A NOVICE'S GUIDE TO SU(2)

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These informal notes provide a brief introduction to the Lie group SU(2). Given the important and fundamental rôle that SU(2) and its Lie algebra has in Lie theory and applications to Physics and other disciplines, it is natural to introduce many ideas of Lie theory through this example, making them as concrete as possible. We hope the interested reader will be motivated to read more complete treatments of the subject, such as those in the bibliography.

1. Basic definitions

Let $\mathrm{M}_2(\mathbb{C})$ be the set of 2×2 complex matrices. Since $\mathrm{M}_2(\mathbb{C}) \cong \mathbb{C}^4 \cong \mathbb{R}^8$, it inherits a natural structure of a metric space. The 2×2 *special unitary group* is

$$\mathrm{SU}(2) = \{A \in \mathrm{M}_2(\mathbb{C}) : A^*A = I = AA^*,\ \det A = 1\} \subset \mathrm{M}_2(\mathbb{C}),$$

viewed as a subspace which is actually a topological subgroup.

Theorem 1.1. $\mathrm{SU}(2) \subset \mathbb{C}^4 \cong \mathbb{R}^8$ *is the zero-set of the following system of polynomial equations in the real and imaginary parts: of* a, b, c, d

$$\begin{cases} |a|^2 + |b|^2 - 1 &= 0, \\ |c|^2 + |d|^2 - 1 &= 0, \\ a\bar{c} + b\bar{d} &= 0, \\ ad - bc - 1 &= 0. \end{cases} \tag{1.1}$$

The derivative matrix of this system has rank 5 *at each point of* SU(2)*. Hence* SU(2) *is a* 3*-dimensional smooth submanifold of* $\mathbb{C}^4 \cong \mathbb{R}^8$*.*
Each of the multiplication and inverse maps

$$\mu\colon \mathrm{SU}(2)\times\mathrm{SU}(2) \longrightarrow \mathrm{SU}(2), \quad \chi\colon \mathrm{SU}(2) \longrightarrow \mathrm{SU}(2),$$

is smooth. Hence, SU(2) *is a* Lie group.

The map

$$\mathrm{SU}(2) \longrightarrow \mathrm{S}^3;\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto (a, b),$$

can be shown to be a diffeomorphism onto the 3-sphere

$$\mathrm{S}^3 = \{\mathbf{z} \in \mathbb{C}^2 \cong \mathbb{R}^4 : |\mathbf{z}| = 1\}.$$

Let

$$T = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{bmatrix} : |\alpha| = 1 \right\} \subset \mathrm{SU}(2),$$

which is an abelian subgroup of SU(2). Let

$$\mathrm{N}_{\mathrm{SU}(2)}(T) = \{P \in \mathrm{SU}(2) : PTP^* = T\}$$

be the *normalizer* of T, i.e., the largest subgroup of SU(2) containing T as a normal subgroup. The *Weyl group* of T in SU(2) is the finite quotient group

$$\mathrm{W}_{\mathrm{SU}(2)}(T) = \mathrm{N}_{\mathrm{SU}(2)}(T)/T.$$

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Proposition 1.2. *T is a maximal abelian subgroup for which*

$$\mathrm{SU}(2) = \bigcup_{P \in \mathrm{SU}(2)} PTP^*; \quad \mathrm{N}_{\mathrm{SU}(2)}(T) = T \cup \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} T \quad \text{(disjoint union)}; \quad \mathrm{W}_{\mathrm{SU}(2)}(T) \cong \mathbb{Z}/2.$$

Notice that for any conjugate PTP^* of T with $P \in \mathrm{SU}(2)$ we have

$$\mathrm{W}_{\mathrm{SU}(2)}(PTP^*) = \mathrm{N}_{\mathrm{SU}(2)}(PTP^*)/PTP^* \cong \mathbb{Z}/2,$$

so up to a group isomorphism the Weyl group is an invariant of the group SU(2). Also, the Weyl group $\mathrm{W}_{\mathrm{SU}(2)}(T)$ acts on T by

$$PT \cdot \begin{bmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{bmatrix} = P \begin{bmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{bmatrix} P^*.$$

In particular,

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} T \cdot \begin{bmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{bmatrix} = \begin{bmatrix} \overline{\alpha} & 0 \\ 0 & \alpha \end{bmatrix}.$$

2. The Lie algebra

To give an explicit chart for SU(2) about the identity I, we proceed as follows. In (1.1), take the variables

$$a = 1 + \delta a, \; b = \delta b, \; c = \delta c, \; d = 1 + \delta d;$$

then working order 1 in $\delta a, \delta b, \delta c, \delta d$, Equation (1.1) gives

$$\begin{cases} \delta a + \overline{\delta a} & \doteq 0 \\ \delta d + \overline{\delta d} & \doteq 0 \\ \overline{\delta c} + \delta b & \doteq 0 \\ \delta a + \delta d & \doteq 0 \end{cases} \implies \begin{cases} \overline{\delta a} & \doteq -\delta a \\ \overline{\delta d} & \doteq -\delta d \\ \delta c & \doteq \overline{\delta b} \\ \delta d & \doteq -\delta a \end{cases}$$

and so we have

$$\begin{cases} \delta a & = i\delta t \\ \delta b & = \delta u + i\delta v \\ \delta c & = -\delta u + i\delta v \\ \delta d & = -i\delta t \end{cases} \quad (\delta t, \delta u, \delta v \in \mathbb{R}).$$

This suggest that the tangent space $\mathrm{T}_I\mathrm{SU}(2)$ to SU(2) at I is the set of all skew-hermitian matrices of trace 0,

$$\text{Sk-H}^0(2) = \{A \in \mathrm{M}_2(C) : A^* = -A, \; \operatorname{tr} A = 0\},$$

which is a 3-dimensional real vector space with basis consisting of the *Pauli matrices*

$$R = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

We can define an *exponential map*

$$\exp\colon \text{Sk-H}^0(2) \longrightarrow \mathrm{SU}(2); \quad \exp(H) = \sum_{k \geqslant 0} \frac{1}{k!} H^k,$$

since it is easy to see that this series converges absolutely with respect to the *supremum* (or *operator*) *norm*

$$\|H\| = \sup_{|\mathbf{x}|=1} |H\mathbf{x}|.$$

The fact that $\exp(H)$ lies in SU(2) is a consequence of the relations

$$\begin{aligned}\exp(H)^* &= (\sum_{k\geqslant 0} \frac{1}{k!} H^k)^* \\ &= \sum_{k\geqslant 0} \frac{1}{k!} H^{*k} \\ &= \sum_{k\geqslant 0} \frac{1}{k!} (-H)^k \\ &= \exp(-H) = \exp(H)^{-1}.\end{aligned}$$

Theorem 2.1. *The exponential map* $\exp\colon \text{Sk-H}^0(2) \longrightarrow \mathrm{SU}(2)$ *is smooth and surjective. Moreover,* $\exp(H) = I$ *if and only if*

$$H = Q \begin{bmatrix} 2\pi i n & 0 \\ 0 & -2\pi i n \end{bmatrix} Q^* \quad (n \in \mathbb{Z},\ Q \in \mathrm{SU}(2)).$$

Hence, there is an open subset $U \subset \text{Sk-H}^0(2)$ *containing* O*, such that* $\exp\colon U \longrightarrow \exp(U)$ *is a diffeomorphism.*

To calculate $\exp(H)$ in practise, we first diagonalise H by finding a $Q \in \mathrm{SU}(2)$ such that

$$QHQ^* = \begin{bmatrix} is & 0 \\ 0 & -is \end{bmatrix} \quad (s \in \mathbb{R}).$$

Then

$$\exp(H) = Q^* \left(\exp(QHQ^*)\right) Q = Q^* \begin{bmatrix} e^{is} & 0 \\ 0 & e^{-is} \end{bmatrix} Q.$$

In particular,

$$\exp(tR) = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}, \quad \exp(tP) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}, \quad \exp(tR) = \begin{bmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{bmatrix}.$$

The product AB of two traceless skew-Hermitian matrices A, B is not skew-Hermitian, however, there is a way to 'multiply' them to obtain another skew-Hermitian matrix, namely by forming their *commutator*

$$[A, B] = AB - BA,$$

which is both skew-Hermitian and traceless. To see how this connects with the group structure of SU(2), let $0 \neq s, t \in \mathbb{R}$ and consider

$$\exp(tA)\exp(sB)\exp(tA)^{-1} = \exp(tA)\exp(sB)\exp(-tA).$$

We find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,s}_{|s=0}\frac{\mathrm{d}}{\mathrm{d}\,t}_{|t=0} \exp(tA)\exp(sB)\exp(-tA) &= \frac{\mathrm{d}}{\mathrm{d}\,s}_{|s=0} \left(A\exp(sB) - \exp(sB)A\right) \\ &= AB - BA = [A, B].\end{aligned} \tag{2.1}$$

So this bracket arises from the conjugation action of SU(2) on itself. The structure we get on the set of traceless skew-Hermitian matrices is that of a *Lie algebra.* This consists of a vector space $\mathfrak{g}$ over a field $\mathbb{k}$, equipped with a $\mathbb{k}$-bilinear map $[\ ,\ ]\colon \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$ such that for $x, y, z \in \mathfrak{g}$,

(skew symmetry) $$[x, y] = -[y, x],$$

(Jacobi identity) $$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Such a structure is called a $\mathbb{k}$-*Lie algebra.* The following examples are basic.

- For a $\mathbb{k}$-algebra A, set $\mathfrak{g} = A$ and $[x, y] = xy - yx$.

- For any $\Bbbk$-vector space V, set $\mathfrak{g} = V$ and $[x, y] = 0$; this is gives a *trivial* or *abelian* Lie algebra.
- The $\mathbb{R}$-Lie algebra associated to $\mathrm{SU}(2)$ is denoted $\mathfrak{su}(2)$. There is a close connection between the representation theory of $\mathrm{SU}(2)$ and that of $\mathfrak{su}(2)$.
- Take $\mathfrak{g} = \mathbb{R}^3$ with standard basis i, j, k and
$$[i, j] = k,\ [j, k] = i,\ [k, i] = j.$$
Then of course $[x, y] = x \times y$. It turns out that this $\mathbb{R}$-Lie algebra is isomorphic to $\mathfrak{su}(2)$.

3. The flag space

The orbit space $\mathrm{SU}(2)/T$ is often called the *flag space* of $\mathrm{SU}(2)$. There is a smooth quotient map $q\colon \mathrm{SU}(2) \longrightarrow \mathrm{SU}(2)/T$. Let $\mathbb{C}\mathrm{P}^1$ denote the 1-point compactification of $\mathbb{C}$, i.e., the Riemann sphere $\mathbb{C} \cup \{\infty\} \cong \mathrm{S}^2$. There is a smooth mapping
$$\varphi\colon \mathrm{SU}(2) \longrightarrow \mathbb{C}\mathrm{P}^1;\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{cases} c/a & \text{if } a \neq 0, \\ \infty & \text{if } a = 0. \end{cases}$$
If $A \in T$ and $B \in \mathrm{SU}(2)$, then $\varphi(BA) = \varphi(B)$, giving a factorisation $\varphi = \overline{\varphi} \circ q$ through a continuous bijection $\overline{\varphi}\colon \mathrm{SU}(2)/T \longrightarrow \mathbb{C}\mathrm{P}^1$, which is a actually a diffeomorphism. Hence, $\mathrm{SU}(2)/T \cong \mathbb{C}\mathrm{P}^1$. The associated map $\mathrm{S}^3 \longrightarrow \mathrm{S}^2$ is the famous *Hopf map* for which inverse images of points are pairwise linked circles.

Notice that $\mathbb{C}\mathrm{P}^1$ has a natural complex structure, whereas $\mathrm{SU}(2)/T$ only appears to be a real manifold. The explanation of this involves the *complexification* of $\mathrm{SU}(2)$, namely $\mathrm{SL}_2(\mathbb{C})$. In fact $\mathrm{SU}(2)/T \cong \mathrm{SL}_2(\mathbb{C})/B$ where $B \subset \mathrm{SL}_2(\mathbb{C})/B$ is the Borel subgroup of upper triangular matrices which is a complex Lie subgroup, hence the quotient space is a complex manifold.

4. Representation theory

In this section we assume that G is a Lie group (possibly complex) and $\mathfrak{g}$ is its Lie algebra, i.e., $\mathfrak{g} = \mathrm{T}_I G$, given a Lie bracket structure defined using an exponential map $\exp\colon \mathrm{T}_I G \longrightarrow G$ as in Equation (2.1). (If G is complex, then $\mathfrak{g}$ is a complex Lie algebra.) Let $\Bbbk = \mathbb{R}, \mathbb{C}$ and let V be a $\Bbbk$-vector space of dimension d. We write $\mathfrak{gl}(V)$ for the Lie algebra of $\Bbbk$-linear maps with bracket defined by
$$[A, B] = AB - BA.$$

Definition 4.1. Let $R\colon G \longrightarrow \mathrm{GL}(V)$ be a homomorphism which is an analytic map of manifolds. Then R is a *representation of G in V*.

Definition 4.2. A k-linear map $r\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a *representation of $\mathfrak{g}$ in V* if
$$r([x, y]) = [r(x), r(y)] = r(x)r(y) - r(y)r(x).$$

We usually write $A \cdot v = R(A)(v)$ and $a \cdot v = r(a)(v)$.

Given such a representation R, then we can obtain a representation of $\mathfrak{g}$ as follows. For $a \in \mathfrak{g}$ and $v \in V$, define
$$a \cdot v = \frac{\mathrm{d}}{\mathrm{d}\, t}_{|t=0} \exp(t)(v) = \lim_{t \longrightarrow 0} \frac{1}{t} \left(\exp(ta)(v) - v\right). \tag{4.1}$$
Using Equation (2.1) we find that r is a representation of $\mathfrak{g}$. In fact, we can also go the other way, provided certain topological restrictions on G are satisfied. For $\mathrm{SU}(2)$, they always are.

If we take a basis $v_1, \ldots, v_d$ for V, then we can express elements of $\mathrm{GL}(V)$ in terms of matrices, i.e., elements of the *general linear group* $\mathrm{GL}_d(\Bbbk)$. Similarly, a representation of $\mathfrak{g}$ takes values in $\mathfrak{gl}_d(\Bbbk)$, the Lie algebra of $d \times d$ matrices with entries in $\Bbbk$.

5. Representations of SU(2)

Now will study complex representations of SU(2). First note that by the remarks in Section 4, we may as well consider its (real) Lie algebra $\mathfrak{su}(2)$. But a complex representation must factor through the complexification

$$\mathfrak{su}(2) \longrightarrow \mathfrak{su}(2) \otimes \mathbb{C} \longrightarrow \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2.$$

Hence we need only consider complex representations of the latter.

The following elements form a $\mathbb{C}$-basis for $\mathfrak{sl}_2$,

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We have the Lie brackets

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h, \tag{5.1}$$

together with the trivial brackets $[h, h] = [e, e] = [f, f] = 0$. The subalgebra generated by h, e is called a *Borel subalgebra* and is denoted $\mathfrak{b}$ and contains the Cartan algebra $\mathfrak{h}$ generated by h. Let $\mathfrak{h}^* = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ be the dual of $\mathfrak{h}$.

Let $r\colon \mathfrak{sl}_2 \longrightarrow \mathfrak{gl}(V)$ be a complex representation of $\mathfrak{sl}_2$.

Definition 5.1. Let $0 \neq w \in V$. Then w is a *weight vector* of *weight* $\lambda \in \mathfrak{h}^*$ if

$$h \cdot w = \lambda(h) w.$$

It is a *highest* (resp. *lowest*) *weight vector* if it also satisfies

$$e \cdot w = 0 \quad (\text{resp. } f \cdot w = 0).$$

Theorem 5.2. *Let $r\colon \mathfrak{sl}_2 \longrightarrow \mathfrak{gl}(V)$ be a non-trivial finite dimensional complex representation.*

(a) *There is highest (resp. lowest) weight vector in V.*

(b) *If r is an irreducible representation, then any highest (resp. lowest) weight vector w generates V over $\mathfrak{sl}_2$, i.e., V is spanned as a $\mathbb{C}$-vector space by the elements $a_k \cdots a_1 \cdot w$ ($a_j \in \mathfrak{sl}_2$).*

(c) *If r is an irreducible representation, and w is a highest weight vector with eigenvalue ℓ, then $0 \leqslant \ell \in \mathbb{Z}$. Moreover, the elements*

$$v_k \quad (k = -\ell, -\ell + 2, \ldots, \ell - 2, \ell)$$

are the only weight vectors and satisfy

$$v_\ell = w, \quad v_k = f \cdot v_{k+2}.$$

Proof. We only prove (a), the remaining proofs can be found in Serre's book [5].
(a) The element h acts on V as a linear transformation and must therefore have an eigenvector w_0 with eigenvalue ℓ_0 say, hence $h \cdot v_0 = \ell_0 v_0$. We can define a corresponding $\lambda_0 \in \mathfrak{h}^*$ by $\lambda_0(th) = t\ell_0$.

Computing Lie brackets, we obtain

$$h \cdot e \cdot v_0 = e \cdot h \cdot v_0 + [h, e] \cdot v_0 = \ell_0 e \cdot v_0 + 2e \cdot v_0,$$
$$h \cdot f \cdot v_0 = f \cdot h \cdot v_0 + [h, f] \cdot v_0 = \ell_0 e \cdot v_0 - 2e \cdot v_0,$$

and so $v_1 = e \cdot v_0$ (resp. $v_{-1} = f \cdot v_0$) is either an eigenvector with eigenvalue $\ell_1 = \ell_0 + 2$ (resp. $\ell_{-1} = \ell_1 - 2$) or 0. But as eigenvectors of h for distinct eigenvalues are linearly independent, if we repeat this procedure we will eventually obtain a highest weight vector v_m say for eigenvalue ℓ_m and a lowest weight vector v_{-n} for eigenvalue ℓ_{-n}. At each stage we can define $\lambda_j \in \mathfrak{h}^*$ by

$$\lambda_j(th) = t\ell_j.$$

□

Example 5.3. If we take one dimensional representation $V = \mathbb{C}$ in which $e \cdot 1 = f \cdot 1 = h \cdot 1$, then we have $\ell = 0$. This is the 1-dimensional *trivial* representation.

Example 5.4. Let $V = \mathbb{C}^2$ with each element of $\mathfrak{sl}_2$ acting as the corresponding 2×2 matrix,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot (x, y) = (ax + by, cx + dy).$$

Then we have $\ell = 1$ with $v_1 = e_1$ and $v_1 = e_2$, the standard basis vectors. This is the *natural* representation.

Example 5.5. Let $\mathrm{P} = \mathbb{C}[u, v]$ be the set of all polynomials in u, v and P_k the set of all homogeneous polynomials of degree k. Then if we set

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot f(u, v) = f(au + bv, cu + dv),$$

this defines an action of $\mathfrak{sl}_2$ on P, which preserves each P_k. It can be shown that P_ℓ is the irreducible representation with highest weight ℓ.

6. The adjoint representation

We now consider the most 'natural' representation of the Lie algebra $\mathfrak{sl}_2$. Namely define $\mathrm{ad}\colon \mathfrak{sl}_2 \longrightarrow \mathfrak{gl}(\mathfrak{sl}_2)$, where

$$\mathrm{ad}(x)(y) = [x, y].$$

That this *is* a representation follows from the Jacobi identity. It is easily checked that $\ell = 2$, $v_2 = e$, $v_0 = h$ and $v_{-2} = f$.

Such a representation

$$\mathrm{ad}\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g});\ \mathrm{ad}(x)(y) = [x, y]$$

exists for *any* Lie algebra $\mathfrak{g}$. Non-zero vectors $w \in \mathfrak{g}$ such that there is some $\lambda \in \mathfrak{h}^*$ with

$$h \cdot w = [h, w] = \lambda(h)w \quad (h \in \mathfrak{h})$$

are called *root vectors* and the functional λ is called a *root*. There is a *root space decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha$$

where $x \in \mathfrak{g}_\alpha$ if and only if

$$h \cdot x = [h, x] = \alpha(h)x \quad (h \in \mathfrak{h}).$$

Moreover, if $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$, then $[x, y] \in \mathfrak{g}_{\alpha+\beta}$. For (semi)simple Lie algebras such as $\mathfrak{sl}_2$, $\mathfrak{g}_0 = \mathfrak{h}$.

The adjoint representation comes from a 3-dimensional real representation of the group SU(2). For if we identify the Lie algebra $\mathfrak{su}(2)$ with the 3-dimensional real vector space $\text{Sk-H}^0(2)$, SU(2) acts linearly by Hermitian conjugation, i.e., $A \in \mathrm{SU}(2)$ acts by sending H to AHA^*. If we choose as a basis $\text{Sk-H}^0(2)$ the vectors $(1/\sqrt{2})R, (1/\sqrt{2})P, (1/\sqrt{2})Q$, then the linear transformation induced by A has a 3×3 matrix which is orthogonal and has determinant 1. There is a corresponding group homomorphism $\mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$ which is surjective and has kernel $\{\pm I\}$. There is an associated isomorphism of real Lie algebras. This double covering homomorphism is related to the idea of *spinors*.

7. The invariant inner product

For $X, Y \in \mathfrak{su}(2)$ let $(X, Y) = -\operatorname{tr}(XY)$. Then $(\ ,\ )$ is a real symmetric bilinear form and in fact is positive definite. The standard basis elements ih, $(e-f)$, $i(e+f)$ satisfy

$$(ih, ih) = (e-f, e-f) = (i(e+f), i(e+f)) = 2.$$

It is also *invariant* in the sense that

$$([Z, X], Y) + (X, [Z, Y]) = 0.$$

The element ih is the *coroot* associated to the natural positive simple root α given by

$$[h, e] = 2e, \quad [h, f] = -2f.$$

8. Bott–Borel–Weil theory

Let K denote a compact, connected, simply connected, (semi)-simple Lie group and G its complexification. Thus the (real) Lie algebra $\mathfrak{k}$ has complexification $\mathfrak{k} \otimes \mathbb{C} = \mathfrak{g}$. A maximal solvable subgroup $B \subset G$ is called a *Borel subgroup*; a closed subgroup $P \subset G$ containing a Borel subgroup is called a *parabolic subgroup*. Such a B contains a maximal complex torus $D \cong (\mathbb{C}^\times)^r$, where r is the *rank* of K and G. Notice that $K \cap B = T$, where $T \cong \mathbb{T}^r$ is a maximal (real) torus for K. We have $K/T \cong G/B$, hence the latter is compact, and the former is a complex manifold; more generally, if P is any parabolic, then $K/P \cap K \cong G/P$ and these remarks are true. In fact, parabolic subgroups are characterised by the fact that the quotient G/P is compact and Kähler.

Let V be a finite dimensional (complex) representation of G, given an inner product so that the restriction to K is unitary. Then we can consider the *projectivisation of* V, $\mathbb{CP}(V)$. This is the space of lines in V, which can be expressed as

$$\mathbb{CP}(V) = (V_0)/\mathbb{C}^\times,$$

where $V_0 = V - \{0\}$. Given a basis for V, we have an equivalence $\mathbb{CP}(V) \cong \mathbb{CP}^{\dim V - 1}$ and see that $\mathbb{CP}(V)$ has a complex structure. The representation gives rise to actions of K and G on $\mathbb{CP}(V)$.

Given a line $[v] \in \mathbb{CP}(V)$, we may consider the orbits $K \cdot [v]$ and $G \cdot [v]$, which in general will be different.

A non-zero vector $v \in V$ is called a *weight vector* for the maximal torus $T \subset K$ (respectively $D \subset G$) if

$$z \cdot v = \lambda(z) v \quad (z \in T \text{ or } z \in D),$$

where $\lambda(z) \in \mathbb{C}^\times$. It is easily verified that the function $\lambda\colon T \longrightarrow \mathbb{C}^\times$ (or $\lambda\colon D \longrightarrow \mathbb{C}^\times$) is a continuous homomorphism, called the *weight* associated to v. Notice that the line $[v]$ is stabilized by T (or D), hence there is a surjection

$$K/T \longrightarrow K \cdot [v] \quad (\text{or } G/D \longrightarrow G \cdot [v]).$$

A weight vector is a *highest weight vector for the Borel subgroup* B if $[v]$ is stabilized by B; equivalently, the weight $\lambda\colon D \longrightarrow \mathbb{C}^\times$ extends to a continuous homomorphism $\lambda\colon B \longrightarrow \mathbb{C}^\times$.

We will refer to the line $[v]$ spanned by a (highest) weight vector v as a (highest) weight ray.

Theorem 8.1. *Let V be an irreducible representation of G.*

(a) *If $[v] \in \mathbb{CP}(V)$ is a highest weight ray, the orbits $K \cdot [v]$ and $G \cdot [v]$, hence are compact submanifolds of $\mathbb{CP}(V)$. Moreover, the stabilizer of $[v]$ in G is a parabolic subgroup P and then*

$$K/P_{[v]} \cap K \cong G/P \cong K \cdot [v] = G \cdot [v].$$

Thus, these orbits are compact Kähler submanifolds of $\mathbb{CP}(V)$.

(b) *Let* $[v] \in \mathbb{CP}(V)$ *be a weight ray for some maximal torus* $T \subset K$*, satisfying the following condition:*

(*) for each root α of K with respect to T, if T_α stabilizes the vector v, then so does $\mathrm{SU}(2)_\alpha$.

Then the orbit $K \cdot [v]$ *is a compact symplectic submanifold of* $\mathbb{CP}(V)$ *which is only a Kähler submanifold if* $[v]$ *is a highest weight ray.*

(c) *The only symplectic orbits* $K \cdot [v]$ *are those described in part (b).*

In this Theorem, the root α has an associated subgroup $\mathrm{SU}(2)_\alpha \subset K$ isomorphic to $\mathrm{SU}(2)$ and having maximal torus $T_\alpha = T \cap \mathrm{SU}(2)_\alpha$.

Notice also that if P stabilizes a ray $[v]$, then the weight $\lambda\colon D \longrightarrow \mathbb{C}^\times$ extends to a homomorphism $\lambda\colon P \longrightarrow \mathbb{C}^\times$.

Given the irreducible representation V of G and hence K, we can actually recover V from G/P, where P is the parabolic subgroup of G stabilizing a highest weight ray $[v] \in \mathbb{CP}(V)$, with weight λ. We see this as follows. First take the dual of V, V^*, which is a G representation with action

$$(g \cdot f)(x) = f(g^{-1} \cdot x) \quad (g \in G,\ f \in V^*,\ x \in V).$$

Let f be a highest weight vector for V^* with respect to B, and let μ be the weight associated to this. In fact, μ^{-1} is a *lowest weight* for V.

Since P acts trivially on $[v]$, the weight λ extends to a homomorphism $\lambda\colon P \longrightarrow \mathbb{C}^\times$. Similarly, there are extensions of μ^{-1} and μ to homomorphisms $P \longrightarrow \mathbb{C}^\times$.

We can form a line bundle

$$\xi_\mu = G \underset{P}{\times} \mathbb{C}_\mu \longrightarrow G/P,$$

where $\mathbb{C}_\mu$ denotes $\mathbb{C}$ with P acting by

$$p \cdot z = \mu(p) z.$$

The projection map is G-equivariant making this into a *homogeneous holomorphic line bundle*. Let $\Gamma^{\mathrm{hol}}(\xi_\mu, G/P)$ denote the space of global holomorphic sections of ξ_μ.

Theorem 8.2. *There is an isomorphism of* G*-modules*

$$\Gamma^{\mathrm{hol}}(\xi_\mu, G/P) \cong V,$$

where given a section $s\colon G/P \longrightarrow G \times_P \mathbb{C}_\mu$ *we set*

$$g \cdot s(xP) = g s(g^{-1} x P).$$

References

[1] J.F. Adams, Lectures on Lie Groups, University of Chicago Press (1969).

[2] A. Baker, Matrix Groups: An Introduction to Lie Group Theory, Springer-Verlag (2002).

[3] M.L. Curtis, Matrix Groups, Springer-Verlag (1984).

[4] V. Guillemin & S. Sternberg, Symplectic techniques in physics, reprinted with corrections, Cambridge University Press (1990).

[5] J-P. Serre, Complex Semisimple Lie Algebras, Springer-Verlag (1987).

[6] S. Sternberg, Group Theory and Physics, Cambridge University Press (1994).

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