G.vFell. 1960. The Dual Spaces of C*–Algebras, Transactions of the American Mathematical Society, 94: 365–403.

[100] F.M. Fernandez and E. A. Castro. 1996. (Lie) *"Algebraic Methods in Quantum Chemistry and Physics."*, Boca Raton: CRC Press, Inc.

[101] Feynman, R. P., 1948, Space–Time Approach to Non–Relativistic Quantum Mechanics, Reviews of Modern Physics, 20: 367387. [It is reprinted in (Schwinger 1958).]

[102] R. H. Fox, *Free differential calculus I: Derivations in the group ring*, Ann. Math., 57, (1953), 547–560.

[103] A. Fröhlich, *Non-Abelian homological algebra. I. Derived functors and satellites.*, Proc. London Math. Soc. (3), 11, (1961), 239–275.

[104] The GAP Group, 2002, *Groups, Algorithms, and Programming, version 4.3*, Technical report, http://www.gap-system.org.

[105] P. Gaucher, *Combinatorics of branchings in higher dimensional automata*, Theory Appl. Categ., 8, (2001), No. 12, 324–376 (electronic), ISSN 1201-561X.

[106] Gel'fand, I. and Naimark, M., 1943, On the Imbedding of Normed Rings into the Ring of Operators in Hilbert Space, Recueil Mathmatique [Matematicheskii Sbornik] Nouvelle Srie, 12 [54]: 197–213. [Reprinted in C*–algebras: 1943–1993, in the series Contemporary Mathematics, 167, Providence, R.I. : American Mathematical Society, 1994.]

[107] Georgescu, G. and C. Vraciu 1970. "On the Characterization of Łukasiewicz Algebras." *J Algebra*, 16 (4), 486-495.

[108] K.A. Hardie, K.H. Kamps and R.W. Kieboom, A homotopy 2-groupoid of a Hausdorff space, *Applied Cat. Structures* **8** (2000), 209-234.

[109] S. W. Hawking and R. Penrose. 2000. *"The Nature of Space and Time"*. Princeton and Oxford: Princeton University Press.

[110] R. Heynman and S. Lifschitz. 1958. *"Lie Groups and Lie Algebras"*., New York and London: Nelson Press.

[111] P. J. Higgins, 'Presentations of Groupoids, with Applications to Groups', *Proc. Camb. Phil. Soc.*60 (1964) 7-20.

[112] P. J. Higgins, 1971, *Categories and Groupoids*, van Nostrand, New York.

[113] P. J. Higgins and J. Taylor, 1982, *The Fundamental Groupoid and Homotopy Crossed Complex of an Orbit Space*, in K. H. Kamps et al., ed., *Category Theory: Proceedings Gummersbach 1981*, volume 962, 115–122.

[114] C. Hog-Angeloni and W. Metzler, eds., 1993, *Two-dimensional homotopy and combinatorial group theory*, volume 197 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge.

[115] J. Huebschmann, *Crossed n-fold extensions of groups and cohomology*, Comment. Math. Helv., 55, (1980), 302–313, ISSN 0010-2571.

[116] J. Huebschmann, *Automorphisms of group extensions and differentials in the Lyndon-Hochschild-Serre spectral sequence*, J. Algebra, 72, (1981), 296–334, ISSN 0021-8693.

[117] J. Huebschmann, *Group extensions, crossed pairs and an eight term exact sequence*, Jour. fur. reine. und ang. Math., 321, (1981), 150–172.

[118] E.Hurewicz. CW–complexes.Trans AMS.1955.

[119] C. J. Isham : A new approach to quantising space–time: I. quantising on a general category, *Adv. Theor. Math. Phys.* **7** (2003), 331–367.

[120] E. H. v. Kampen, *On the Connection Between the Fundamental Groups of some Related Spaces*, Amer. J. Math., 55, (1933), 261–267.

[121] S. Kobayashi and K. Nomizu: "Foundations of Differential Geometry" Vol I., Wiley Interscience, New York–London 1963. S. Kobayashi and K. Nomizu:

[122] H. Krips : Measurement in Quantum Theory, *The Stanford Encyclopedia of Philosophy* (Winter 1999 Edition), Edward N. Zalta (ed.),URL=¡http://plato.stanford.edu/archives/win1999/entries@ /qt–meas@ ureme@ nt/¿

[123] J.-P. Labesse, *Cohomologie, stabilisation et changement de base*, Astérisque, vi+161, ISSN 0303-1179, appendix A by Laurent Clozel and Labesse, and Appendix B by Lawrence Breen.

[124] E. C. Lance. *Hilbert C*–Modules. London Math. Soc. Lect. Notes* **210**, *Cambridge Univ. Press.* 1995.

[125] N. P. Landsman: *"Mathematical topics between classical and quantum mechanics"*. New York: *Springer Verlag*, 1998.

[126] N. P. Landsman: *Compact quantum groupoids*, arXiv:math–ph/9912006

[127] N. P. Landsman: Operator algebras and Poisson Manifolds Associated to Groupoids., *Commun. Math. Phys.* 222, 97-116 (2001).

[128] N. P. Landsman and B. Ramazan: Quantization of Poisson algebras associated to Lie algebroids. In R. Kaminker et al, eds., *Proc. Conf. on Groupoids in Physics, Analysis and Geometry*, Contemporary Mathematics series, *E-print: math-ph/001005.*

[129] J.-L. Loday, *Spaces with finitely many homotopy groups*, J.Pure Appl. Alg., 24, (1982), 179–202.

[130] A. S. T. Lue, *Cohomology of Groups relative to a variety*, J. Alg., 69, (1981), 155 – 174.

[131] K. C. H. Mackenzie : Lie Groupoids and Lie Algebroids in Differential Geometry, LMS Lect. Notes **124**, Cambridge University Press, 1987.

[132] M. Mackaay and R. F. Picken, *Holonomy and parallel transport for abelian gerbes*, arXiv:math.DG/0007053.
[133] S. MacLane, 1967, *Homology*, number 114 in Grundlehren, Springer.
[134] S. MacLane and J. H. C. Whitehead, *On the 3-type of a complex*, Proc. Nat. Acad. Sci. U.S.A., 36, (1950), 41–48.
[135] S. Maclane and I. Moerdijk: "Sheaves in Geometry and Logic - A first Introduction to Topos Theory", Springer Verlag, New York 1992.
[136] S. Magid: Foundations of Quantum Group Theory, Cambridge Univ. Press 1995.
[137] J.P. May: A Concise Course in Algebraic Topology. Chicago and London: The Chicago University Press. 1999.
[138] I. Moerdijk : Classifying toposes and foliations, *Ann. Inst. Fourier, Grenoble* **41**, 1 (1991) 189–209.
[139] I. Moerdijk: Introduction to the language of stacks and gerbes, arXiv:math.AT/0212266 (2002).
[140]
[141] E. J. Moore, 2001, *Graphs of Groups: Word Computations and Free Crossed Resolutions*, Ph.D. thesis, University of Wales, Bangor.
[142] M. A. Mostow : The differentiable space structure of Milnor classifying spaces, simplicial complexes, and geometric realizations, *J. Diff. Geom.* **14** (1979) 255–293.
[143] G. Nan Tie, *A Dold-Kan theorem for crossed complexes*, J. Pure Appl. Algebra,, 56, (1989.), 177–194.
[144] R. Peiffer, *Uber Identitaten zwischen Relationen*, Math. Ann., 121, (1949), 67 – 99.
[145] P. Olum, *Mappings of manifolds and the notion of degree*, Ann. of Math. (2), 58, (1953), 458–480.
[146] R. J. Plymen and P. L. Robinson: Spinors in Hilbert Space. *Cambridge Tracts in Math.* **114**, *Cambridge Univ. Press* 1994.
[147] N. Popescu. 1968. *"The Theory of Abelian Categories with Applications to Rings and Modules".* New York and London: Academic Press.
[148] T. Porter, *n-types of simplicial groups and crossed n-cubes*, Topology, 32, (1993), 5 – 24.
[149] S. Pride, 1991, *Identities among relations*, in A. V. E.Ghys, A.Haefliger, ed., *Proc. Workshop on Group Theory from a Geometrical Viewpoint*, 687–716, International Centre of Theoretical Physics, Trieste, 1990, World Scientific.
[150] I. Raptis : Algebraic quantisation of causal sets, *Int. Jour. Theor. Phys.* **39** (2000), 1233.
[151] I. Raptis : Quantum space–time as a quantum causal set, arXiv:gr-qc/0201004.
[152] I. Raptis and R. R. Zapatrin : Quantisation of discretized spacetimes and the correspondence principle, *Int. Jour. Theor. Phys.* **39** (2000), 1.
[153] K. Reidemeister, *Uber Identitaten von Relationen*, Abh. Math. Sem. Hamburg, 16, (1949), 114 – 118.
[154] M. A. Rieffel : Group C*–algebras as compact quantum metric spaces, *Documenta Math.* **7** (2002), 605–651.
[155] G.–C. Rota : On the foundation of combinatorial theory, I. The theory of Möbius functions, *Zetschrif für Wahrscheinlichkeitstheorie* **2** (1968), 340.
[156] J. E. Roberts : More lectures on algebraic quantum field theory (in A. Connes, et al 'non–commutative Geometry'), Springer (2004).
[157] C. Rovelli : Loop quantum gravity (1997), arXiv:gr–qc/9710008.
[158] Schwartz, L., 1945, Generalisation de la Notion de Fonction, de Derivation, de Transformation de Fourier et Applications Mathematiques et Physiques, Annales de l'Universite de Grenoble, 21: 57–74.
———-, 1951–1952, Theorie des Distributions, Publications de l'Institut de Mathematique de l'Universite de Strasbourg, Vols 9–10, Paris: Hermann.
[159] Segal, I. E., 1947a, Irreducible Representations of Operator Algebras, Bulletin of the American Mathematical Society, 53: 73–88.
[160] ———-, 1947b, Postulates for General Quantum Mechanics, Annals of Mathematics, 4: 930–948.
[161] Jan Smit. 2002. "*Quantum Field Theory on a Lattice*".
[162] R. D. Sorkin : Finitary substitute for continuous topology, *Int. J. Theor. Phys.* **30** No. 7. (1991), 923–947.
[163] E. H. Spanier: "Algebraic Topology", New York: McGraw Hill, 1966.
[164] J. Taylor, *Quotients of Groupoids by the Action of a Group*, Math. Proc. Camb. Phil. Soc., 103, (1988), 239–249.

[165] A. P. Tonks, 1993, *Theory and applications of crossed complexes*, Ph.D. thesis, University of Wales, Bangor.
[166] E. H. van Kampen, *On the Connection Between the Fundamental Groups of some Related Spaces*, Amer. J. Math., 55, (1933), 261–267.
[167] J. C. Varilly : An introduction to non–commutative geometry, arXiv:physics/9709045
[168] J. Virsik : On the holonomity of higher order connections, *Cah. Top. Geom. Diff. Cat.* **12** (1971), 197–212.
[169] C. T. C. Wall, *Finiteness conditions for CW-complexes II*, Proc. Roy. Soc. Ser. A, 295, (1966), 149–166.
[170] S. Weinberg.1995–2000. *The Quantum Theory of Fields.* Cambridge, New York and Madrid: Cambridge University Press, Vols. 1 to 3.
[171] A. Weinstein : Groupoids: unifying internal and external symmetry, *Notices of the Amer. Math. Soc.* **43**, 744–752.
[172] Wess and Bagger. 2000. Supergravity. (Weinberg)
[173] J. Wheeler and W. Zurek, Eds."Quantum theory and Measurement", Princeton 1983.
[174] J. H. C. Whitehead, *On adding relations to homotopy groups*, Ann. of Math. (2), 42, (1941), 409–428.
[175] J. H. C. Whitehead, *Combinatorial Homotopy I*, Bull. Amer. Math. Soc., 55, (1949), 213–245.
[176] J. H. C. Whitehead, *Combinatorial Homotopy II*, Bull. Amer. Math. Soc., 55, (1949), 453–496.
[177] J. H. C. Whitehead, *Simple homotopy types*, Amer. J. Math., 72, (1950), 1–57.
[178] Wightman, A. S., 1956, Quantum Field Theory in Terms of Vacuum Expectation Values, Physical Review, 101: 860–866.
[179] ———-, 1976, Hilbert's Sixth Problem: Mathematical Treatment of the Axioms of Physics, Proceedings of Symposia in Pure Mathematics, 28: 147–240. Wightman, A.S. and Grding, L., 1964, Fields as Operator–Valued Distributions in Relativistic Quantum Theory, Arkiv fur Fysik, 28: 129–184.
[180] S. L. Woronowicz : Twisted SU(2) group : An example of a non–commutative differential calculus, RIMS, Kyoto University **23** (1987), 613–665.

*Department of Mathematics and Computer Science, Eastern Illinois University, Charleston IL 61920 USA [J. F. Glazebrook]cfjfg@eiu.edu

**University of Illinois at Urbana–Champaign, FSHN and NPRE Departments, NMR and NIR Microspectroscopy Facility , Urbana IL 61801 USA[I. C. Baianu]ibaianu@uiuc.edu

***School of Informatics, University of Wales, Dean St. Bangor, Gwynedd LL57 1UT UK [R. Brown]r.brown@bangor.ac.uk

Acknowledgments:

The work of R. Brown was partially suported by a Leverhulme Emeritus Fellowship (2002–2004).

Research reported in this publication by I.C. Baianu was supported in part by a Research Grant from Renessen Biotechnology Corp., Springfield, IL, USA.

Appendix *The Lagrangian and the Path Integral Approach of QED.*

New info3

=

Feynman's Path Integral Method in Quantum Electrodynamics (QED)

.Therefore, one cannot consistently define either ‘quantum computation’, or ‘entropy’, ‘information’, etc., for such singular regions for two fundamental reasons: one has no means to make any quantum or classical observations whatsoever inside the singular regions, and one cannot obtain any information about their algebraic topology as they appear to be *closed* with respect to our observable universe beyond the ’event horizon’. Even though energy may escape near the event horizon from such regions, perhaps through QSS fluctuations allowed by the Heisenberg Uncertainty Principle