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BIQUANTIZATION OF LIE BIALGEBRAS

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For any finite-dimensional Lie bialgebra $\mathfrak{g}$, we construct a bialgebra $A_{u,v}(\mathfrak{g})$ over the ring $\mathbf{C}[u][[v]]$, which quantizes simultaneously the universal enveloping bialgebra $U(\mathfrak{g})$, the bialgebra dual to $U(\mathfrak{g}^*)$, and the symmetric bialgebra $S(\mathfrak{g})$. Following Turaev, we call $A_{u,v}(\mathfrak{g})$ a biquantization of $S(\mathfrak{g})$. We show that the bialgebra $A_{u,v}(\mathfrak{g}^*)$ quantizing $U(\mathfrak{g}^*)$, $U(\mathfrak{g})^*$, and $S(\mathfrak{g}^*)$ is essentially dual to the bialgebra obtained from $A_{u,v}(\mathfrak{g})$ by exchanging u and v. Thus, $A_{u,v}(\mathfrak{g})$ contains all information about the quantization of $\mathfrak{g}$. Our construction extends Etingof and Kazhdan's one-variable quantization of $U(\mathfrak{g})$.

Résumé. *Etant donné une bigèbre de Lie $\mathfrak{g}$ de dimension finie, nous construisons une $\mathbf{C}[u][[v]]$-bigèbre $A_{u,v}(\mathfrak{g})$ qui quantifie simultanément la bigèbre enveloppante $U(\mathfrak{g})$, la bigèbre duale de $U(\mathfrak{g}^*)$ et la bigèbre symétrique $S(\mathfrak{g})$. Suivant Turaev, nous appelons $A_{u,v}(\mathfrak{g})$ une biquantification de $S(\mathfrak{g})$. Nous montrons que la bigèbre $A_{u,v}(\mathfrak{g}^*)$ qui quantifie $U(\mathfrak{g}^*)$, $U(\mathfrak{g})^*$ et $S(\mathfrak{g}^*)$ est en dualité avec la bigèbre obtenue à partir de $A_{u,v}(\mathfrak{g})$ en échangeant u et v. La bigèbre $A_{u,v}(\mathfrak{g})$ contient ainsi toutes les informations sur la quantification de $\mathfrak{g}$. Notre construction généralise la quantification en une variable de $U(\mathfrak{g})$ par Etingof et Kazhdan.*

Introduction.

The notion of a Lie bialgebra was introduced by Drinfeld [**Dri82**], [**Dri87**] in the framework of his algebraic formalism for the quantum inverse scattering method. A Lie bialgebra is a Lie algebra $\mathfrak{g}$ provided with a Lie cobracket $\mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$ which is related to the Lie bracket by a certain compatibility condition. The notion of a Lie bialgebra is self-dual: If $\mathfrak{g}$ is a finite-dimensional Lie bialgebra over a field, then the dual $\mathfrak{g}^*$ is also a Lie bialgebra.

Drinfeld raised the question of quantizing Lie bialgebras (see loc. cit. and [**Dri92**]). For any Lie bialgebra $\mathfrak{g}$, its universal enveloping algebra $U(\mathfrak{g})$ is a co-Poisson bialgebra. The quantization problem for $\mathfrak{g}$ consists in finding a (topological) bialgebra structure on the module of formal power series $U(\mathfrak{g})[[h]]$ which induces the given bialgebra structure and Poisson cobracket on $U(\mathfrak{g}) = U(\mathfrak{g})[[h]]/(h)$. This problem is solved in the theory of quantum groups for certain semisimple $\mathfrak{g}$. Recently, P. Etingof and D. Kazhdan [**EK96**] quantized an arbitrary Lie bialgebra $\mathfrak{g}$ over a field $\mathbf{C}$ of characteristic zero. Their construction is based on a delicate analysis of Drinfeld associators.

Besides $U(\mathfrak{g})$, there are other Poisson and co-Poisson bialgebras associated with a Lie bialgebra $\mathfrak{g}$. One can consider, for instance, the (appropriately defined) Poisson bialgebra $U(\mathfrak{g})^*$ dual to $U(\mathfrak{g})$, as well as similar bialgebras $U(\mathfrak{g}^*), U(\mathfrak{g}^*)^*$ associated with $\mathfrak{g}^*$. Note also that the symmetric algebra $S(\mathfrak{g}) = \bigoplus_{n\geq 0} S^n(\mathfrak{g})$ is a bialgebra with Poisson bracket and cobracket extending the Lie bracket and cobracket in $\mathfrak{g}$. The Etingof-Kazhdan theory provides us with quantizations of $U(\mathfrak{g})$ and $U(\mathfrak{g}^*)$ in the category of topological bialgebras. It is essentially clear that, taking the dual bialgebras, we obtain quantizations of $U(\mathfrak{g})^*$ and $U(\mathfrak{g}^*)^*$. The bialgebras $S(\mathfrak{g})$ and $S(\mathfrak{g}^*)$ stay apart and need to be considered separately. At this point, the relationship between all these bialgebras and their quantizations looks a little messy and needs clarification.

The aim of our paper is to sort out and unify these quantizations. We shall show that there is a bialgebra $A(\mathfrak{g})$ quantizing simultaneously $U(\mathfrak{g})$, $U(\mathfrak{g}^*)^*$, and $S(\mathfrak{g})$. Moreover, the bialgebra $A(\mathfrak{g}^*)$ quantizing $U(\mathfrak{g}^*)$, $U(\mathfrak{g})^*$, $S(\mathfrak{g}^*)$ is essentially dual to $A(\mathfrak{g})$. Thus, we can view $A(\mathfrak{g})$ as a "master" bialgebra containing all information about the quantization of $\mathfrak{g}$.

To formalize our results, we appeal to the notion of biquantization introduced in [**Tur89**], [**Tur91**]. It was inspired by a topological study of skein classes of links in the cylinder over a surface. The idea consists in introducing two independent quantization variables, u and v, responsible for the quantization of multiplication and comultiplication, respectively. Let us illustrate this idea with the following construction. Let A be a bialgebra over the ring of formal power series $\mathbf{C}[[u,v]]$. Assume that A is topologically free as a $\mathbf{C}[[u,v]]$-module, commutative modulo u, and cocommutative modulo v. It is clear that A/uA is a commutative bialgebra with Poisson bracket

$$\{p_u(a), p_u(b)\} = p_u\Big(\frac{ab-ba}{u}\Big),$$

where $a, b \in A$ and $p_u : A \to A/uA$ is the projection. The morphism p_u is a quantization of the Poisson bialgebra A/uA. Similarly, the comultiplication Δ in A induces on A/vA the structure of a cocommutative bialgebra with Poisson cobracket

$$\delta(p_v(a)) = (p_v\otimes p_v)\Big(\frac{\Delta(a)-\Delta^{\mathrm{op}}(a)}{v}\Big),$$

where $a \in A$ and $p_v : A \to A/vA$ is the projection. The morphism $p_v : A \to A/vA$ is a quantization of the co-Poisson bialgebra A/vA. By similar formulas, the quotient $A/(u,v) = A/(uA+vA)$ acquires both a Poisson bracket and a Poisson cobracket, and becomes a bi-Poisson bialgebra.

The projections of A/uA and A/vA onto $A/(u,v)$ quantize the comultiplication and the multiplication in $A/(u,v)$, respectively. We sum up these observations in the following commutative diagram of projections

$$\begin{array}{ccc} A & \longrightarrow & A/uA \\ \Big\downarrow & & \Big\downarrow \\ A/vA & \longrightarrow & A/(u,v) \end{array} \tag{0.1}$$

called a biquantization square. This square involves four bialgebras and four bialgebra morphisms quantizing either the multiplication or the comultiplication in their targets. The bialgebra A appears as the summit of the square, quantizing three other bialgebras. We say that A is a biquantization of the bi-Poisson bialgebra $A/(u,v)$. The notion of a biquantization allows us to combine four quantizations of three bialgebras in a single bialgebra. Note that instead of the ring $\mathbf{C}[[u,v]]$ one can use subrings containing u and v. In this paper, as a ground ring for biquantization, we use the ring $\mathbf{C}[u][[v]]$ consisting of the formal power series in v with coefficients in the ring of polynomials $\mathbf{C}[u]$.

Our main result is that, for any finite-dimensional Lie bialgebra $\mathfrak{g}$ over a field $\mathbf{C}$ of characteristic zero, the bi-Poisson bialgebra $S(\mathfrak{g})$ admits a biquantization. More precisely, we construct a topological $\mathbf{C}[u][[v]]$-bialgebra $A_{u,v}(\mathfrak{g})$ biquantizing $S(\mathfrak{g})$. Specifically, $A_{u,v}(\mathfrak{g})$ is free as a topological $\mathbf{C}[u][[v]]$-module, is commutative modulo u and cocommutative modulo v, and $A_{u,v}(\mathfrak{g})/(u,v) = S(\mathfrak{g})$ as bi-Poisson bialgebras. This gives us a biquantization square (0.1) with $A = A_{u,v}(\mathfrak{g})$.

Our second result computes the left-bottom corner A/vA of the biquantization square (0.1), where $A = A_{u,v}(\mathfrak{g})$. Consider the $\mathbf{C}[u]$-algebra $V_u(\mathfrak{g})$ defined in the same way as the universal enveloping algebra $U(\mathfrak{g})$, except that the identity $xy - yx = [x,y]$ is replaced by $xy - yx = u[x,y]$, where $x, y \in \mathfrak{g}$. We view $V_u(\mathfrak{g})$ as a parametrized version of $U(\mathfrak{g})$; note that $V_u(\mathfrak{g})/(u-1) = U(\mathfrak{g})$. Similarly to $U(\mathfrak{g})$, we provide $V_u(\mathfrak{g})$ with the structure of a co-Poisson bialgebra. We prove that $A_{u,v}(\mathfrak{g})/vA_{u,v}(\mathfrak{g}) = V_u(\mathfrak{g})$ as co-Poisson bialgebras. According to the remarks above, the projection $A_{u,v}(\mathfrak{g}) \to A_{u,v}(\mathfrak{g})/vA_{u,v}(\mathfrak{g}) = V_u(\mathfrak{g})$ is a quantization of $V_u(\mathfrak{g})$. This is a refined version of the Etingof-Kazhdan quantization of $U(\mathfrak{g})$. Indeed, quotienting both $A_{u,v}(\mathfrak{g})$ and $V_u(\mathfrak{g})$ by $u-1$, we obtain the Etingof-Kazhdan quantization of $U(\mathfrak{g})$ (cf. Remark 8.4).

Our third result concerns the right-top corner A/uA of the biquantization square for $A = A_{u,v}(\mathfrak{g})$. Namely, we prove that A/uA is isomorphic to a topological dual of $V_v(\mathfrak{g}^*)$ consisting of $\mathbf{C}[v]$-linear maps $V_v(\mathfrak{g}^*) \to \mathbf{C}[[v]]$ continuous with respect to the v-adic topology in $\mathbf{C}[[v]]$ and a suitable topology in $V_v(\mathfrak{g}^*)$. This dual is a Poisson bialgebra over $\mathbf{C}[[v]]$. It is isomorphic to the Poisson bialgebra $E_v(\mathfrak{g})$ of functions on the Poisson-Lie group associated with $\mathfrak{g}^*\otimes_{\mathbf{C}}\mathbf{C}[[v]]$, cf. [**Tur91**, Sections 11-12]. (As an algebra, $E_v(\mathfrak{g}) = S(\mathfrak{g})[[v]]$.) According to the remarks above, the projection $A_{u,v}(\mathfrak{g}) \to A_{u,v}(\mathfrak{g})/uA_{u,v}(\mathfrak{g}) \cong E_v(\mathfrak{g})$ is a quantization of $E_v(\mathfrak{g})$.

To sum up, the $\mathbf{C}[u][[v]]$-bialgebra $A_{u,v}(\mathfrak{g})$ quantizes $S(\mathfrak{g})$, $V_u(\mathfrak{g})$, and the topological dual $E_v(\mathfrak{g})$ of $V_v(\mathfrak{g}^*)$.

We can apply the same constructions to the dual Lie bialgebra $\mathfrak{g}^*$. It is convenient to exchange u and v, i.e., to consider the $\mathbf{C}[v][[u]]$-bialgebra $A_{v,u}(\mathfrak{g}^*)$ obtained from $A_{u,v}(\mathfrak{g}^*)$ via an appropriate tensoring with $\mathbf{C}[v][[u]]$. As above, $A_{v,u}(\mathfrak{g}^*)$ quantizes $S(\mathfrak{g}^*)$, $V_v(\mathfrak{g}^*)$, and the topological dual $E_u(\mathfrak{g}^*)$ of $V_u(\mathfrak{g})$. Observe that the three lower level corners of the biquantization square for $A_{v,u}(\mathfrak{g}^*)$ are dual to the lower level corners of the biquantization square for $A_{u,v}(\mathfrak{g})$. We prove that the bialgebras $A_{u,v}(\mathfrak{g})$ and $A_{v,u}(\mathfrak{g}^*)$ are essentially dual to each other.

Our definition of $A_{u,v}(\mathfrak{g})$ is obtained by an elaboration of Etingof and Kazhdan's quantization of $U(\mathfrak{g})$ and can be regarded as an extension of their work. The definition goes in two steps. First we replace the variable h by the product uv, which allows us to introduce two variables into the game. In particular, the universal R-matrix R_h constructed in [**EK96**] gives rise to a two-variable universal R-matrix $R_{u,v}$. Then we separate the variables u, v in an expression for $R_{u,v}$ by collecting all powers of u (resp. v) in the first (resp. second) tensor factor. The algebra $A_{u,v}(\mathfrak{g})$ is generated by the first tensor factors appearing in such an expression.

The plan of the paper is as follows. In Section 1 we recall the notions of Poisson, co-Poisson, and bi-Poisson bialgebras, as well as the definitions of quantizations and biquantizations. In Section 2 we formulate the main results of the paper (Theorems 2.3, 2.6, 2.9, and 2.11). In Section 3 we recall a construction due to Drinfeld producing certain linear maps out of a bialgebra comultiplication. We use these maps to show that every bialgebra over $\mathbf{C}[[u]]$ has a canonical subalgebra that is commutative modulo u. In Section 4 we collect several useful facts concerning $\mathbf{C}[[u,v]]$-modules. In Section 5 we recall the basic facts concerning Etingof and Kazhdan's quantization $U_h(\mathfrak{g})$ of a Lie bialgebra $\mathfrak{g}$. In Section 6 we define $A_{u,v}(\mathfrak{g})$ and show that it is a topologically free module. The proof that $A_{u,v}(\mathfrak{g})$ is an algebra is also given in Section 6; it uses Lemma 6.10 whose proof is postponed to Section 7. In Section 7 we introduce a completion $\widehat{A}_{u,v}(\mathfrak{g})$ of $A_{u,v}(\mathfrak{g})$ and define a bialgebra structure on $A_{u,v}(\mathfrak{g})$. Section 8 is devoted to the proofs of Theorems 2.3 and 2.6, and the first part of Theorem 2.9. In Section 9 we investigate the two-variable universal R-matrix $R_{u,v}$ and construct a nondegenerate bialgebra pairing between $A_{u,v}(\mathfrak{g})$ and a certain bialgebra A_-^{cop}. In Section 10, using the pairing of Section 9, we relate $S(\mathfrak{g})[[v]]$ to the topological dual of $V_v(\mathfrak{g}^*)$, which allows us to complete the proof of Theorem 2.9. In Section 11 we compare Etingof and Kazhdan's quantization for a Lie bialgebra and the dual Lie bialgebra. In Section 12 we use the results of

Section 11 to show that $A_-^{\rm cop} \cong A_{v,u}(\mathfrak{g}^*)$ and prove Theorem 2.11. In the appendix we describe explicitly the biquantization of a trivial Lie bialgebra.

We fix once and for all a field $\mathbf{C}$ of characteristic zero.

1. Poisson bialgebras and their quantizations.

We introduce the basic notions used throughout the paper. All objects will be considered over a field $\mathbf{C}$ of characteristic zero. Given a commutative $\mathbf{C}$-algebra κ, we recall that a κ-bialgebra is an associative, unital κ-algebra A equipped with morphisms of algebras $\Delta : A \to A \otimes_\kappa A$, the comultiplication, and $\varepsilon : A \to \kappa$, the counit, such that

$$(\Delta \otimes \mathrm{id}_A)\Delta = (\mathrm{id}_A \otimes \Delta)\Delta \quad \text{and} \quad (\varepsilon \otimes \mathrm{id}_A)\Delta = (\mathrm{id}_A \otimes \varepsilon)\Delta = \mathrm{id}_A,$$

where id_A denotes the identity map of A. We shall also consider topological bialgebras. A topological bialgebra A is defined in terms of a two-sided ideal $I \subset A$. The definition is the same as for a κ-bialgebra, except that the comultiplication takes values in the completed tensor product

$$A \,\widehat{\otimes}_\kappa\, A = \varprojlim_n \Big(A/I^n \otimes_\kappa A/I^n \Big).$$

The topological bialgebra A is equipped with the I-adic topology, namely the linear topology for which the powers of I form a fundamental system of neighbourhoods of 0 (see [**Bou61**, Chap. 3]).

1.1. Poisson Bialgebras.

A Poisson bracket on a commutative algebra B over the field $\mathbf{C}$ is a Lie bracket $\{\ ,\ \} : B \times B \to B$ satisfying the Leibniz rule, i.e., such that for all $a, b, c \in B$ we have

$$\{ab, c\} = a\{b, c\} + b\{a, c\}. \tag{1.1}$$

A Poisson bracket on B defines a Poisson bracket on $B \otimes B$ by

$$\{a \otimes a', b \otimes b'\} = ab \otimes \{a', b'\} + \{a, b\} \otimes a'b' \tag{1.2}$$

where a, a', b, $b' \in B$.

A *Poisson bialgebra* is a commutative $\mathbf{C}$-bialgebra B equipped with a Poisson bracket such that the comultiplication $\Delta : B \to B \otimes B$ preserves the Poisson bracket:

$$\Delta(\{a, b\}) = \{\Delta(a), \Delta(b)\} \tag{1.3}$$

for all $a, b \in B$.

The following well-known construction yields examples of Poisson bialgebras. Let A be a bialgebra over the ring $\mathbf{C}[u]$ of polynomials in a variable u. Assume that A is commutative modulo u in the sense that $ab - ba \in uA$ for all $a, b \in A$. If the multiplication by u is injective on A, then the quotient bialgebra A/uA is a Poisson bialgebra with Poisson bracket defined for all a, $b \in A$ by

$$\{p(a), p(b)\} = p\Big(\frac{ab - ba}{u}\Big), \tag{1.4}$$

where $p : A \to A/uA$ is the projection.

The inverse of this construction is called quantization. More precisely, a *quantization* of a Poisson $\mathbf{C}$-bialgebra B is a $\mathbf{C}[u]$-bialgebra A which is isomorphic as a $\mathbf{C}[u]$-module to the module $B[u]$ of polynomials in u with coefficients in B, is commutative modulo u, and such that A/uA is isomorphic to B as a Poisson bialgebra. The latter condition implies that Equality (1.4) holds for all $a, b \in A$, where $p : A \to A/uA \cong B$ is the projection and $\{\ ,\ \}$ is the Poisson bracket in B.

One can similarly define quantization over the ring $\mathbf{C}[[u]]$ of formal power series. To shorten, we call $\mathbf{C}[[u]]$-bialgebra a topological $\mathbf{C}[[u]]$-algebra A where the topology is the u-adic topology, i.e., is defined by the ideal uA. In this case,

$$A \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u]]} A = \varprojlim_n \Big(A/u^n A \otimes_{\mathbf{C}[[u]]/(u^n)} A/u^n A\Big). \tag{1.5}$$

A *quantization* over $\mathbf{C}[[u]]$ of a Poisson $\mathbf{C}$-bialgebra B is a (topological) $\mathbf{C}[[u]]$-bialgebra A which is isomorphic as a $\mathbf{C}[[u]]$-module to the module $B[[u]]$ of formal power series with coefficients in B, is commutative modulo u, and such that $A/uA = B$ as Poisson bialgebras.

1.2. Co-Poisson Bialgebras. It is straightforward to dualize the definitions of Section 1.1. A Poisson cobracket on a cocommutative $\mathbf{C}$-coalgebra B is a Lie cobracket $\delta : B \to B \otimes B$ satisfying the Leibniz rule, i.e., such that

$$(\mathrm{id} \otimes \Delta)\delta = \big(\delta \otimes \mathrm{id} + (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \delta)\big)\Delta, \tag{1.6}$$

where $\Delta : B \to B \otimes B$ is the comultiplication of B and σ is the permutation $a \otimes b \mapsto b \otimes a$ in $B \otimes B$. Recall the notation $\Delta^{\mathrm{op}} = \sigma\Delta$ for the opposite comultiplication.

A *co-Poisson bialgebra* is a cocommutative $\mathbf{C}$-bialgebra B equipped with a Poisson cobracket δ such that

$$\delta(ab) = \delta(a)\Delta(b) + \Delta(a)\delta(b) \tag{1.7}$$

for all $a, b \in B$.

We obtain co-Poisson bialgebras by dualizing the constructions of Section 1.1. Here again we have the choice between the ring $\mathbf{C}[v]$ of polynomials and the ring $\mathbf{C}[[v]]$ of formal power series in a variable v. In the context of co-Poisson bialgebras, it will be more relevant to work with formal power series. So let A be a bialgebra over $\mathbf{C}[[v]]$ in the sense of Section 1.1. Assume that A is cocommutative modulo v, i.e., for all $a \in A$ we have

$\Delta(a) - \Delta^{\mathrm{op}}(a) \in vA\widehat{\otimes}_{\mathbf{C}[[v]]}A$, where Δ denotes the comultiplication and Δ^{op} the opposite comultiplication of A. If v acts injectively on $A\,\widehat{\otimes}_{\mathbf{C}[[v]]}\,A$, then the quotient bialgebra A/vA is a co-Poisson bialgebra with cobracket

$$\delta(p(a)) = (p\otimes p)\Big(\frac{\Delta(a) - \Delta^{\mathrm{op}}(a)}{v}\Big) \tag{1.8}$$

for $a \in A$, where $p : A \to A/vA$ is the projection.

A *coquantization* of a co-Poisson $\mathbf{C}$-bialgebra B is a $\mathbf{C}[[v]]$-bialgebra A which is isomorphic to $B[[v]]$ as a $\mathbf{C}[[v]]$-module, is cocommutative modulo v, and such that A/vA is isomorphic to B as a co-Poisson bialgebra. This implies that Formula (1.8) holds for any $a \in A$, where $p : A \to A/vA \cong B$ is the projection and δ is the Poisson cobracket in B.

1.3. Bi-Poisson Bialgebras. Following [**Tur89**, **Tur91**], we combine the definitions given above and define the concepts of bi-Poisson bialgebras and their biquantizations. A *bi-Poisson bialgebra* is a commutative and cocommutative bialgebra B equipped with Poisson bracket $\{\ ,\ \}$ and Poisson cobracket δ turning B into a Poisson and co-Poisson bialgebra, and satisfying the additional condition:

$$\delta(\{a, b\}) = \{\delta(a), \Delta(b)\} + \{\Delta(a), \delta(b)\} \tag{1.9}$$

for all $a, b \in B$.

In order to introduce biquantization, we use two variables u and v and the ring $\mathbf{C}[u][[v]]$ which consists of formal power series in v whose coefficients are polynomials in u. The following definitions can easily be adapted to the rings $\mathbf{C}[u, v]$, $\mathbf{C}[[u, v]]$, and $\mathbf{C}[v][[u]]$.

By a $\mathbf{C}[u][[v]]$-bialgebra A we mean a topological $\mathbf{C}[u][[v]]$-algebra A, where the topology is defined by the ideal vA, so that the comultiplication takes values in

$$A\,\widehat{\otimes}_{\mathbf{C}[u][[v]]}\,A = \varprojlim_n \Big(A/v^nA \otimes_{\mathbf{C}[u][[v]]/(v^n)} A/v^nA\Big). \tag{1.10}$$

Let A be a $\mathbf{C}[u][[v]]$-bialgebra that is commutative modulo u and cocommutative modulo v. If u and v act injectively on A, then the quotient bialgebra $A/(uA + vA)$ is a bi-Poisson bialgebra over $\mathbf{C}$ with Poisson bracket given by (1.4) and Poisson cobracket given by (1.8), where $p : A \to A/(uA + vA)$ is the projection. Inverting this construction, we obtain the following notion of biquantization.

Definition 1.4. A biquantization of a bi-Poisson $\mathbf{C}$-bialgebra B is a $\mathbf{C}[u][[v]]$-bialgebra A which is isomorphic to $B[u][[v]]$ as a $\mathbf{C}[u][[v]]$-module, is commutative modulo u and cocommutative modulo v, and such that $A/(uA + vA) = B$ as bi-Poisson bialgebras.

Any biquantization A gives rise to a "biquantization square" as follows. Observe that A/vA is a cocommutative co-Poisson bialgebra over $\mathbf{C}[u]$ and that A/uA is a commutative Poisson bialgebra over $\mathbf{C}[[v]]$. We form the commutative square

$$\begin{array}{ccc} A & \xrightarrow{p_u} & A/uA \\ {\scriptstyle p_v}\downarrow & & \downarrow{\scriptstyle q_v} \\ A/vA & \xrightarrow{q_u} & B \end{array} \tag{1.11}$$

where p_u, p_v, q_u, q_v are the natural projections. The morphisms p_u and q_u are quantizations whereas p_v and q_v are coquantizations. The projection $p : A \to B$ can therefore be factored in two ways as a composition of a quantization and a coquantization: $p = q_v p_u = q_u p_v$.

2. Statement of the main results.

Any Lie bialgebra $\mathfrak{g}$ gives rise to a bi-Poisson bialgebra $S(\mathfrak{g})$. In this section, after recalling the necessary facts on Lie bialgebras, we state our main theorems concerning a biquantization of $S(\mathfrak{g})$.

2.1. Lie Bialgebras (cf. [**Dri82**])**.** A *Lie cobracket* on a vector space $\mathfrak{g}$ over $\mathbf{C}$ is a linear map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ such that

$$\sigma\delta = -\delta \quad \text{and} \quad (\mathrm{id} + \tau + \tau^2)(\delta \otimes \mathrm{id}) = 0 \tag{2.1}$$

where σ (resp. τ) is the automorphism of $\mathfrak{g} \otimes \mathfrak{g}$ (resp. of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$) given by $\sigma(x \otimes y) = y \otimes x$ (resp. $\tau(x \otimes y \otimes z) = y \otimes z \otimes x$). It is clear that the transpose map $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \subset (\mathfrak{g} \otimes \mathfrak{g})^* \to \mathfrak{g}^*$ is a Lie bracket in the dual space $\mathfrak{g}^* = \mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, \mathbf{C})$.

A *Lie bialgebra* is a vector space over $\mathbf{C}$ equipped with a Lie bracket $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and a Lie cobracket $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ such that

$$\delta([x, y]) = x\delta(y) - y\delta(x) \tag{2.2}$$

for all $x, y \in \mathfrak{g}$. Here $\mathfrak{g}$ acts on $\mathfrak{g} \otimes \mathfrak{g}$ by the adjoint action ($x, z, z' \in \mathfrak{g}$):

$$x(z \otimes z') = [x, z] \otimes z' + z \otimes [x, z'].$$

Let $\mathfrak{g}$ be a Lie bialgebra with Lie bracket $[\ ,\ ]$ and Lie cobracket δ. It is easy to check that, if we replace $[\ ,\ ]$ by $-[\ ,\ ]$ without changing the Lie cobracket, then we obtain a new Lie bialgebra, which we denote $\mathfrak{g}^{\mathrm{op}}$. If we leave the Lie bracket in $\mathfrak{g}$ unaltered and replace δ by $-\delta$, then we obtain another Lie bialgebra denoted $\mathfrak{g}^{\mathrm{cop}}$. The opposite $-\mathrm{id}_{\mathfrak{g}}$ of the identity map of $\mathfrak{g}$ is an isomorphism of Lie bialgebras $\mathfrak{g}^{\mathrm{op}} \to \mathfrak{g}^{\mathrm{cop}}$ and $\mathfrak{g} \to (\mathfrak{g}^{\mathrm{op}})^{\mathrm{cop}}$.

When the Lie bialgebra $\mathfrak{g}$ is finite-dimensional, then the dual vector space $\mathfrak{g}^*$ with the transpose bracket and cobracket is also a Lie bialgebra. Clearly, $(\mathfrak{g}^*)^{\mathrm{op}} = (\mathfrak{g}^{\mathrm{cop}})^*$ and $(\mathfrak{g}^*)^{\mathrm{cop}} = (\mathfrak{g}^{\mathrm{op}})^*$.

2.2. A Bi-Poisson Bialgebra Associated to $\mathfrak{g}$ (cf. [**Tur89**, **Tur91**])**.** For any vector space $\mathfrak{g}$, the symmetric algebra $S(\mathfrak{g}) = \bigoplus_{n\geq 0} S^n(\mathfrak{g})$ has a structure of bialgebra with comultiplication Δ determined by $\Delta(x) = x\otimes 1 + 1\otimes x$ for all $x \in \mathfrak{g} = S^1(\mathfrak{g})$. If $\mathfrak{g}$ is a Lie algebra with Lie bracket $[\ ,\ ]$, then $S(\mathfrak{g})$ is a Poisson bialgebra with Poisson bracket determined by

$$\{x, y\} = [x, y] \tag{2.3}$$

for all $x, y \in \mathfrak{g}$. If $\mathfrak{g}$ is a Lie coalgebra, then $S(\mathfrak{g})$ is a co-Poisson bialgebra with the unique Poisson cobracket such that its restriction to $S^1(\mathfrak{g}) = \mathfrak{g}$ is the Lie cobracket of $\mathfrak{g}$. If, furthermore, $\mathfrak{g}$ is a Lie bialgebra, then $S(\mathfrak{g})$ is a bi-Poisson bialgebra ([**Tur91**, Theorem 16.2.4]).

We now state our first main theorem.

Theorem 2.3. *Given a finite-dimensional Lie bialgebra $\mathfrak{g}$, there exists a biquantization $A_{u,v}(\mathfrak{g})$ for $S(\mathfrak{g})$.*

The construction of $A_{u,v}(\mathfrak{g})$ will be given in Section 6. It is an extension of Etingof and Kazhdan's quantization of $U(\mathfrak{g})$, as constructed in [**EK96**]. As in loc. cit., our definition of $A_{u,v}(\mathfrak{g})$ is based on the choice of a Drinfeld associator. We nevertheless believe that it is unique up to isomorphism. We shall not discuss this point in this paper.

The fundamental feature of our construction is that the bialgebras in the lower left and the upper right corners in the biquantization square (1.11) when $A = A_{u,v}(\mathfrak{g})$ are closely related to the universal enveloping bialgebra $U(\mathfrak{g})$ of $\mathfrak{g}$ and to the dual of $U(\mathfrak{g}^*)$. We shall give precise statements in the remaining part of this section. We begin with a short discussion of $U(\mathfrak{g})$ and its parametrized version $V_u(\mathfrak{g})$.

2.4. The Bialgebra $V_u(\mathfrak{g})$. Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{C}$. Consider the $\mathbf{C}[u]$-algebra $T(\mathfrak{g})[u]$ of polynomials with coefficients in the tensor algebra $T(\mathfrak{g}) = \bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}$. Let $V_u(\mathfrak{g})$ be the quotient of $T(\mathfrak{g})[u]$ by the two-sided ideal generated by the elements

$$x \otimes y - y \otimes x - u[x, y],$$

where $x, y \in \mathfrak{g}$. The composition of the natural linear maps $\mathfrak{g} = T^1(\mathfrak{g}) \subset T(\mathfrak{g}) \subset T(\mathfrak{g})[u] \to V_u(\mathfrak{g})$ is an embedding whose image generates $V_u(\mathfrak{g})$ as a $\mathbf{C}[u]$-algebra. The algebra $V_u(\mathfrak{g})$ is a bialgebra with comultiplication Δ determined by

$$\Delta(x) = x \otimes 1 + 1 \otimes x \tag{2.4}$$

for all $x \in \mathfrak{g}$. Clearly, $V_u(\mathfrak{g})/(u-1)V_u(\mathfrak{g}) = U(\mathfrak{g})$ and $V_u(\mathfrak{g})/uV_u(\mathfrak{g}) = S(\mathfrak{g})$.

In this paper, we will use the fact that $V_u(\mathfrak{g})$ embeds in the polynomial algebra $U(\mathfrak{g})[u]$. The algebra $U(\mathfrak{g})[u]$ is equipped with a $\mathbf{C}[u]$-bialgebra structure whose comultiplication Δ is also given by (2.4). Let $i : V_u(\mathfrak{g}) \to U(\mathfrak{g})[u]$ be the morphism of $\mathbf{C}[u]$-bialgebras defined by $i(x) = ux$ for all

$x \in \mathfrak{g} \subset V_u(\mathfrak{g})$. Using the Poincaré-Birkhoff-Witt theorem (cf. [**Dix74**, Chap. 2]), we see that $V_u(\mathfrak{g})$ is a free $\mathbf{C}[u]$-module and that i is injective. To describe its image, recall the standard filtration $U^0(\mathfrak{g}) = \mathbf{C} \subset U^1(\mathfrak{g}) \subset U^2(\mathfrak{g}) \subset \cdots$ of $U(\mathfrak{g})$: The subspace $U^m(\mathfrak{g})$ is the image of $\bigoplus_{k=0}^m \mathfrak{g}^{\otimes k}$ under the projection $T(\mathfrak{g}) \to U(\mathfrak{g})$. Then

$$i(V_u(\mathfrak{g})) = \left\{ \sum_{m\geq 0} a_m u^m \in U(\mathfrak{g})[u] \;\middle|\; a_m \in U^m(\mathfrak{g}) \text{ for all } m \geq 0 \right\}.$$

We also have $U^m(\mathfrak{g})/U^{m-1}(\mathfrak{g}) = S^m(\mathfrak{g})$ for all $m \geq 0$. From now on, we identify $V_u(\mathfrak{g})$ with $i(V_u(\mathfrak{g}))$ and $S(\mathfrak{g})$ with the graded algebra

$$\bigoplus_{m\geq 0} U^m(\mathfrak{g}) \,/\, U^{m-1}(\mathfrak{g}).$$

Under these identifications, the natural projection $q_u : V_u(\mathfrak{g}) \to S(\mathfrak{g})$ sends any element $\sum_{m\geq 0} a_m u^m \in V_u(\mathfrak{g})$ to $\sum_{m\geq 0} \bar{a}_m \in S(\mathfrak{g})$, where $\bar{a}_m \in S^m(\mathfrak{g})$ is the class of $a_m \in U^m(\mathfrak{g})$ modulo $U^{m-1}(\mathfrak{g})$. These observations lead to the following easy fact.

Lemma 2.5. *The $\mathbf{C}[u]$-bialgebra $V_u(\mathfrak{g})$ is a quantization of the Poisson bialgebra $S(\mathfrak{g})$.*

Suppose now that $\mathfrak{g}$ is a Lie bialgebra with Lie cobracket δ. It was shown in [**Tur91**, Theorem 7.4] that δ induces a co-Poisson bialgebra structure on $V_u(\mathfrak{g})$ with Poisson cobracket δ_u determined for all $x \in \mathfrak{g}$ by

$$\delta_u(ux) = u^2\delta(x) \in u\mathfrak{g}\otimes u\mathfrak{g} \subset V_u(\mathfrak{g}) \otimes_{\mathbf{C}[u]} V_u(\mathfrak{g}). \tag{2.5}$$

The projection $q_u : V_u(\mathfrak{g}) \to S(\mathfrak{g})$ preserves the co-Poisson structure; in other words, $V_u(\mathfrak{g})$ is a quantization of $S(\mathfrak{g})$ in the category of co-Poisson bialgebras.

Theorem 2.6. *For the bialgebra $A_{u,v}(\mathfrak{g})$ of Theorem 2.3, there is an isomorphism of co-Poisson $\mathbf{C}[u]$-bialgebras*

$$A_{u,v}(\mathfrak{g})/vA_{u,v}(\mathfrak{g}) = V_u(\mathfrak{g}).$$

Theorem 2.6 will be proved in Section 8.

2.7. The Bialgebra $E_v(\mathfrak{g})$. Let $\mathfrak{g}$ be a finite-dimensional Lie coalgebra with Lie cobracket δ. By Section 2.2 the cobracket δ induces a co-Poisson bialgebra structure on $S(\mathfrak{g})$.

Turaev ([**Tur89**, Sections 4-5] and [**Tur91**, Sections 11-12]) constructed a (topological) $\mathbf{C}[[v]]$-bialgebra $E_v(\mathfrak{g})$ which may be viewed as the bialgebra of functions on the simply-connected Lie group associated to the dual Lie algebra $\mathfrak{g}^*$. As an algebra, $E_v(\mathfrak{g})$ is the algebra of formal power series with coefficients in $S(\mathfrak{g})$:

$$E_v(\mathfrak{g}) = S(\mathfrak{g})[[v]].$$

To define the comultiplication in $E_v(\mathfrak{g})$, consider the Campbell-Hausdorff series

$$\begin{aligned}(2.6)\quad \mu(X,Y) &= \log(e^X e^Y)\\ &= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\big([X,[X,Y]] + [[X,Y],Y]\big) + \cdots\end{aligned}$$

where $X, Y \in \mathfrak{g}^*$. Let us multiply all Lie brackets of length n by v^n. This yields the modified Campbell-Hausdorff series

$$\begin{aligned}(2.7)\quad \mu_v(X,Y) &= \frac{1}{v}\log(e^{vX} e^{vY})\\ &= X + Y + \frac{v}{2}[X,Y] + \frac{v^2}{12}\big([X,[X,Y]] + [[X,Y],Y]\big) + \cdots .\end{aligned}$$

The comultiplication Δ' in $E_v(\mathfrak{g})$ is given by $a \mapsto a \circ \mu_v$, which makes sense when we identify elements of $E_v(\mathfrak{g})$ with $\mathbf{C}[[v]]$-valued polynomial functions on $\mathfrak{g}^*$. For $x \in \mathfrak{g} \subset E_v(\mathfrak{g})$ we have

$$(2.8)\quad \Delta'(x) = x \otimes 1 + 1 \otimes x + \frac{v}{2}\delta(x) + \frac{v^2}{12}\sum_i (x_i' x_i'' \otimes x_i''' + x_i''' \otimes x_i' x_i'') + \cdots ,$$

where $(\mathrm{id} \otimes \delta)\delta(x) = \sum_i x_i' \otimes x_i'' \otimes x_i'''$. For details, see loc. cit.

Let $q_v : E_v(\mathfrak{g}) \to S(\mathfrak{g})$ be the algebra morphism sending an element of $E_v(\mathfrak{g})$ to its class modulo $vE_v(\mathfrak{g})$. Formula (2.8) implies that the induced map $E_v(\mathfrak{g})/vE_v(\mathfrak{g}) \to S(\mathfrak{g})$ is an isomorphism of co-Poisson bialgebras. This leads to the following.

Lemma 2.8. *The $\mathbf{C}[[v]]$-bialgebra $E_v(\mathfrak{g})$ is a coquantization of the co-Poisson bialgebra $S(\mathfrak{g})$.*

If the Lie coalgebra $\mathfrak{g}$ has a Lie bracket $[\ ,\ ]$ turning it into a Lie bialgebra, then $E_v(\mathfrak{g})$ carries a structure of a Poisson bialgebra whose Poisson bracket $\{\ ,\ \}$ is uniquely determined by the condition

$$(2.9)\qquad \{x_1, x_2\} \equiv [x_1, x_2] \mod \Big(\bigoplus_{n\geq 2} S^n(\mathfrak{g})\Big)[[v]],$$

for all $x_1, x_2 \in \mathfrak{g}$ (cf. [**Tur91**, Theorem 11.4 and Remark 11.7]).

Theorem 2.9. *For the bialgebra $A_{u,v}(\mathfrak{g})$ of Theorem 2.3, there is an isomorphism of Poisson $\mathbf{C}[[v]]$-bialgebras*

$$A_{u,v}(\mathfrak{g})/uA_{u,v}(\mathfrak{g}) = E_v(\mathfrak{g}).$$

Theorem 2.9 will be proved in two steps: In Section 8.2 we prove that $A_{u,v}(\mathfrak{g})/uA_{u,v}(\mathfrak{g}) = S(\mathfrak{g})[[v]]$ as an algebra; in Section 10.7 we determine its coalgebra structure.

2.10. Duality. By Theorem 2.3 we have a biquantization square

$$\begin{array}{ccc} A_{u,v}(\mathfrak{g}) & \xrightarrow{p_u} & A_{u,v}(\mathfrak{g})/uA_{u,v}(\mathfrak{g}) \\ {\scriptstyle p_v}\downarrow & & \downarrow{\scriptstyle q_v} \\ A_{u,v}(\mathfrak{g})/vA_{u,v}(\mathfrak{g}) & \xrightarrow{q_u} & S(\mathfrak{g}). \end{array} \tag{2.10a}$$

Replacing $\mathfrak{g}$ by the Lie bialgebra $\mathfrak{g}' = (\mathfrak{g}^*)^{\mathrm{cop}}$ (see Section 2.1 for the notation) and exchanging u and v, we obtain the biquantization square

$$\begin{array}{ccc} A_{v,u}(\mathfrak{g}') & \xrightarrow{p_v} & A_{v,u}(\mathfrak{g}')/vA_{v,u}(\mathfrak{g}') \\ {\scriptstyle p_u}\downarrow & & \downarrow{\scriptstyle q_u} \\ A_{v,u}(\mathfrak{g}')/uA_{v,u}(\mathfrak{g}') & \xrightarrow{q_v} & S(\mathfrak{g}'). \end{array} \tag{2.10b}$$

We prove that these squares are in duality as follows.

Let K be a commutative $\mathbf{C}$-algebra together with two subalgebras K_1 and K_2. Given a K_1-module A and a K_2-module B, a $\mathbf{C}$-bilinear map $(\ ,\ ) : A \times B \to K$ will be called a *pairing* if

$$(\lambda_1 a, \lambda_2 b) = \lambda_1 \lambda_2 \, (a, b)$$

for all $\lambda_1 \in K_1 \subset K$, $\lambda_2 \in K_2 \subset K$, $a \in A$, and $b \in B$. We say that the pairing $(\ ,\ )$ is *nondegenerate* if both annihilators

$$\big\{a \in A \mid (a,b) = 0 \ \text{ for all } b \in B\big\} \quad \text{and} \quad \big\{b \in B \mid (a,b) = 0 \ \text{ for all } a \in A\big\}$$

vanish. The pairing $A \times B \to K$ induces a pairing $(\ ,\ ) : (A \otimes_{K_1} A) \times (B \otimes_{K_2} B) \to K$ by

$$(a \otimes a', b \otimes b') = (a, b)\,(a', b')$$

for all $a, a' \in A$ and $b, b' \in B$. Suppose, in addition, that A and B are bialgebras over K_1 and K_2, respectively. The pairing $(\ ,\ ) : A \times B \to K$ is a *bialgebra pairing* if

$$\begin{aligned} (a, bb') &= (\Delta(a), b \otimes b'), \\ (aa', b) &= (a \otimes a', \Delta(b)), \\ (a, 1) &= \varepsilon(a), \\ (1, b) &= \varepsilon(b) \end{aligned} \tag{2.11}$$

for all $a, a' \in A$ and $b, b' \in B$, where Δ denotes the comultiplication and ε the counit.

Theorem 2.11. *Let $\mathfrak{g}$ be a finite-dimensional Lie bialgebra and $\mathfrak{g}' = (\mathfrak{g}^*)^{\mathrm{cop}}$. Then there is a nondegenerate bialgebra pairing*

$$A_{u,v}(\mathfrak{g}) \times A_{v,u}(\mathfrak{g}') \to \mathbf{C}[[u, v]],$$

which induces the standard bialgebra pairing

$$S(\mathfrak{g}) \times S(\mathfrak{g}') = A_{u,v}(\mathfrak{g})/(u,v) \times A_{v,u}(\mathfrak{g}')/(u,v) \to \mathbf{C},$$

uniquely determined by $(x, y) = \langle x, y\rangle$ *for all* $x \in \mathfrak{g}$ *and* $y \in \mathfrak{g}' = \mathfrak{g}^*$*, where* $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g}^* \to \mathbf{C}$ *is the evaluation pairing.*

Theorem 2.11 will be proved in Section 12. Note that, quotienting by u (resp. v), we obtain nondegenerate bialgebra pairings

$$E_v(\mathfrak{g}) \times V_v(\mathfrak{g}') \to \mathbf{C}[[v]] \quad \text{and} \quad V_u(\mathfrak{g}) \times E_u(\mathfrak{g}') \to \mathbf{C}[[u]].$$

3. The maps δ^n.

Let A be a $\mathbf{C}[[u]]$-bialgebra in the sense of Section 1.1. In [**Dri87**, Section 7] Drinfeld used a general procedure to construct a $\mathbf{C}[[u]]$-subalgebra A' of A. In Drinfeld's terms, if A is a quantized universal enveloping algebra, then A' is a quantized formal series Hopf algebra. The subalgebra A' is defined using a family of linear maps $(\delta^n : A \to A^{\widehat{\otimes} n})_{n\geq 0}$, whose definition will be recalled below.

In this section, we prove that A' is commutative modulo u. To this end, we establish some properties of the maps δ^n.

3.1. Definition of δ^n**.** Starting from a bialgebra A over a commutative ring κ with comultiplication Δ and counit ε, we define for each $n \geq 0$ a morphism of algebras $\Delta^n : A \to A^{\otimes n}$ as follows: $\Delta^0 = \varepsilon : A \to \kappa$, $\Delta^1 = \mathrm{id}_A : A \to A$, the map $\Delta^2 : A \to A^{\otimes 2}$ is the comultiplication Δ and, for $n \geq 3$,

$$\Delta^n = (\Delta \otimes \mathrm{id}_A^{\otimes (n-2)})\Delta^{n-1}.$$

Let us embed $A^{\otimes n}$ into $A^{\otimes (n+1)}$ by tensoring on the right by the unit $1 \in A$. We thus get a direct system of algebras

$$A \to A^{\otimes 2} \to A^{\otimes 3} \to \cdots$$

whose limit we denote by $A^{\otimes \infty}$. In this way, each $A^{\otimes n}$ is naturally embedded in $A^{\otimes \infty}$.

Let I be a finite subset of the set of positive integers $\mathbf{N}' = \{1, 2, 3, \dots\}$. If $n = |I|$ is the cardinality of I, we define an algebra morphism $j_I : A^{\otimes n} \to A^{\otimes \infty}$ as follows. If $I = \{i_1, \dots, i_n\}$ with $i_1 < \dots < i_n$, then $j_I(a_1 \otimes \cdots \otimes a_n) = b_1 \otimes b_2 \otimes \cdots$, where $b_i = 1$ if $i \notin I$ and $b_{i_p} = a_p$ for $p = 1, \dots, n$. If $I = \emptyset$, then $j_I : \kappa \to A^{\otimes \infty}$ is the κ-linear map sending the unit of κ to the unit of $A^{\otimes \infty}$.

Suppose we have a κ-linear map $f : A \to A^{\otimes n}$ for some $n \geq 0$. For any set $I \subset \mathbf{N}'$ of cardinality n, we define a linear map $f_I : A \to A^{\otimes \infty}$ by $f_I = j_I \circ f$. If $I = \{1, \dots, n\}$, then f_I is equal to f composed with the standard embedding of $A^{\otimes n}$ in $A^{\otimes \infty}$. This shows that knowing the linear map $f : A \to A^{\otimes n}$ is equivalent to knowing the family of maps $f_I : A \to A^{\otimes \infty}$ indexed by the subsets I of $\mathbf{N}'$ of cardinality n. In particular, from each $\Delta^n : A \to A^{\otimes n}$ we obtain the family of linear maps (Δ_I) indexed by the sets $I \subset \mathbf{N}'$ of cardinality n and defined by $\Delta_I = (\Delta^n)_I : A \to A^{\otimes \infty}$.

After these preliminaries, we define the maps $\delta^n : A \to A^{\otimes n}$ for $n \geq 0$ by the following relation in terms of finite sets $I \subset \mathbf{N}'$:

$$\delta_I = \sum_{J \subset I} (-1)^{|I|-|J|} \, \Delta_J. \tag{3.1}$$

By the inclusion-exclusion principle, we have the equivalent relation

$$\Delta_I = \sum_{J \subset I} \delta_J. \tag{3.2}$$

It follows immediately from (3.1) that

$$\delta_I(1) = \begin{cases} 1 & \text{if } I = \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Lemma 3.2. *Let $a, b \in A$ and K be a finite subset of $\mathbf{N}'$. Then*

$$\delta_K(ab) = \sum_{\substack{I,J \subset K \\ I \cup J = K}} \delta_I(a)\delta_J(b). \tag{3.4}$$

Moreover, if $K \neq \emptyset$, then

$$\delta_K(ab - ba) = \sum_{\substack{I,J \subset K \\ I \cup J = K, I \cap J \neq \emptyset}} \big(\delta_I(a)\delta_J(b) - \delta_J(b)\delta_I(a)\big). \tag{3.5}$$

Proof. In order to prove (3.4), we first observe that by (3.2),

$$\sum_{K' \subset K} \delta_{K'}(ab) = \Delta_K(ab) = \Delta_K(a)\Delta_K(b) = \sum_{I,J \subset K} \delta_I(a)\delta_J(b). \tag{3.6}$$

We rewrite (3.6) as follows:

$$\sum_{K' \subset K} \delta_{K'}(ab) = \sum_{K' \subset K} \left(\sum_{\substack{I,J \subset K' \\ I \cup J = K'}} \delta_I(a)\delta_J(b) \right). \tag{3.7}$$

Let us prove (3.4) by induction on the cardinality of K. If $K = \emptyset$, then $\delta_K = j_\emptyset \circ \varepsilon$, which is a morphism of algebras. Suppose now that (3.4) holds for all sets of cardinality $< |K|$, in particular for all proper subsets K' of K. Thus, the right-hand side of (3.7) equals

$$\sum_{\substack{K' \subset K \\ K' \neq K}} \delta_{K'}(ab) + \sum_{\substack{I,J \subset K \\ I \cup J = K}} \delta_I(a)\delta_J(b).$$

We get the desired formula by substracting the summands corresponding to the proper subsets K' of K from both sides of (3.7).

Formula (3.5) follows from (3.4) and the fact that $\delta_I(a)$ and $\delta_J(b)$ commute when $I \cap J = \emptyset$.

3.3. Remark. Note that, if I and $J \subset \mathbf{N}'$ are disjoint finite sets, then

$$(\delta_I \otimes \delta_J) \circ \Delta = \delta_{I \cup J}. \tag{3.8}$$

Eric Müller observed (private communication) that $\delta^n : A \to A^{\otimes n}$ can also be defined as $\delta^n = (\mathrm{id}_A - \varepsilon)^{\otimes n} \circ \Delta^n$.

3.4. Definition of A'. Let A be a bialgebra over $\mathbf{C}[[u]]$ in the sense of Section 1.1. Using the comultiplication $\Delta : A \to A \widehat{\otimes}_{\mathbf{C}[[u]]} A$, we define $\mathbf{C}[[u]]$-linear maps $\delta^n : A \to A^{\widehat{\otimes} n}$ as in Section 3.1. Observe that Formulas (3.1)-(3.5) hold in this setting as well. Following Drinfeld [**Dri87**, Section 7], we introduce the submodule A' of A by

$$A' = \left\{ a \in A \ \mid \ \delta^n(a) \in u^n A^{\widehat{\otimes} n} \ \text{ for all } n > 0 \right\}. \tag{3.9}$$

It follows from (3.3) and (3.4) that A' is a subalgebra of A.

Proposition 3.5. *If the multiplication by u is injective on $A^{\widehat{\otimes} n}$ for all $n \geq 1$, then the algebra A' is commutative modulo u, i.e., $ab - ba \in uA'$ for all a, $b \in A'$.*

Proof. Let us first observe that there exists $a_1 \in A$ such that $a = ua_1 + \varepsilon(a)1$. This follows from the fact that $\mathrm{id}_A = \Delta^1 = \delta^1 + \delta^0 = \delta^1 + \varepsilon\, 1$ and $\delta^1(a) \in uA$. Similarly, there exists $b_1 \in A$ such that $b = ub_1 + \varepsilon(b)1$. Hence, $ab - ba = uc$, where $c = u(a_1 b_1 - b_1 a_1)$. It suffices to show that $c \in A'$. To this end, it is enough to check that $\delta_K(c)$ is divisible by $u^{|K|}$ for any nonempty finite subset K of $\mathbf{N}'$. Since the multiplication by u is injective on $A^{\widehat{\otimes}|K|}$, it is enough to check that $\delta_K(ab - ba)$ is divisible by $u^{|K|+1}$. We apply Formula (3.5). Let I and J be subsets of K such that $I \cup J = K$ and $I \cap J \neq \emptyset$. Then $|I| + |J| \geq |K| + 1$. Since $\delta_I(a)$ is divisible by $u^{|I|}$ and $\delta_J(b)$ is divisible by $u^{|J|}$, it follows from (3.5) that $\delta_K(ab - ba)$ is divisible by $u^{|I|+|J|}$, hence by $u^{|K|+1}$. $\square$

3.6. Remark. If A is topologically free, i.e., isomorphic to $V[[u]]$ as a $\mathbf{C}[[u]]$-module for some vector space V, then so is A'. A similar, but more complicated statement will be proved in Lemma 7.2.

3.7. Example. Consider a Lie algebra $\mathfrak{g}$ and its universal enveloping bialgebra $U(\mathfrak{g})$. Let $U(\mathfrak{g})[[u]]$ be the $\mathbf{C}[[u]]$-bialgebra consisting of the formal power series over $U(\mathfrak{g})$, with comultiplication Δ given by (2.4). Using the notation of Section 2.4, we introduce a subalgebra $\widehat{V}_u(\mathfrak{g})$ of $U(\mathfrak{g})[[u]]$ by

$$\widehat{V}_u(\mathfrak{g}) = \left\{ \sum_{m \geq 0} a_m\, u^m \in U(\mathfrak{g})[[u]] \ \mid \ a_m \in U^m(\mathfrak{g}) \text{ for all } m \geq 0 \right\}. \tag{3.10}$$

Clearly, $V_u(\mathfrak{g}) \subset \widehat{V}_u(\mathfrak{g})$. Let I_u be the two-sided ideal of $V_u(\mathfrak{g})$ generated by $uV_u(\mathfrak{g})$ and by $u\mathfrak{g} \subset uU^1(\mathfrak{g}) \subset V_u(\mathfrak{g})$; it is the kernel of the morphism of algebras

$$V_u(\mathfrak{g}) \xrightarrow{q_u} S(\mathfrak{g}) \longrightarrow S(\mathfrak{g})/\Big(\bigoplus_{n\geq 1} S^n(\mathfrak{g})\Big) = \mathbf{C},$$

cf. Section 2.4. It is easy to check that $\widehat{V}_u(\mathfrak{g})$ is the I_u-adic completion of $V_u(\mathfrak{g})$.

Proposition 3.8. *If $A = U(\mathfrak{g})[[u]]$, then $A' = \widehat{V}_u(\mathfrak{g})$.*

Proof. Let $a = \sum_{m\geq 0} a_m u^m$ be a formal power series with coefficients in $U(\mathfrak{g})$. For $n \geq 1$, the condition $\delta^n(a) \in u^n U(\mathfrak{g})^{\otimes n}[[u]]$ implies that $\delta^n(a_{n-1}) = 0$. We claim that

$$\operatorname{Ker}\big(\delta^n : U(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes n}\big) = U^{n-1}(\mathfrak{g}) \tag{3.11}$$

for all $n \geq 1$. It follows from this claim that $a_{n-1} \in U^{n-1}(\mathfrak{g})$, hence, $a \in \widehat{V}_u(\mathfrak{g})$.

Equality (3.11) is probably well known, but we give a proof for the sake of completeness. The standard symmetrization map $\eta : S(\mathfrak{g}) \to U(\mathfrak{g})$ is known to be an isomorphism of coalgebras (cf. [**Dix74**, Chap. 2]). Hence, $\eta^{\otimes n}\delta^n = \delta^n\eta$, where δ^n stands for the corresponding maps both on $S(\mathfrak{g})$ and $U(\mathfrak{g})$. Moreover, $\eta^{-1}(U^{n-1}(\mathfrak{g})) = \oplus_{k=0}^{n-1} S^k(\mathfrak{g})$. Therefore, Equality (3.11) is equivalent to

$$\operatorname{Ker}\big(\delta^n : S(\mathfrak{g}) \to S(\mathfrak{g})^{\otimes n}\big) = \bigoplus_{k=0}^{n-1} S^k(\mathfrak{g}).$$

If $(x_i)_i$ is a totally ordered basis of $\mathfrak{g}$, we get a basis of $S(\mathfrak{g})$ by taking all words $w = x_{i_1}\dots x_{i_p}$ such that $x_{i_1} \leq \cdots \leq x_{i_p}$. We call subword of a word w any word obtained from w by deleting some letters. With this convention, the comultiplication Δ of $S(\mathfrak{g})$ is given on a basis element w by $\Delta(w) = \sum w_1 \otimes w_2$, where the sum is over all subwords w_1, w_2 of w such that $w = w_1 w_2$. Iterating Δ, we get for all $n \geq 1$

$$\Delta^n(w) = \sum w_1 \otimes \cdots \otimes w_n,$$

where the sum is over all subwords $w_1, \dots, w_n$ of w such that $w = w_1 \dots w_n$. This, together with (3.1) or (3.2), implies that

$$\delta^n(w) = \sum w_1 \otimes \cdots \otimes w_n, \tag{3.12}$$

where the sum is now over all *nonempty* subwords $w_1, \dots, w_n$ of w such that $w = w_1 \dots w_n$. This shows that, if w is of length $< n$, then the right-hand side of (3.12) is empty and $\delta^n(w) = 0$. Therefore,

$$\bigoplus_{k=0}^{n-1} S^k(\mathfrak{g}) \subset \operatorname{Ker}(\delta^n).$$

To prove the opposite inclusion, it is enough to check that the restriction of δ^n to the subspace $\oplus_{k\geq n} S^k(\mathfrak{g})$ is injective. This is a consequence of the following observation: If w is a basis element of length $\geq n$ and μ is the multiplication in $S(\mathfrak{g})$, then (3.12) implies that $\mu\delta^n(w) = \|w\|\, w$, where $\|w\| > 0$ is the number of summands on the right-hand side of (3.12). $\square$

4. Topologically free $\mathbf{C}[[u, v]]$-modules.

In this section, we establish a few technical results on modules over the ring $\mathbf{C}[[u, v]]$ of formal power series in two commuting variables u and v with coefficients in $\mathbf{C}$. They are modelled on similar results for modules over the ring $\mathbf{C}[[h]]$ of formal power series in h.

4.1. Modules over $\mathbf{C}[[h]]$. We recall a few facts about $\mathbf{C}[[h]]$-modules (see, e. g., [**Kas95**, Sections XVI.2-3]). A $\mathbf{C}[[h]]$-module M is called *topologically free* if it is isomorphic to a module $V[[h]]$ consisting of all formal power series with coefficients in the vector space V. A $\mathbf{C}[[h]]$-module M is topologically free if and only if there is no nonzero element $m \in M$ such that $hm = 0$ and the natural map $M \to \varprojlim_n M/h^nM$ is an isomorphism. We define a topological tensor product $\widehat{\otimes}_{\mathbf{C}[[h]]}$ for $\mathbf{C}[[h]]$-modules M and N by

$$M \,\widehat{\otimes}_{\mathbf{C}[[h]]}\, N = \varprojlim_{n} \big(M/h^nM \otimes_{\mathbf{C}[[h]]/(h^n)} N/h^nN \big).$$

For all vector spaces V, W, we have $V[[h]] \,\widehat{\otimes}_{\mathbf{C}[[h]]}\, W[[h]] \cong (V \otimes_{\mathbf{C}} W)[[h]]$.

Let us extend these considerations to $\mathbf{C}[[u, v]]$-modules.

4.2. Basic Definitions. Let M be a $\mathbf{C}[[u, v]]$-module. We say that M is *u-torsion-free* (resp. *v-torsion-free*) if there is no nonzero element $m \in M$ such that $um = 0$ (resp. such that $vm = 0$).

We say that M is *admissible* if any element divisible by both u and v in M is divisible by uv in M. In other words, M is admissible if, for any $m \in M$ such that there exists $m_1, m_2 \in M$ with $m = um_1 = vm_2$, there exists $m_0 \in M$ such that $m = uvm_0$.

Observe that, if M is admissible and u-torsion-free, then any element of M divisible by u^n and by v is divisible by u^nv, where $n > 0$.

We denote by $\widehat{M}_{(u,v)}$ the (u, v)-adic completion of M: It is the projective limit of the projective system $(M/(u, v)^nM)_{n\geq 1}$, where $(u, v)M = uM + vM$. The projections $M \to M/(u, v)^nM$ induce a natural $\mathbf{C}[[u, v]]$-linear map $i : M \to \widehat{M}_{(u,v)}$. The kernel of i is the intersection of the submodules $((u, v)^nM)_{n\geq 1}$. We say that the module M is *separated* (resp. *complete*) if the map $i : M \to \widehat{M}_{(u,v)}$ is injective (resp. surjective).

Given a vector space V over $\mathbf{C}$, consider the vector space $V[[u, v]]$ consisting of formal power series $\sum_{m,n\geq 0} x_{mn}\, u^m v^n$, where the coefficients x_{mn} $(m, n \geq 0)$ are elements of V. The standard multiplication of formal power series endows $V[[u,v]]$ with a $\mathbf{C}[[u,v]]$-module structure. A $\mathbf{C}[[u,v]]$-module M isomorphic to a module of the form $V[[u,v]]$ will be called *topologically free*.

It is easy to check that a topologically free $\mathbf{C}[[u,v]]$-module is u-torsion-free, v-torsion-free, admissible, separated, and complete. We now prove the converse.

Lemma 4.3. *Any u-torsion-free, v-torsion-free, admissible, separated, complete $\mathbf{C}[[u,v]]$-module M is topologically free.*

Proof. Let V be a vector subspace of M supplementary to the submodule $(u,v)M$. We claim that for all $n \geq 0$ we have the direct sum decomposition of vector spaces

$$(u,v)^n M = (u,v)^{n+1} M \oplus \bigoplus_{\substack{k,\ell \geq 0 \\ k+\ell = n}} u^k v^\ell V. \tag{4.1}$$

From (4.1) we derive

$$M = (u,v)^{n+1} M \oplus \bigoplus_{\substack{k,\ell \geq 0 \\ k+\ell \leq n}} u^k v^\ell V.$$

Consequently,

$$M/(u,v)^{n+1} M = \bigoplus_{\substack{k,\ell \geq 0 \\ k+\ell \leq n}} u^k v^\ell V = V[[u,v]]/(u,v)^{n+1} V[[u,v]].$$

Using the hypotheses, we get the following chain of $\mathbf{C}[[u,v]]$-linear isomorphisms:

$$M \cong \widehat{M}_{(u,v)} \cong \widehat{V[[u,v]]}_{(u,v)} \cong V[[u,v]].$$

It remains to check (4.1). We shall prove it by induction on n. If $n = 0$, the identity (4.1) holds by definition of V. If $n > 0$, let us first show that

$$(u,v)^n M = (u,v)^{n+1} M + \sum_{\substack{k,\ell \geq 0 \\ k+\ell = n}} u^k v^\ell V. \tag{4.2}$$

Indeed, any element of $(u,v)^n M$ is of the form $um' + vm''$, where $m', m'' \in (u,v)^{n-1} M$. By the induction hypothesis, m' and m'' belong to

$$(u,v)^n M + \sum_{\substack{k,\ell \geq 0 \\ k+\ell = n-1}} u^k v^\ell V.$$

This implies (4.2).

Suppose now that we have elements $m \in (u,v)^{n+1} M$ and $x_0, x_1, \ldots, x_n \in V$ such that

$$m + \sum_{k=0}^{n} u^k v^{n-k} x_{n-k} = 0. \tag{4.3}$$

We have to show that $m = x_0 = x_1 = \cdots = x_n = 0$. The element $m \in (u,v)^{n+1}M$ is of the form $m = u^{n+1}m_0 + vm''$, where $m_0 \in M$ and $m'' \in (u,v)^n M$. The element $u^n x_0 + u^{n+1}m_0 = u^n(x_0 + um_0)$ is divisible by u^n. It follows from (4.3) that it is also divisible by v. Since M is admissible and u-torsion-free, there exists $m_1 \in M$ such that $u^n(x_0 + um_0) = u^n v m_1$. Hence, $x_0 + um_0 - vm_1 = 0$. Now, $x_0 \in V$ and $um_0 - vm_1 \in (u,v)M$ belong to supplementary subspaces. Therefore, $x_0 = um_0 - vm_1 = 0$ and $m = u^{n+1}m_0 + vm'' = vm'$, where $m' = u^n m_1 + m'' \in (u,v)^n M$. Now, (4.3) becomes $v\big(m' + \sum_{k=0}^{n-1} u^k v^{n-1-k} x_{n-k}\big) = 0$. Since M is v-torsion-free, we get $m' + \sum_{k=0}^{n-1} u^k v^{n-1-k} x_{n-k} = 0$. By the induction hypothesis, $m' = x_1 = \cdots = x_n = 0$. □

4.4. Topological Tensor Product. Given $\mathbf{C}[[u,v]]$-modules M and N, we define their topological tensor product over $\mathbf{C}[[u,v]]$ by

$$M \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} N = \varprojlim_n \Big(M/(u,v)^n M \otimes_{\mathbf{C}[[u,v]]/(u,v)^n} N/(u,v)^n N \Big).$$

For example, $M \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} \mathbf{C}[[u,v]] = \widehat{M}_{(u,v)}$.

Lemma 4.5. (a) *If $M \cong V[[u,v]]$ and $N \cong W[[u,v]]$ are topologically free $\mathbf{C}[[u,v]]$-modules, then $M \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} N$ is topologically free:*

$$M \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} N \cong (V \otimes_{\mathbf{C}} W)[[u,v]].$$

(b) *If $i : M' \to M$ and $j : N' \to N$ are injective $\mathbf{C}[[u,v]]$-maps of topologically free modules, then so is the map $i \otimes j : M' \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} N' \to M \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} N$.*

Proof. (a) Proceed as in the proof of [**Kas95**, Proposition XVI.3.2].

(b) Since $i \otimes j = (\mathrm{id} \otimes j)(i \otimes \mathrm{id})$, it is enough to prove Part (b) when $N = N'$ or $M = M'$. We give a proof for $N = N'$.

Let V, V', W be vector spaces such that $M = V[[u,v]]$, $M' = V'[[u,v]]$, and $N = W[[u,v]]$. Take a basis $(f_m)_m$ of W. By Part (a), any element Y of $M \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} N$ can be uniquely written as $Y = \sum_m X_m \otimes f_m$, where $X_m \in M$. Set $j_m(Y) = X_m$. This defines for all m a $\mathbf{C}[[u,v]]$-linear map $j_m : M \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} N \to M$. Using the same basis of W, we define a linear map $j'_m : M' \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} N \to M'$ similarly. Clearly, $j_m \circ (i \otimes \mathrm{id}) = i \circ j'_m$ for all m. Now, take $Y' \in M' \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} N$ such that $(i \otimes \mathrm{id})(Y') = 0$. By the previous equality, we have $i(j'_m(Y')) = 0$ for all m. The map i being injective, we get $j'_m(Y') = 0$ for all m. Therefore, $Y' = \sum_m j'_m(Y') \otimes f_m = 0$ and $i \otimes \mathrm{id}$ is injective. □

4.6. From One Variable to Two Variables. One of the crucial steps in our constructions will be to transform a module N over $\mathbf{C}[[h]]$ into a module $\widetilde{N}$ over $\mathbf{C}[[u,v]]$. This is done as follows.

Let $\iota : \mathbf{C}[[h]] \to \mathbf{C}[[u,v]]$ be the algebra morphism sending h to the product uv. Observe that ι factors through the subalgebras $\mathbf{C}[u][[v]]$ and $\mathbf{C}[v][[u]]$. The morphism ι sends the ideal (h^n) into the ideal $(u,v)^{2n}$. Given a $\mathbf{C}[[h]]$-module N, we consider the projective system of $\mathbf{C}[[u,v]]$-modules

$$N/(h^n) \otimes_{\mathbf{C}[[h]]/(h^n)} \mathbf{C}[[u,v]]/(u,v)^{2n}$$

where $n = 1, 2, 3, \ldots$ and set

$$\widetilde{N} = \varprojlim_n \Big(N/(h^n) \otimes_{\mathbf{C}[[h]]/(h^n)} \mathbf{C}[[u,v]]/(u,v)^{2n} \Big). \tag{4.4}$$

Clearly, for any $x \in N$, there is defined a corresponding element $\widetilde{x} \in \widetilde{N}$.

Lemma 4.7. (a) *If $N = V[[h]]$ for some vector space V over $\mathbf{C}$, then $\widetilde{N} = V[[u,v]]$.*

(b) *If N and N' are topologically free $\mathbf{C}[[h]]$-modules, then*

$$\big(N \widehat{\otimes}_{\mathbf{C}[[h]]} N'\big)^{\sim} \cong \widetilde{N} \widehat{\otimes}_{\mathbf{C}[[u,v]]} \widetilde{N'}.$$

(c) *Let $i : N' \to N$ be an injective map of topologically free $\mathbf{C}[[h]]$-modules. Then the induced $\mathbf{C}[[u,v]]$-map $\widetilde{\imath} : \widetilde{N'} \to \widetilde{N}$ is also injective.*

Proof. (a) We have the following chain of $\mathbf{C}[[u,v]]$-linear isomorphisms

$$\begin{aligned}\widetilde{N} &= \varprojlim_n V[[h]]/(h^n) \otimes_{\mathbf{C}[[h]]/(h^n)} \mathbf{C}[[u,v]]/(u,v)^{2n} \\ &= \varprojlim_n V \otimes_{\mathbf{C}} \mathbf{C}[[h]]/(h^n) \otimes_{\mathbf{C}[[h]]/(h^n)} \mathbf{C}[[u,v]]/(u,v)^{2n} \\ &= \varprojlim_n V \otimes_{\mathbf{C}} \mathbf{C}[[u,v]]/(u,v)^{2n} \\ &= \varprojlim_n V[[u,v]]/(u,v)^{2n} \\ &= V[[u,v]].\end{aligned}$$

The first isomorphism follows from the definition of $\widetilde{N}$, the second and the fourth ones from the finite-dimensionality of $\mathbf{C}[[h]]/(h^n)$ and of $\mathbf{C}[[u,v]]/(u,v)^{2n}$.

(b) This is an easy exercise which follows from Part (a) and the properties of the topological tensor products over $\mathbf{C}[[h]]$ and $\mathbf{C}[[u,v]]$ stated in Section 4.1 and in Lemma 4.5 (a).

(c) We assume that $N = V[[h]]$ and $N' = V'[[h]]$ for some vector spaces V and V'. Let $(e_k)_k$ be a basis of V' and $(f_j)_j$ a basis of V. The $\mathbf{C}[[h]]$-linear map $i : N' \to N$ is determined by $i(e_k) = \sum_{\ell \geq 0;\, j} x^j_{k,\ell}\, f_j\, h^\ell$, where $(x^j_{k,\ell})_{j,k,\ell}$ is a family of scalars such that for each couple (k,ℓ) the set of j with $x^j_{k,\ell} \neq 0$

is finite. Any element $X \in N'$ is of the form $X = \sum_{n\geq 0;\, k} \alpha_n^k\, e_k\, h^n$, where $(\alpha_n^k)_{k,n}$ is a family of scalars such that for each $n \geq 0$ the set of k with $\alpha_n^k \neq 0$ is finite. We have

$$i(X) = \sum_{\ell,n\geq 0;\, j,k} x_{k,\ell}^j \alpha_n^k\, f_j\, h^{\ell+n} = \sum_{p\geq 0} \left(\sum_{\substack{\ell,n\geq 0;\, j,k \\ \ell+n=p}} x_{k,\ell}^j \alpha_n^k\, f_j \right) h^p.$$

The coefficient of $f_j h^p$ in $i(X)$ is

$$\sum_{\substack{\ell,n\geq 0;\, k \\ \ell+n=p}} x_{k,\ell}^j \alpha_n^k = \sum_{\substack{\ell,k \\ 0\leq \ell\leq p}} x_{k,\ell}^j \alpha_{p-\ell}^k.$$

This allows us to reformulate the injectivity of i as follows: The equations on a family of scalars $(\alpha_n^k)_{k;\, n\geq 0}$

$$\sum_{\substack{\ell,k \\ 0\leq \ell\leq p}} x_{k,\ell}^j \alpha_{p-\ell}^k = 0 \tag{4.5}$$

holding for all j and $p \geq 0$ imply that $\alpha_n^k = 0$ for all k and $n \geq 0$.

By Part (a) we have $\widetilde{N} = V[[u,v]]$ and $\widetilde{N'} = V'[[u,v]]$. On the basis $(e_k)_k$ the map $\widetilde{\imath}$ is defined by $\widetilde{\imath}(e_k) = \sum_{\ell\geq 0;\, j} x_{k,\ell}^j\, f_j\, u^\ell v^\ell$. Any element $Y \in \widetilde{N'}$ is of the form $Y = \sum_{m,n\geq 0;\, k} \beta_{mn}^k\, e_k\, u^m v^n$, where $(\beta_{mn}^k)_{k,m,n}$ is a family of scalars such that for each $m, n \geq 0$ the set of k with $\beta_{mn}^k \neq 0$ is finite. We have

$$\begin{aligned} \widetilde{\imath}(Y) &= \sum_{\ell,m,n\geq 0;\, j,k} x_{k,\ell}^j \beta_{mn}^k\, f_j\, u^{\ell+m} v^{\ell+n} \\ &= \sum_{p,q\geq 0} \left(\sum_{\substack{\ell,m,n\geq 0;\, j,k \\ \ell+m=p,\ell+n=q}} x_{k,\ell}^j \beta_{mn}^k\, f_j \right) u^p v^q. \end{aligned}$$

Note that the sum in the brackets is finite. Suppose that $\widetilde{\imath}(Y) = 0$. For all $p, q \geq 0$ and all j we have

$$\sum_{\substack{\ell,m,n\geq 0;\, k \\ \ell+m=p,\ell+n=q}} x_{k,\ell}^j \beta_{mn}^k = \sum_{\substack{\ell,k \\ 0\leq \ell\leq \min(p,q)}} x_{k,\ell}^j \beta_{p-\ell,q-\ell}^k = 0.$$

Fixing $q \geq p \geq 0$ and setting $\alpha_n^k = \beta_{n,q-p+n}^k$, we get (4.5) for all j. This implies that $\beta_{n,q-p+n}^k = \alpha_n^k = 0$ for all k, n, p, q. If $p > q \geq 0$, we set $\alpha_n^k = \beta_{p-q+n,n}^k$ and we conclude likewise. Therefore, $Y = 0$. □

We define a $\mathbf{C}[[u,v]]$-*bialgebra* as a topological $\mathbf{C}[[u,v]]$-bialgebra A with respect to the ideal $(u,v) = uA + vA$. As a consequence of Lemma 4.7, we have the following:

Corollary 4.8. *If A is a $\mathbf{C}[[h]]$-bialgebra that is topologically free as a $\mathbf{C}[[h]]$-module, then $\widetilde{A}$ is a $\mathbf{C}[[u,v]]$-bialgebra that is topologically free as a $\mathbf{C}[[u,v]]$-module.*

Proof. The $\mathbf{C}[[u,v]]$-module $\widetilde{A}$ is topologically free by Lemma 4.7 (a). It is a $\mathbf{C}[[u,v]]$-bialgebra as a consequence of Lemma 4.7 (b). ☐

5. On Etingof and Kazhdan's quantization of a Lie bialgebra.

In this section, we recall the results from Etingof and Kazhdan's work [**EK96**] needed in the sequel.

5.1. The Co-Poisson Bialgebra $U(\mathfrak{g})$. Let $\mathfrak{g}$ be a Lie bialgebra with Lie cobracket δ. Consider the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ with standard cocommutative comultiplication given by (2.4). By [**Dri87**], the bialgebra $U(\mathfrak{g})$ has a unique co-Poisson bialgebra structure with a Poisson cobracket whose restriction to $\mathfrak{g} \subset U(\mathfrak{g})$ is the Lie cobracket δ. Recall from Section 1.2 that a coquantization A of $U(\mathfrak{g})$ is a $\mathbf{C}[[h]]$-bialgebra A such that $A \cong U(\mathfrak{g})[[h]]$ as a $\mathbf{C}[[h]]$-module and $A/hA = U(\mathfrak{g})$ as co-Poisson bialgebras.

In [**EK96**] Etingof and Kazhdan constructed a coquantization $U_h(\mathfrak{g})$ of $U(\mathfrak{g})$ in this sense. To this end, they first constructed a coquantization $U_h(\mathfrak{d})$ of $U(\mathfrak{d})$, where $\mathfrak{d}$ is the double of $\mathfrak{g}$. We recall the definition of $\mathfrak{d}$.

5.2. Double of a Lie Bialgebra. Let $\mathfrak{g} = \mathfrak{g}_+$ be a finite-dimensional Lie bialgebra over $\mathbf{C}$ with Lie bracket $[\ ,\ ]$ and cobracket δ. Let $\mathfrak{g}_- = (\mathfrak{g}_+^{\mathrm{op}})^* = (\mathfrak{g}_+^*)^{\mathrm{cop}}$ be the dual Lie bialgebra modified as in Section 2.1.

Consider the direct sum $\mathfrak{d} = \mathfrak{g}_+ \bigoplus \mathfrak{g}_-$. Drinfeld [**Dri82**, **Dri87**] showed that there is a unique structure of Lie bialgebra on $\mathfrak{d}$, which he called the *double* of $\mathfrak{g}_+$, such that

(a) the inclusions of $\mathfrak{g}_+$ and $\mathfrak{g}_-$ into $\mathfrak{d}$ are morphisms of Lie bialgebras and

(b) the Lie bracket $[x,y]$ for $x \in \mathfrak{g}_+$ and $y \in \mathfrak{g}_-$ is given by

$$[x,y] = (y \otimes 1)\,\delta(x) + x \cdot y, \tag{5.1}$$

where $x \cdot y \in \mathfrak{g}_- \subset \mathfrak{d}$ is defined by $(x\cdot y)(x') = -y([x,x'])$ for $x' \in \mathfrak{g}_+$.

The Lie cobracket on $\mathfrak{d}$ (hence on $\mathfrak{g}_\pm$) is given by

$$\delta(X) = [X \otimes 1 + 1 \otimes X, r] = \sum_{i=1}^{d} \Big([X, x_i] \otimes y_i + x_i \otimes [X, y_i] \Big) \tag{5.2}$$

for $X \in \mathfrak{d}$. Here $r = \sum_{i=1}^d x_i \otimes y_i$ is the canonical element of $\mathfrak{g}_+ \otimes \mathfrak{g}_- \subset \mathfrak{d} \otimes \mathfrak{d}$, where $(x_i)_{i=1}^d$ is a basis of $\mathfrak{g}_+$ and $(y_i)_{i=1}^d$ is the dual basis of $\mathfrak{g}_-$.

5.3. The bialgebra $U_h\mathfrak{d}$. By [**EK96**, Section 3] there exists a $\mathbf{C}[[h]]$-bialgebra $U_h(\mathfrak{d})$ with the following features:

(i) As a $\mathbf{C}[[h]]$-algebra, $U_h(\mathfrak{d}) = U(\mathfrak{d})[[h]]$, i.e., the multiplication is the standard formal power series product.

(ii) There exists an invertible element $J_h \in (U\mathfrak{d} \otimes U\mathfrak{d})[[h]]$ with constant term $1 \otimes 1$ such that the comultiplication Δ_h of $U_h(\mathfrak{d})$ is given for all $a \in U(\mathfrak{d})$ by

$$\Delta_h(a) = J_h^{-1} \Delta(a) J_h, \tag{5.3}$$

where Δ is the standard comultiplication in $U(\mathfrak{d})$. The first terms of the formal power series J_h are given by

$$J_h \equiv 1 \otimes 1 + \frac{h}{2}\, r \mod h^2 \tag{5.4}$$

where $r \in \mathfrak{d} \otimes \mathfrak{d}$ was defined in Section 5.2. From (5.2–5.4) it follows that for $x \in \mathfrak{d} \subset U_h(\mathfrak{d})$ we have

$$\Delta_h(x) - \Delta_h^{\mathrm{op}}(x) \equiv h\, \delta(x) \mod h^2, \tag{5.5}$$

where Δ_h^{op} is the opposite comultiplication and δ is the Lie cobracket (5.2).

(iii) If we set $t = r + r_{21} = \sum_{i=1}^{d} (x_i \otimes y_i + y_i \otimes x_i)$, then the element

$$R_h = (J_h^{-1})_{21} \exp\Big(\frac{ht}{2}\Big) J_h \in (U\mathfrak{d} \otimes U\mathfrak{d})[[h]] = U_h(\mathfrak{d}) \,\widehat{\otimes}_{\mathbf{C}[[h]]}\, U_h(\mathfrak{d}) \tag{5.6}$$

defines a quasitriangular structure on $U_h(\mathfrak{d})$. This means that $\Delta_h^{\mathrm{op}}(a) = R_h \Delta_h(a) R_h^{-1}$ for all $a \in U_h(\mathfrak{d})$ and that

$$(\Delta_h \otimes \mathrm{id})(R_h) = (R_h)_{13}(R_h)_{23} \quad \text{and} \quad (\mathrm{id} \otimes \Delta_h)(R_h) = (R_h)_{13}(R_h)_{12}. \tag{5.7}$$

Formula (5.4) implies

$$R_h = 1 \otimes 1 + h R_h', \tag{5.8}$$

where $R_h' \in U_h(\mathfrak{d}) \,\widehat{\otimes}_{\mathbf{C}[[h]]}\, U_h(\mathfrak{d})$ such that $R_h' \equiv r \mod h$.

From (i) and (ii) it is clear that $U_h(\mathfrak{d})$ is a coquantization of the co-Poisson bialgebra $U(\mathfrak{d})$.

5.4. The bialgebras $U_h(\mathfrak{g}_\pm)$. In [**EK96**, Section 4] Etingof and Kazhdan constructed a $\mathbf{C}[[h]]$-bialgebra $U_h(\mathfrak{g}_\pm)$ (with h-adic topology) with the following properties:

(i) As a $\mathbf{C}[[h]]$-module, $U_h(\mathfrak{g}_\pm)$ is isomorphic to $U(\mathfrak{g}_\pm)[[h]]$.

(ii) $U_h(\mathfrak{g}_\pm)$ is a $\mathbf{C}[[h]]$-subbialgebra of $U_h(\mathfrak{d})$. The map $p_h : U_h(\mathfrak{g}_\pm) \subset U_h(\mathfrak{d}) = U(\mathfrak{d})[[h]] \to U(\mathfrak{d}) = U(\mathfrak{d})[[h]]/hU(\mathfrak{d})[[h]]$ induces a bialgebra isomorphism

$$U_h(\mathfrak{g}_\pm)/hU_h(\mathfrak{g}_\pm) = U(\mathfrak{g}_\pm) \subset U(\mathfrak{d}).$$

(iii) The element $R_h' \in U_h(\mathfrak{d}) \widehat{\otimes}_{\mathbf{C}[[h]]} U_h(\mathfrak{d})$ of (5.8) belongs to $U_h(\mathfrak{g}_+) \widehat{\otimes}_{\mathbf{C}[[h]]} U_h(\mathfrak{g}_-)$. So does the universal R-matrix R_h.

(iv) The coalgebra structure on $U_h(\mathfrak{g}_\pm)$ induces an algebra structure on the dual module $U_h^*(\mathfrak{g}_\pm) = \mathrm{Hom}_{\mathbf{C}[[h]]}(U_h(\mathfrak{g}_\pm), \mathbf{C}[[h]])$. By (iii) we can define linear maps $\rho_\pm : U_h^*(\mathfrak{g}_\mp) \to U_h(\mathfrak{g}_\pm)$ by

$$\rho_+(f) = (\mathrm{id}\otimes f)(R_h) \quad \text{and} \quad \rho_-(g) = (g\otimes \mathrm{id})(R_h) \tag{5.9}$$

for all $f \in U_h^*(\mathfrak{g}_-)$ and $g \in U_h^*(\mathfrak{g}_+)$. In [**EK96**, Propositions 4.8 and 4.10] it was shown that ρ_+ is an injective antimorphism of algebras and ρ_- is an injective morphism of algebras.

The construction of $U_h(\mathfrak{d})$ and $U_h(\mathfrak{g}_\pm)$ depends on a Drinfeld associator, see Sections 11.2-11.4. Nevertheless, it was shown in [**EK97**] (and in Section 10 of the revised version of [**EK96**]) that the assignment $(\mathfrak{g}_+, \mathfrak{d}, \mathfrak{g}_-) \mapsto \Big(U_h(\mathfrak{g}_+) \hookrightarrow U_h(\mathfrak{d}) \hookleftarrow U_h(\mathfrak{g}_-)\Big)$ is functorial when the Drinfeld associator is fixed.

5.5. The Linear Forms f_x. Choose a $\mathbf{C}[[h]]$-linear isomorphism $\alpha_- : U_h(\mathfrak{g}_-) \to U(\mathfrak{g}_-)[[h]]$ such that $\alpha_-(1) = 1$ and $\alpha_- \equiv \mathrm{id}$ modulo h. Choose also a $\mathbf{C}$-linear projection $\pi_- : U(\mathfrak{g}_-) \to U^1(\mathfrak{g}_-) = \mathbf{C}\oplus\mathfrak{g}_-$ that is the identity on $U^1(\mathfrak{g}_-)$. For any $x \in \mathfrak{g}_+$ we define a $\mathbf{C}$-linear form $\langle x, -\rangle : U^1(\mathfrak{g}_-) \to \mathbf{C}$ extending the evaluation map $\langle x, -\rangle : \mathfrak{g}_- \to \mathbf{C}$ and such that $\langle x, 1\rangle = 0$.

Given $x \in \mathfrak{g}_+$ we define a $\mathbf{C}[[h]]$-linear form $f_x : U_h(\mathfrak{g}_-) \to \mathbf{C}[[h]]$ by

$$f_x(b) = \langle x, \pi_-\alpha_-(b)\rangle = \sum_{n\geq 0} \langle x, \pi_-(b_n)\rangle\, h^n, \tag{5.10}$$

where $b \in U_h(\mathfrak{g}_-)$ and the elements $b_n \in U(\mathfrak{g}_-)$ are defined by $\alpha_-(b) = \sum_{n\geq 0} b_n h^n$. It follows from the definition that $f_x(1) = 0$.

Applying the map ρ_+ of (5.9) to $f_x \in U_h^*(\mathfrak{g}_-)$, we get an element $\rho_+(f_x) \in U_h(\mathfrak{g}_+)$. Fix a basis $(x_1, \ldots, x_d)$ of $\mathfrak{g}_+$. Given a d-tuple $\underline{j} = (j_1, \ldots, j_d)$ of nonnegative integers, we set $|\underline{j}| = j_1 + \cdots + j_d$ and $x_{\underline{j}} = x_1^{j_1} \ldots x_d^{j_d} \in U(\mathfrak{g}_+)$. Note that $(x_{\underline{j}})_{\underline{j}}$ is a basis of $U(\mathfrak{g}_+)$.

Lemma 5.6. (a) *For any d-tuple $\underline{j} = (j_1, \ldots, j_d)$ of nonnegative integers, there exists an element $t_{\underline{j}} \in U_h(\mathfrak{g}_+)$ such that*

$$\rho_+(f_{x_1})^{j_1} \ldots \rho_+(f_{x_d})^{j_d} = h^{|\underline{j}|}\, t_{\underline{j}} \quad \textit{and} \quad p_h(t_{\underline{j}}) = x_{\underline{j}},$$

where $p_h : U_h(\mathfrak{g}_+) \to U_h(\mathfrak{g}_+)/hU_h(\mathfrak{g}_+) = U(\mathfrak{g}_+)$ is the canonical projection.

(b) *For any $a \in U_h(\mathfrak{g}_+)$, there is a unique family of scalars $\lambda_{\underline{j}}^{(n)} \in \mathbf{C}$ indexed by a nonnegative integer n and a finite sequence $\underline{j} = (j_1, \ldots, j_d)$ of nonnegative integers such that*

$$a = \sum_{n\geq 0} \left(\sum_{|\underline{j}|\leq c(n)} \lambda_{\underline{j}}^{(n)} t_{\underline{j}} \right) h^n,$$

where $c(n)$ is an integer depending on a and n.

(c) *If $a \in \operatorname{Im} \rho_+$, then $c(n) = n$, that is, $\lambda_{\underline{j}}^{(n)} = 0$ whenever $n < |\underline{j}|$, where $\lambda_{\underline{j}}^{(n)}$ are the scalars above.*

Proof. (a) For any $x \in \mathfrak{g}_+$, we have $\rho_+(f_x) = ht_x$ for some $t_x \in U_h(\mathfrak{g}_+)$ such that $p_h(t_x) = x$. This follows from (5.8) (cf. [**EK96**, Lemma 4.6]). We set $t_{\underline{j}} = t_{x_1}^{j_1} \dots t_{x_d}^{j_d}$.

(b) The proof of Proposition 4.5 of [**EK96**] implies that any $a \in U_h(\mathfrak{g}_+)$ can be expanded as above. Let us check that such an expression is unique. If

$$\sum_{n \geq 0} \Big(\sum_{\underline{j};\, |\underline{j}| \leq c(n)} \lambda_{\underline{j}}^{(n)} t_{\underline{j}} \Big) h^n = 0, \tag{5.11}$$

then $\sum_{|\underline{j}| \leq c(0)} \lambda_{\underline{j}}^{(0)} x_{\underline{j}} = 0$ by application of the projection p_h. Since the elements $(x_{\underline{j}})_{\underline{j}}$ form a basis of $U(\mathfrak{g}_+)$, we conclude that $\lambda_{\underline{j}}^{(0)} = 0$ for all $\underline{j}$. We may then divide the left-hand side of (5.11) by h and start again. This implies the vanishing of $\lambda_{\underline{j}}^{(1)} = 0$ for all $\underline{j}$, and so on.

(c) Clearly, $U_h^*(\mathfrak{g}_-) = U(\mathfrak{g}_-)^*[[h]]$ where $U(\mathfrak{g}_-)^* = \operatorname{Hom}_{\mathbf{C}}(U(\mathfrak{g}_-), \mathbf{C})$. We provide $U_h^*(\mathfrak{g}_-)$ with the multiplication induced by the comultiplication of $U_h(\mathfrak{g}_-)$. We claim that the family of linear forms $(f_{x_d}^{j_d} \dots f_{x_1}^{j_1})_{\underline{j}} \in U_h^*(\mathfrak{g}_-)$ is linearly independent and that the $\mathbf{C}[[h]]$-module it spans is dense in $U_h^*(\mathfrak{g}_-)$ for the I_h^*-adic topology, where I_h^* is the two-sided ideal of $U_h^*(\mathfrak{g}_-)$ generated by h and f_{x_k} $(k = 1, \dots, d)$. It suffices to prove that the images $\theta_{x_d}^{j_d} \dots \theta_{x_1}^{j_1} \in U(\mathfrak{g}_-)^*$ of $f_{x_d}^{j_d} \dots f_{x_1}^{j_1}$ under the algebra morphism $U_h^*(\mathfrak{g}_-) \to U_h^*(\mathfrak{g}_-)/hU_h^*(\mathfrak{g}_-) = U(\mathfrak{g}_-)^*$ are linearly independent and that their linear span is dense in $U(\mathfrak{g}_-)^*$ for the I_0^*-adic topology, where I_0^* is the two-sided ideal of $U(\mathfrak{g}_-)^*$ generated by θ_{x_k} $(k = 1, \dots, d)$. Now, by definition of f_{x_i}, we have $\theta_{x_i} = \langle x_i, \pi_-(-) \rangle$. This implies that, for all $i, j = 1, \dots, d$, we have

$$\theta_{x_i}(1) = 0 \quad \text{and} \quad \theta_{x_i}(y_j) = \delta_{ij}, \tag{5.12}$$

where $(y_1, \dots, y_d)$ is the dual basis of the basis $(x_1, \dots, x_d)$. We compute the values of the linear form $\theta_{x_d}^{j_d} \dots \theta_{x_1}^{j_1}$ on the basis $(y_d^{k_d} \dots y_1^{k_1})_{k_1, \dots, k_d \geq 0}$ of $U(\mathfrak{g}_-)$:

$$(\theta_{x_d}^{j_d} \dots \theta_{x_1}^{j_1})(y_d^{k_d} \dots y_1^{k_1}) = (\theta_{x_d}^{\otimes j_d} \otimes \dots \otimes \theta_{x_1}^{\otimes j_1})\big(\Delta^{|\underline{j}|}(y_d^{k_d} \dots y_1^{k_1})\big).$$

A simple computation, using (5.12) and the definition of Δ (cf. the proof of Proposition 3.8), shows that

(5.13)

$$(\theta_{x_d}^{j_d} \dots \theta_{x_1}^{j_1})(y_d^{k_d} \dots y_1^{k_1}) = \begin{cases} 0 & \text{if } k_1 + \dots + k_d < j_1 + \dots + j_d, \\ \delta_{j_1,k_1} \dots \delta_{j_d,k_d} & \text{if } k_1 + \dots + k_d = j_1 + \dots + j_d. \end{cases}$$

The claim about the linear forms $\theta_{x_d}^{j_d} \dots \theta_{x_1}^{j_1} \in U(\mathfrak{g}_-)^*$ follows immediately from (5.13).

Part (a) of this lemma and the claim established above imply that the $\mathbf{C}[[h]]$-linear span of the set $(\rho_+(f_{x_1})^{j_1} \dots \rho_+(f_{x_d})^{j_d})_{\underline{j}}$ is dense in $\operatorname{Im}\rho_+$ for the h-adic topology. It is enough to prove (c) for an element a in this span. By Part (a), $a = \sum_{n\geq 0,\, \underline{j}} P_{\underline{j}}\, h^{|\underline{j}|}\, t_{\underline{j}}$ with $P_{\underline{j}} \in \mathbf{C}[[h]]$. By Part (b), the element a can be written uniquely as $a = \sum_{n\geq 0,\, \underline{j}} \lambda_{\underline{j}}^{(n)}\, h^n\, t_{\underline{j}}$. Hence, for any $\underline{j}$, the formal power series $\sum_{n\geq 0} \lambda_{\underline{j}}^{(n)} h^n$ is divisible by $h^{|\underline{j}|}$, which implies the vanishing of $\lambda_{\underline{j}}^{(n)}$ for $n < |\underline{j}|$. □

6. The algebra $A_+ = A_{u,v}(\mathfrak{g}_+)$.

We first define a two-variable version $U_{u,v}(\mathfrak{g}_\pm)$ of Etingof and Kazhdan's quantization. Then we construct the algebra $A_+ = A_{u,v}(\mathfrak{g}_+)$ appearing in Theorem 2.3. We use the notation $\mathfrak{g}_\pm$, $\mathfrak{d}$ defined in Section 5.

6.1. The bialgebras $U_{u,v}(\mathfrak{d})$ and $U_{u,v}(\mathfrak{g}_\pm)$. Applying the construction of Section 4.6 to the $\mathbf{C}[[h]]$-bialgebras $U_h(\mathfrak{d})$ and $U_h(\mathfrak{g}_\pm)$, we obtain $\mathbf{C}[[u,v]]$-modules

$$U_{u,v}(\mathfrak{d}) = \widetilde{U_h(\mathfrak{d})} \quad \text{and} \quad U_{u,v}(\mathfrak{g}_\pm) = \widetilde{U_h(\mathfrak{g}_\pm)}. \tag{6.1}$$

As a consequence of Lemma 4.5, Lemma 4.7, Corollary 4.8, and of the results summarized in Sections 5.3 and 5.4, we get the following proposition.

Proposition 6.2. (a) *The $\mathbf{C}[[u,v]]$-modules $U_{u,v}(\mathfrak{d})$ and $U_{u,v}(\mathfrak{g}_\pm)$ are topologically free.*

(b) *$U_{u,v}(\mathfrak{d})$ has a bialgebra structure whose underlying algebra is the algebra $U(\mathfrak{d})[[u,v]]$ of formal power series with coefficients in $U(\mathfrak{d})$.*

(c) *$U_{u,v}(\mathfrak{g}_\pm)$ has a bialgebra structure such that the $\mathbf{C}[[u,v]]$-linear map $U_{u,v}(\mathfrak{g}_\pm) \to U_{u,v}(\mathfrak{d})$ induced by $U_h(\mathfrak{g}_\pm) \subset U_h(\mathfrak{d})$ is an embedding of bialgebras.*

(d) *There are canonical isomorphisms of bialgebras*

$$U_{u,v}(\mathfrak{d})/(u,v)U_{u,v}(\mathfrak{d}) = U(\mathfrak{d}) \quad \textit{and} \quad U_{u,v}(\mathfrak{g}_\pm)/(u,v)U_{u,v}(\mathfrak{g}_\pm) = U(\mathfrak{g}_\pm).$$

By Proposition 6.2 (c) we may view $U_{u,v}(\mathfrak{g}_\pm)$ as a subset (in fact, a sub-bialgebra) of $U_{u,v}(\mathfrak{d})$. We denote the comultiplication in $U_{u,v}(\mathfrak{d})$ and in $U_{u,v}(\mathfrak{g}_\pm)$ by $\Delta_{u,v}$. To Etingof and Kazhdan's universal R-matrix $R_h \in U_h(\mathfrak{d}) \mathbin{\widehat{\otimes}}_{\mathbf{C}[[h]]} U_h(\mathfrak{d})$ corresponds an element $R_{u,v} \in U_{u,v}(\mathfrak{d}) \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} U_{u,v}(\mathfrak{d})$. By Section 5.4 (iii) and Lemma 4.5 (b), we have

$$R_{u,v} \in U_{u,v}(\mathfrak{g}_+) \mathbin{\widehat{\otimes}}_{\mathbf{C}[[u,v]]} U_{u,v}(\mathfrak{g}_-).$$

The following is a consequence of (5.7) and (5.8).

Lemma 6.3. (a) *We have*

$$(\Delta_{u,v}\otimes\mathrm{id})(R_{u,v}) = (R_{u,v})_{13}(R_{u,v})_{23} \quad \text{and}$$
$$(\mathrm{id}\otimes\Delta_{u,v})(R_{u,v}) = (R_{u,v})_{13}(R_{u,v})_{12}.$$

(b) *There is a unique* $R' \in U_{u,v}(\mathfrak{g}_+)\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\,U_{u,v}(\mathfrak{g}_-)$ *such that* $R_{u,v} = 1\otimes 1 + uvR'$. *The image of* R' *under the projection*

$$U_{u,v}(\mathfrak{g}_+)\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\,U_{u,v}(\mathfrak{g}_-) \to \big(U_{u,v}(\mathfrak{g}_+)\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\,U_{u,v}(\mathfrak{g}_-)\big)/(u,v)$$
$$= U(\mathfrak{g}_+)\otimes_{\mathbf{C}} U(\mathfrak{g}_-)$$

is the element $r = \sum_{i=1}^{d} x_i\otimes y_i$ *defined in Section* 5.2.

Following 5.4, consider the dual spaces

$$U^*_{u,v}(\mathfrak{g}_\pm) = \mathrm{Hom}_{\mathbf{C}[[u,v]]}(U_{u,v}(\mathfrak{g}_\pm), \mathbf{C}[[u,v]]),$$

and define $\mathbf{C}[[u,v]]$-linear maps $\rho_+ : U^*_{u,v}(\mathfrak{g}_-) \to U_{u,v}(\mathfrak{g}_+)$ and $\rho_- : U^*_{u,v}(\mathfrak{g}_+) \to U_{u,v}(\mathfrak{g}_-)$ by

$$\rho_+(f) = (\mathrm{id}\otimes f)(R_{u,v}) \quad \text{and} \quad \rho_-(g) = (g\otimes\mathrm{id})(R_{u,v}) \tag{6.2}$$

for $f \in U^*_{u,v}(\mathfrak{g}_-)$ and $g \in U^*_{u,v}(\mathfrak{g}_+)$. The dual space $U^*_{u,v}(\mathfrak{g}_\pm)$ carries a $\mathbf{C}[[u,v]]$-algebra structure. The map ρ_+ is an antimorphism of algebras and ρ_- is a morphism of algebras. This follows by a standard argument from Lemma 6.3 (a) (cf. [**EK96**, Proposition 4.8]).

6.4. The Linear Forms $\widetilde{f}_x$. In Section 5.5 we constructed a $\mathbf{C}[[h]]$-linear form $f_x : U_h(\mathfrak{g}_-) \to \mathbf{C}[[h]]$ for all $x \in \mathfrak{g}_+$. The construction depends on the choice of an isomorphism $\alpha_- : U_h(\mathfrak{g}_-) \to U(\mathfrak{g}_-)[[h]]$ and a projection $\pi_- : U(\mathfrak{g}_-) \to U^1(\mathfrak{g}_-)$. By extension of scalars, we obtain a $\mathbf{C}[[u,v]]$-linear form $\widetilde{f}_x : U_{u,v}(\mathfrak{g}_-) \to \mathbf{C}[[u,v]]$. We have $\widetilde{f}_x(1) = 0$.

Let us apply $\rho_+ : U^*_{u,v}(\mathfrak{g}_-) \to U_{u,v}(\mathfrak{g}_+)$ to $\widetilde{f}_x$. The following is a consequence of Lemma 6.3 (b).

Lemma 6.5. *The element* $\rho_+(\widetilde{f}_x) \in U_{u,v}(\mathfrak{g}_+)$ *is divisible by* uv.

6.6. Definition of A_+. Let $(x_1, \dots, x_d)$ be the basis of $\mathfrak{g}_+$ fixed in Section 5.5. The set $(u^{|\underline{j}|}\, x_{\underline{j}})$, where $\underline{j} = (j_1, \dots, j_d)$ runs over all d-tuples of nonnegative integers, is a basis of the free $\mathbf{C}[u]$-module $V_u(\mathfrak{g}_+)$ introduced in Section 2.4. In view of Lemma 6.5, we can define a $\mathbf{C}[u]$-linear map $\psi_+ : V_u(\mathfrak{g}_+) \to U_{u,v}(\mathfrak{g}_+)$ by $\psi_+(1) = 1$ and

$$\psi_+(u^{|\underline{j}|}\, x_{\underline{j}}) = v^{-|\underline{j}|}\, \rho_+(\widetilde{f}_{x_1})^{j_1} \dots \rho_+(\widetilde{f}_{x_d})^{j_d}, \tag{6.3}$$

where $\underline{j} = (j_1, \dots, j_d)$ is a d-tuple of nonnegative integers with $|\underline{j}| \geq 1$. This map extends uniquely to a $\mathbf{C}[u][[v]]$-linear map, still denoted ψ_+, from

$V_u(\mathfrak{g}_+)[[v]]$ to $U_{u,v}(\mathfrak{g}_+)$ by

$$\psi_+\left(\sum_{n\geq 0} w_n v^n\right) = \sum_{n\geq 0} \psi_+(w_n)v^n,$$

where $w_0, w_1, w_2, \ldots \in V_u(\mathfrak{g}_+)$. We define the $\mathbf{C}[u][[v]]$-module A_+ by

$$A_+ = \psi_+(V_u(\mathfrak{g}_+)[[v]]) \subset U_{u,v}(\mathfrak{g}_+). \tag{6.4}$$

The remaining part of Section 6 is concerned with the study of A_+. The relevant results are stated in Theorem 6.9.

We choose a $\mathbf{C}[[h]]$-linear isomorphism $\alpha_+ : U_h(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[h]]$ such that $\alpha_+(1) = 1$ and $\alpha_+ \equiv \mathrm{id}$ modulo h. Such an isomorphism exists by Section 5.4 (ii). Extending the scalars, we get a $\mathbf{C}[[u,v]]$-linear isomorphism $\widetilde{\alpha}_+ : U_{u,v}(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[u,v]]$ such that $\widetilde{\alpha}_+ \equiv \mathrm{id}$ modulo uv. Let us consider the composed map

$$p_v : U_{u,v}(\mathfrak{g}_+) \xrightarrow{\widetilde{\alpha}_+} U(\mathfrak{g}_+)[[u,v]] \to U(\mathfrak{g}_+)[[u]],$$

where the second map is the projection $v \mapsto 0$. We equip $U(\mathfrak{g}_+)[[u]]$ with the power series multiplication and the comultiplication (2.4).

Lemma 6.7. *The map $p_v : U_{u,v}(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[u]]$ is a morphism of bialgebras.*

Proof. The multiplication and the comultiplication of $U_h(\mathfrak{g}_+)$ transfer, via the $\mathbf{C}[[h]]$-linear isomorphism $\alpha_+ : U_h(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[h]]$, to a multiplication μ_h and a comultiplication Δ_h on $U(\mathfrak{g}_+)[[h]]$. Expanding μ_h and Δ_h into formal power series, we obtain

$$\begin{aligned} \mu_h &= \mu_0 + h\mu_1 + h^2\mu_2 + \cdots \quad \text{and} \\ \Delta_h &= \Delta_0 + h\Delta_1 + h^2\Delta_2 + \cdots , \end{aligned} \tag{6.5}$$

where $\mu_i : U(\mathfrak{g}_+)^{\otimes 2} \to U(\mathfrak{g}_+)$ and $\Delta_i : U(\mathfrak{g}_+) \to U(\mathfrak{g}_+)^{\otimes 2}$ are linear maps for all $i = 0, 1, \ldots$ Since $U_h(\mathfrak{g}_+)/hU_h(\mathfrak{g}_+) = U(\mathfrak{g}_+)$ as bialgebras, we see that μ_0 and Δ_0 are the standard multiplication and comultiplication of $U(\mathfrak{g}_+)$.

The multiplication and the comultiplication of $U_{u,v}(\mathfrak{g}_+)$ give rise, via $\widetilde{\alpha}_+$, to a multiplication $\mu_{u,v}$ and a comultiplication $\Delta_{u,v}$ on $U(\mathfrak{g}_+)[[u,v]]$ of the form

$$\begin{aligned} \mu_{u,v} &= \mu_0 + uv\mu_1 + u^2v^2\mu_2 + \cdots \quad \text{and} \\ \Delta_{u,v} &= \Delta_0 + uv\Delta_1 + u^2v^2\Delta_2 + \cdots , \end{aligned} \tag{6.6}$$

where the maps μ_i and Δ_i are the same as in (6.5). It follows that p_v is a morphism of bialgebras, where $U(\mathfrak{g}_+)[[u]]$ is equipped with μ_0 and Δ_0. □

The following result is an elaboration of Lemma 5.6 (a).

Lemma 6.8. (a) *For any d-tuple $\underline{j} = (j_1, \dots , j_d)$, the element $\psi_+(u^{|\underline{j}|}\, x_{\underline{j}})$ defined by (6.3) belongs to $u^{|\underline{j}|}\, U_{u,v}(\mathfrak{g}_+)$ and*

$$p_v\big(\psi_+(u^{|\underline{j}|}\, x_{\underline{j}})\big) = u^{|\underline{j}|}\, x_{\underline{j}} \in U(\mathfrak{g}_+)[[u]].$$

(b) *We have $p_v(A_+) = V_u(\mathfrak{g}_+)$ and $p_v \circ \psi_+ : V_u(\mathfrak{g}_+)[[v]] \to V_u(\mathfrak{g}_+)$ is the projection sending v to 0.*

Proof. (a) By multiplicativity of p_v, it suffices to prove that $v^{-1}\rho_+(\widetilde{f}_x)$ belongs to $u\, U_{u,v}(\mathfrak{g}_+)$ and that $p_v\big(v^{-1}\rho_+(\widetilde{f}_x)\big) = ux$ for any $x \in \mathfrak{g}_+$. The first assertion follows from Lemma 6.5.

Let us compute $p_v\big(v^{-1}\rho_+(\widetilde{f}_x)\big)$. Recall the isomorphism $\alpha_- : U_h(\mathfrak{g}_-) \to U(\mathfrak{g}_-)[[h]]$ from Section 5.5 and the isomorphism $\alpha_+ : U_h(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[h]]$ defined above. Let $X_i \in U_h(\mathfrak{g}_+)$ be defined by $X_i = \alpha_+^{-1}(x_i)$ and $Y_i \in U_h(\mathfrak{g}_-)$ by $Y_i = \alpha_-^{-1}(y_i)$, where $(x_1, \dots , x_d)$ is the fixed basis of $\mathfrak{g}_+$ and $(y_1, \dots , y_d)$ is the dual basis. By (5.10),

$$f_x(Y_i) = \langle x, \pi_-\alpha_-(Y_i)\rangle = \langle x, \pi_-(y_i)\rangle = \langle x, y_i\rangle. \tag{6.7}$$

It follows from (5.8) that

$$R_h = 1\otimes 1 + h \sum_{i=1}^{d} X_i\otimes Y_i + h^2 Z, \tag{6.8}$$

where $Z \in U_h(\mathfrak{g}_+)\,\widehat{\otimes}_{\mathbf{C}[[h]]}\, U_h(\mathfrak{g}_-)$. By extension of scalars from $\mathbf{C}[[h]]$ to $\mathbf{C}[[u, v]]$, we get

$$R_{u,v} = 1\otimes 1 + uv \sum_{i=1}^{d} \widetilde{X}_i\otimes \widetilde{Y}_i + u^2v^2 \widetilde{Z}, \tag{6.9}$$

where $\widetilde{X}_i \in U_{u,v}(\mathfrak{g}_+)$, $\widetilde{Y}_i \in U_{u,v}(\mathfrak{g}_-)$, and $\widetilde{Z} \in U_{u,v}(\mathfrak{g}_+)\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-)$. Moreover, using the definition of p_v and Formula (6.7), we have

$$p_v(\widetilde{X}_i) = x_i, \quad \text{and} \quad \widetilde{f}_x(\widetilde{Y}_i) = \langle x, y_i\rangle. \tag{6.10}$$

Applying $\mathrm{id}\otimes\widetilde{f}_x$ to $R_{u,v}$ and using (6.9) and (6.10), we obtain

$$\begin{aligned}
\rho_+(\widetilde{f}_x) &= (\mathrm{id}\otimes\widetilde{f}_x)(R_{u,v}) \\
&= \widetilde{f}_x(1) + uv \sum_{i=1}^{d} \widetilde{X}_i \widetilde{f}_x(\widetilde{Y}_i) + u^2v^2 (\mathrm{id}\otimes\widetilde{f}_x)(\widetilde{Z}) \\
&= uv \sum_{i=1}^{d} \langle x, y_i\rangle\, \widetilde{X}_i + u^2v^2 (\mathrm{id}\otimes\widetilde{f}_x)(\widetilde{Z}).
\end{aligned}$$

Therefore,

$$v^{-1}\rho_+(\widetilde{f}_x) = u\sum_{i=1}^{d} \langle x, y_i\rangle \, \widetilde{X}_i + u^2 v(\mathrm{id}\otimes\widetilde{f}_x)(\widetilde{Z}).$$

This implies, in view of (6.10),

$$p_v\big(v^{-1}\rho_+(\widetilde{f}_x)\big) = u\sum_{i=1}^{d} \langle x, y_i\rangle \, p_v(\widetilde{X}_i) = u\sum_{i=1}^{d} \langle x, y_i\rangle \, x_i = ux.$$

(b) It follows from Part (a) and the definition of A_+. □

Theorem 6.9. (a) *The map $\psi_+ : V_u(\mathfrak{g}_+)[[v]] \to A_+$ is an isomorphism of $\mathbf{C}[u][[v]]$-modules.*

(b) *A_+ is a subalgebra of $U_{u,v}(\mathfrak{g}_+)$.*

(c) *The algebra A_+ is independent of the choices made in Section 5.5.*

Proof. (a) The map ψ_+ is surjective by definition of A_+. Let us check that it is injective. Let $w = \sum_{n\geq 0} w_n v^n \in V_u(\mathfrak{g}_+)[[v]]$ with $w_0, w_1, w_2, \ldots \in V_u(\mathfrak{g}_+)$. Assume that $w \neq 0$. Take the minimal $N \geq 0$ such that $w_N \neq 0$ and define w' by $w = v^N w'$. By Lemma 6.8, we have $p_v(\psi_+(w')) = w_N \neq 0$, hence $\psi_+(w') \neq 0$. As $A_+ \subset U_{u,v}(\mathfrak{g}_+)$ has no v-torsion, we see that $\psi_+(w) = v^N\psi_+(w') \neq 0$.

(b) Let us check that $\psi_+(u^{|\underline{i}|}x_{\underline{i}})\,\psi_+(u^{|\underline{j}|}x_{\underline{j}}) \in A_+$ for all d-tuples $\underline{i} = (i_1, \ldots, i_d)$ and $\underline{j} = (j_1, \ldots, j_d)$. Since $\rho_+ : U_h^*(\mathfrak{g}_-) \to U_h(\mathfrak{g}_+)$ is an antimorphism of algebras, the product

$$\rho_+(f_{x_1})^{i_1} \ldots \rho_+(f_{x_d})^{i_d}\rho_+(f_{x_1})^{j_1} \ldots \rho_+(f_{x_d})^{j_d}$$

belongs to the image of ρ_+. Therefore, by Lemma 5.6 (b-c), it can be expanded as

$$\rho_+(f_{x_1})^{i_1} \ldots \rho_+(f_{x_d})^{i_d}\rho_+(f_{x_1})^{j_1} \ldots \rho_+(f_{x_d})^{j_d} = \sum_{n\geq 0}\left(\sum_{|\underline{k}|\leq n} \lambda_{\underline{k}}^{(n)} t_{\underline{k}}\right) h^n,$$

where $\lambda_{\underline{k}}^{(n)} \in \mathbf{C}$. By Lemma 5.6 (a),

$$\begin{aligned}&\rho_+(f_{x_1})^{i_1} \ldots \rho_+(f_{x_d})^{i_d}\rho_+(f_{x_1})^{j_1} \ldots \rho_+(f_{x_d})^{j_d}\\ &\quad = \sum_{n\geq 0;\, \underline{k},\, |\underline{k}|\leq n} \lambda_{\underline{k}}^{(n)}\rho_+(f_{x_1})^{k_1} \ldots \rho_+(f_{x_d})^{k_d} h^{n-|\underline{k}|}.\end{aligned}$$

By extension of scalars from $\mathbf{C}[[h]]$ to $\mathbf{C}[[u,v]]$, we have $\widetilde{\rho_+(f_{x_i})} = \rho_+(\widetilde{f}_{x_i})$. Therefore,

$$\rho_+(\widetilde{f}_{x_1})^{i_1} \ldots \rho_+(\widetilde{f}_{x_d})^{i_d}\rho_+(\widetilde{f}_{x_1})^{j_1} \ldots \rho_+(\widetilde{f}_{x_d})^{j_d}$$

$$= \sum_{n\geq 0;\, \underline{k},\, |\underline{k}|\leq n} \lambda^{(n)}_{\underline{k}} \rho_+(\widetilde{f}_{x_1})^{k_1} \dots \rho_+(\widetilde{f}_{x_d})^{k_d} u^{n-|\underline{k}|} v^{n-|\underline{k}|}.$$

Using (6.3), we obtain

$$\begin{aligned} v^{|\underline{i}|+|\underline{j}|}\, \psi_+(u^{|\underline{i}|}x_{\underline{i}})\, \psi_+(u^{|\underline{j}|}x_{\underline{j}}) &= \sum_{n\geq 0;\, \underline{k},\, |\underline{k}|\leq n} \lambda^{(n)}_{\underline{k}} \psi_+(u^{|\underline{k}|}x_{\underline{k}})\, u^{n-|\underline{k}|}\, v^n \\ &= \sum_{n\geq 0} \left(\sum_{\underline{k};\, |\underline{k}|\leq n} \lambda^{(n)}_{\underline{k}} u^{n-|\underline{k}|}\, \psi_+(u^{|\underline{k}|}x_{\underline{k}}) \right) v^n. \end{aligned}$$

Thus, $v^{|\underline{i}|+|\underline{j}|}\, \psi_+(u^{|\underline{i}|}x_{\underline{i}})\, \psi_+(u^{|\underline{j}|}x_{\underline{j}})$ is a formal power series in v whose coefficients belong to the $\mathbf{C}[u]$-linear span of the elements $\psi_+(u^{|\underline{k}|}x_{\underline{k}})$. Hence, $v^{|\underline{i}|+|\underline{j}|}\, \psi_+(u^{|\underline{i}|}x_{\underline{i}})\, \psi_+(u^{|\underline{j}|}x_{\underline{j}}) \in A_+$. Applying Lemma 6.10 below $|\underline{i}| + |\underline{j}|$ times, we obtain $\psi_+(u^{|\underline{i}|}x_{\underline{i}})\, \psi_+(u^{|\underline{j}|}x_{\underline{j}}) \in A_+$.

(c) The definition of A_+ in Section 6.6 was based on the choice of a $\mathbf{C}[[h]]$-linear isomorphism $\alpha_- : U_h(\mathfrak{g}_-) \to U(\mathfrak{g}_-)[[h]]$ such that $\alpha_-(1) = 1$ and $\alpha_- \equiv \mathrm{id}$ modulo h, of a $\mathbf{C}$-linear projection $\pi_- : U(\mathfrak{g}_-) \to U^1(\mathfrak{g}_-)$ that restricts to the identity on $U^1(\mathfrak{g}_-)$, and of a basis $(x_1, \dots , x_d)$ of $\mathfrak{g}_+$. We have to check that A_+ is independent of these choices as a subset of $U_{u,v}(\mathfrak{g}_+)$.

(i) Suppose that we take another $\mathbf{C}[[h]]$-linear isomorphism $\alpha'_- : U_h(\mathfrak{g}_-) \to U(\mathfrak{g}_-)[[h]]$ such that $\alpha'_-(1) = 1$ and $\alpha'_- \equiv \mathrm{id}$ modulo h. This gives us a new linear form $f'_x : U_h(\mathfrak{g}_-) \to \mathbf{C}[[h]]$ and, by extension of scalars, a new linear form $\widetilde{f}'_x : U_{u,v}(\mathfrak{g}_-) \to \mathbf{C}[[u, v]]$ for all $x \in \mathfrak{g}_+$. Lemma 6.5 also holds for $\widetilde{f}'_x$. By Part (b) it is enough to check that $v^{-1}\, \rho_+(\widetilde{f}'_x)$ belongs to A_+.

Since $\alpha'_- \equiv \alpha_-$ modulo h, we have $f'_x \equiv f_x$ modulo h. By the proof of Lemma 5.6 (c), we see that

$$f'_x = f_x + \sum_{n\geq 1} h^n \left(\sum_{\underline{j}} \lambda^{(n)}_{\underline{j}}\, f^{j_d}_{x_d} \dots f^{j_1}_{x_1} \right), \tag{6.11}$$

where $\lambda^{(n)}_{\underline{j}} \in \mathbf{C}$ are indexed by a nonnegative integer n and a d-tuple $\underline{j} = (j_1, \dots , j_d)$ of nonnegative integers. Applying ρ_+, we get

$$\rho_+(f'_x) = \rho_+(f_x) + \sum_{n\geq 1} h^n \left(\sum_{\underline{j}} \lambda^{(n)}_{\underline{j}}\, \rho_+(f_{x_1})^{j_1} \dots \rho_+(f_{x_d})^{j_d} \right).$$

By extension of scalars, we have

$$\rho_+(\widetilde{f}'_x) = \rho_+(\widetilde{f}_x) + \sum_{n\geq 1} u^n\, v^n \left(\sum_{\underline{j}} \lambda^{(n)}_{\underline{j}}\, \rho_+(\widetilde{f}_{x_1})^{j_1} \dots \rho_+(\widetilde{f}_{x_d})^{j_d} \right).$$

Using (6.3), we obtain

$$\begin{aligned} v^{-1}\,\rho_+(\widetilde{f}'_x) &= v^{-1}\,\rho_+(\widetilde{f}_x) + \sum_{n\geq 1} u^n\, v^{n-1} \left(\sum_{\underline{j}} \lambda^{(n)}_{\underline{j}}\, \rho_+(\widetilde{f}_{x_1})^{j_1} \dots \rho_+(\widetilde{f}_{x_d})^{j_d} \right) \\ &= \psi_+(ux) + \sum_{n\geq 1} u^n\, v^{n-1} \left(\sum_{\underline{j}} \lambda^{(n)}_{\underline{j}}\, v^{|\underline{j}|}\, \psi_+(u^{|\underline{j}|} x_{\underline{j}}) \right) \\ &= \psi_+(ux) + \sum_{k\geq 1} v^k \left(\sum_{\underline{j};\, |\underline{j}|\leq k} \lambda^{(n)}_{\underline{j}}\, u^{k-|\underline{j}|+1}\, \psi_+(u^{|\underline{j}|} x_{\underline{j}}) \right). \end{aligned}$$

This shows that $v^{-1}\,\rho_+(\widetilde{f}'_x)$ is a formal power series in v whose coefficients belong to the $\mathbf{C}[u]$-linear span of the elements $\psi_+(u^{|\underline{j}|} x_{\underline{j}})$. Hence, $v^{-1}\,\rho_+(\widetilde{f}'_x) \in A_+$.

(ii) Suppose now that we take another projection $\pi'_- : U(\mathfrak{g}_-) \to U^1(\mathfrak{g}_-)$ whose restriction to $U^1(\mathfrak{g}_-)$ is the identity. We denote by f'_x the new linear form $U_h(\mathfrak{g}_-) \to \mathbf{C}[[h]]$ obtained by using π'_-. By extension of scalars, we obtain a new linear form $\widetilde{f}'_x : U_{u,v}(\mathfrak{g}_-) \to \mathbf{C}[[u,v]]$ for $x \in \mathfrak{g}_+$.

Since $\pi'_- - \pi_- = 0$ on $U^1(\mathfrak{g}_-)$, it follows from the proof of Lemma 5.6 (c) that

$$f'_x = f_x + \sum_{|\underline{j}|\geq 2} \lambda^{(0)}_{\underline{j}}\, f^{j_d}_{x_d} \dots f^{j_1}_{x_1} + \sum_{n\geq 1} h^n \left(\sum_{\underline{j}} \lambda^{(n)}_{\underline{j}}\, f^{j_d}_{x_d} \dots f^{j_1}_{x_1} \right), \tag{6.12}$$

where $\lambda^{(n)}_{\underline{j}} \in \mathbf{C}$ are scalars. Note the difference with (6.11): In (6.12) there are extra terms of degree 0 in h. Nevertheless, the same arguments as in Part (i) allow us to conclude.

(iii) Since $x \mapsto f_x$ is linear, it follows that A_+ is independent of the basis in $\mathfrak{g}_+$. □

Lemma 6.10. *We have* $A_+ \cap vU_{u,v}(\mathfrak{g}_+) = vA_+$.

Lemma 6.10 will be proved in Section 7.7.

7. Bialgebra structure on A_+.

In this section we establish that A_+ has a $\mathbf{C}[u][[v]]$-bialgebra structure. We begin with a $\mathbf{C}[[u,v]]$-subalgebra $\widehat{A}_+$ of $U_{u,v}(\mathfrak{g}_+)$ in which A_+ sits as a dense subalgebra.

7.1. The Algebra $\widehat{A}_+$. Using the comultiplication $\Delta_{u,v}$ of $U_{u,v}(\mathfrak{g}_+)$ and proceeding as in Section 3.1, we obtain $\mathbf{C}[[u,v]]$-linear maps $\delta^n : U_{u,v}(\mathfrak{g}_+) \to$

$U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}$ for all $n \geq 1$. Formulas (3.1)–(3.5) hold in this setting. We define a $\mathbf{C}[[u,v]]$-submodule $\widehat{A}_+$ of $U_{u,v}(\mathfrak{g}_+)$ by

$$(7.1) \qquad \widehat{A}_+ = \left\{ a \in U_{u,v}(\mathfrak{g}_+) \ \middle|\ \delta^n(a) \in u^n U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n} \ \text{ for all } n \geq 1 \right\}.$$

It follows from (3.3) and (3.4) that $\widehat{A}_+$ is a subalgebra of $U_{u,v}(\mathfrak{g}_+)$.

Lemma 7.2. *$\widehat{A}_+$ is a topologically free $\mathbf{C}[[u,v]]$-module.*

Proof. By Lemma 4.3 it is enough to check that $\widehat{A}_+$ is a u-torsion-free, v-torsion-free, admissible, separated, and complete $\mathbf{C}[[u,v]]$-module.

We use the fact that $\widehat{A}_+$ is a submodule of the topologically free module $U_{u,v}(\mathfrak{g}_+)$. Since the latter is separated, u-torsion-free, and v-torsion-free, so is any of its submodules. We are left with checking admissibility and completeness.

Admissibility: Let $a, a_1, a_2 \in \widehat{A}_+$ be such that $a = ua_1 = va_2$. Since $U_{u,v}(\mathfrak{g}_+)$ is topologically free, hence admissible, there exists $a_0 \in U_{u,v}(\mathfrak{g}_+)$ such that $a = uva_0$. We shall prove that $a_0 \in \widehat{A}_+$, i.e., that $\delta^n(a_0) \in u^n U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}$. Since $u(va_0 - a_1) = 0$ and $U_{u,v}(\mathfrak{g}_+)$ has no u-torsion, we have $a_1 = va_0$. Therefore, $v\delta^n(a_0) \in u^n U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}$. In other words, $v\delta^n(a_0)$ is divisible both by v and by u^n in $U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}$, which is topologically free. By an observation in Section 4.2, $v\delta^n(a_0) = u^n v Z$ for some $Z \in U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}$. Since $U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}$ has no v-torsion, $\delta^n(a_0) = u^n Z$.

Completeness: Let $(a_n)_{n \geq 0}$ be a sequence of elements of $\widehat{A}_+$ such that for all $n \geq 0$ the image of a_{n+1} in $\widehat{A}_+/(u,v)^{n+1}$ maps onto the image of a_n in $\widehat{A}_+/(u,v)^n$. Since $U_{u,v}(\mathfrak{g}_+)$ is complete, it contains an element a such that $a - a_n \in (u,v)^n U_{u,v}(\mathfrak{g}_+)$ for all $n \geq 0$. We shall show that $a \in \widehat{A}_+$, i.e., that $\delta^p(a)$ is divisible by u^p for all $p \geq 1$. For any $n \geq p$,

$$\delta^p(a) - \delta^p(a_n) \in (u,v)^n U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} p} \quad \text{and} \quad \delta^p(a_n) \in u^p U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} p},$$

which implies that $\delta^p(a) \in u^p U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} p} + (u,v)^n U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} p}$. Consequently, $\delta^p(a)$ is divisible by u^p in $\varprojlim_n U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} p}/(u,v)^n = U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} p}$. □

Consider the morphism $p_v : U_{u,v}(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[u]]$ of Lemma 6.7. Recall from (3.10) the algebra

$$\widehat{V}_u(\mathfrak{g}_+) = \left\{ \sum_{m \geq 0} a_m \, u^m \ \middle|\ a_m \in U^m(\mathfrak{g}_+) \text{ for all } m \geq 0 \right\} \subset U(\mathfrak{g}_+)[[u]].$$

Lemma 7.3. (a) *The morphism p_v sends $\widehat{A}_+$ into $\widehat{V}_u(\mathfrak{g}_+)$.*

(b) *We have* $\operatorname{Ker}\big(p_v : \widehat{A}_+ \to \widehat{V}_u(\mathfrak{g}_+)\big) = \widehat{A}_+ \cap v\, U_{u,v}(\mathfrak{g}_+) = v\, \widehat{A}_+$.

Proof. (a) By (3.1) and (6.6) the map δ^n for $U_{u,v}(\mathfrak{g}_+)$ is of the form

$$\delta^n = \delta_0^n + uv\delta_1^n,$$

where δ_0^n is obtained by (3.1) from the standard comultiplication Δ of $U(\mathfrak{g}_+)[[u]]$. Hence, $p_v^{\otimes n}\delta^n = \delta_0^n p_v$. Therefore, Part (a) follows from the definitions and Proposition 3.8.

(b) Let $a \in \widehat{A}_+$ and $b \in U_{u,v}(\mathfrak{g}_+)$ be such that $a = vb$. We have to check that $b \in \widehat{A}_+$. For any $n \geq 1$, the element $\delta^n(a) = v\delta^n(b)$ is divisible both by v and by u^n in $U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}$. Since the latter is topologically free, there exists $Z \in U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}$ such that $v\delta^n(b) = u^n vZ$. Hence, $\delta^n(b) = u^n Z$, which shows that $b \in \widehat{A}_+$. □

Lemma 7.4. *We have $A_+ \subset \widehat{A}_+$.*

Proof. Let us first prove that $\psi_+(ux) = v^{-1}\rho_+(\widetilde{f}_x)$ belongs to $\widehat{A}_+$ for all $x \in \mathfrak{g}_+$. Given $n \geq 1$, we have to check that $\delta^n(v^{-1}\rho_+(\widetilde{f}_x))$ is divisible by u^n. Formula $(\Delta_{u,v}\otimes\mathrm{id})(R) = R_{13}R_{23}$ for $R = R_{u,v}$ implies

$$(\Delta^n_{u,v}\otimes\mathrm{id})(R) = R_{1,n+1}R_{2,n+1}\cdots R_{n-1,n+1}R_{n,n+1}.$$

Therefore,

$$(\delta^n\otimes\mathrm{id})(R) = (R_{1,n+1}-1)(R_{2,n+1}-1)\cdots(R_{n-1,n+1}-1)(R_{n,n+1}-1).$$

Since $R = 1\otimes 1 + uvR'$, we have

$$(\delta^n\otimes\mathrm{id})(R) = u^n v^n\, R'_{1,n+1}R'_{2,n+1}\cdots R'_{n-1,n+1}R'_{n,n+1}.$$

It follows that

$$\begin{aligned}
&\delta^n(\rho_+(\widetilde{f}_x))\\
&= \delta^n\big((\mathrm{id}\otimes\widetilde{f}_x)(R)\big)\\
&= (\delta^n\otimes\widetilde{f}_x)(R)\\
&= (\mathrm{id}\otimes\widetilde{f}_x)\big((\delta^n\otimes\mathrm{id})(R)\big)\\
&= u^n v^n(\mathrm{id}\otimes\widetilde{f}_x)(R'_{1,n+1}R'_{2,n+1}\cdots R'_{n-1,n+1}R'_{n,n+1}) \in u^n U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}.
\end{aligned}$$

Hence, for $n \geq 1$,

$$\begin{aligned}
&\delta^n(v^{-1}\rho_+(\widetilde{f}_x))\\
&= u^n v^{n-1}(\mathrm{id}\otimes\widetilde{f}_x)(R'_{1,n+1}R'_{2,n+1}\cdots R'_{n-1,n+1}R'_{n,n+1}) \in u^n U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}.
\end{aligned}$$

Since $\widehat{A}_+$ is a subalgebra of $U_{u,v}(\mathfrak{g}_+)$, $\psi_+(u^{|\underline{j}|}\, x_{\underline{j}}) \in \widehat{A}_+$ for any d-tuple $\underline{j}$. Since $\widehat{A}_+$ is topologically free (hence complete) by Lemma 7.2, the map

$$\psi_+ : V_u(\mathfrak{g}_+)[[v]] \to U_{u,v}(\mathfrak{g}_+)$$

takes its values in $\widehat{A}_+$. We conclude with Formula (6.4). □

Lemma 7.5. *The* $\mathbf{C}[u][[v]]$*-linear map* $\psi_+ : V_u(\mathfrak{g}_+)[[v]] \to U_{u,v}(\mathfrak{g}_+)$ *extends to a* $\mathbf{C}[[u,v]]$*-linear map* $\widehat{\psi}_+ : \widehat{V}_u(\mathfrak{g}_+)[[v]] \to U_{u,v}(\mathfrak{g}_+)$*. The map* $\widehat{\psi}_+$ *is injective, its image is* $\widehat{A}_+$*:*

$$\widehat{\psi}_+(\widehat{V}_u(\mathfrak{g}_+)[[v]]) = \widehat{A}_+,$$

and $p_v \circ \widehat{\psi}_+ : \widehat{V}_u(\mathfrak{g}_+)[[v]] \to \widehat{V}_u(\mathfrak{g}_+)$ *is the projection sending* v *to* 0*.*

Proof. Any element of $\widehat{V}_u(\mathfrak{g}_+)$ is of the form $w = \sum_{m\geq 0} a_m\, u^m$, where

$$a_m = \sum_{\underline{j};\, |\underline{j}|\leq m} \nu_{\underline{j}}^{(m)} x_{\underline{j}}$$

and $\nu_{\underline{j}}^{(m)} \in \mathbf{C}$. By Lemma 6.8 (a), the element $\psi_+(a_m\, u^m)$ belongs to $u^m U_{u,v}(\mathfrak{g}_+)$. Since $U_{u,v}(\mathfrak{g}_+)$ is topologically free over $\mathbf{C}[[u,v]]$, the series $\sum_{m\geq 0} \psi_+(a_m\, u^m)$ converges in $U_{u,v}(\mathfrak{g}_+)$, so that we can define

$$\widehat{\psi}_+(w) = \sum_{m\geq 0} \psi_+(a_m\, u^m).$$

By Lemma 7.4 and (7.1), for each $m \geq 0$, $\delta^n(\psi_+(a_m\, u^m))$ is divisible by u^n for all $n \geq 1$. It follows that $\delta^n(\widehat{\psi}_+(w))$ is also divisible by u^n for all $n \geq 1$. Therefore, $\widehat{\psi}_+(w) \in \widehat{A}_+$. Now any element of $\widehat{V}_u(\mathfrak{g}_+)[[v]]$ is of the form $\sum_{n\geq 0} w_n v^n$, where $w_n \in \widehat{V}_u(\mathfrak{g}_+)$ for all $n \geq 0$. Clearly, $\sum_{n\geq 0} \widehat{\psi}_+(w_n)v^n$ converges in $\widehat{A}_+$. We set $\widehat{\psi}_+\Big(\sum_{n\geq 0} w_n v^n\Big) = \sum_{n\geq 0} \widehat{\psi}_+(w_n)v^n$.

Lemma 6.8 (b) implies that $p_v \circ \widehat{\psi}_+$ is the identity on $\widehat{V}_u(\mathfrak{g}_+)$. Proceeding as in the proof of Theorem 6.9 (a), we see that $\widehat{\psi}_+$ is injective on $\widehat{V}_u(\mathfrak{g}_+)[[v]]$.

It remains to prove that the image of $\widehat{\psi}_+$ is $\widehat{A}_+$. For $a \in \widehat{A}_+$, set $w_0 = p_v(a) \in \widehat{V}_u(\mathfrak{g}_+)$, cf. Lemma 7.3 (a). Viewing w_0 as a constant formal power series in $\widehat{V}_u(\mathfrak{g}_+)[[v]]$, we consider the element $a - \widehat{\psi}_+(w_0) \in \widehat{A}_+$; it clearly sits in the kernel of p_v, which is $v\widehat{A}_+$ by Lemma 7.3 (b). Therefore, there exists $a_1 \in \widehat{A}_+$ such that $a - \widehat{\psi}_+(w_0) = va_1$. Similarly, there exist $w_1 \in \widehat{V}_u(\mathfrak{g}_+)$ and $a_2 \in \widehat{A}_+$ such that $a_1 - \widehat{\psi}_+(w_1) = va_2$. Repeating this construction and using the separatedness of $\widehat{A}_+$, we obtain an element $w = \sum_{n\geq 0} w_n v^n \in \widehat{V}_u(\mathfrak{g}_+)[[v]]$ such that $a = \widehat{\psi}_+(w)$. □

Corollary 7.6. *We have*

$$A_+ \cap v\widehat{A}_+ = vA_+ \quad \textit{and} \quad A_+ \cap u\widehat{A}_+ = uA_+.$$

Proof. By Theorem 6.9 (a) and Lemma 7.5, it is enough to check that

$$V_u(\mathfrak{g}_+)[[v]] \cap v\widehat{V}_u(\mathfrak{g}_+)[[v]] = vV_u(\mathfrak{g}_+)[[v]]$$

and

$$V_u(\mathfrak{g}_+)[[v]] \cap u\widehat{V}_u(\mathfrak{g}_+)[[v]] = uV_u(\mathfrak{g}_+)[[v]].$$

The former is clear; the latter is a consequence of $V_u(\mathfrak{g}_+) \cap u\widehat{V}_u(\mathfrak{g}_+) = uV_u(\mathfrak{g}_+)$, which is easy to check. □

7.7. Proof of Lemma 6.10. It is a consequence of Lemmas 7.3 (b) and 7.4, and the first inclusion of Corollary 7.6. □

We can now show that A_+ has a bialgebra structure. (For the definition of $\widehat{\otimes}_{\mathbf{C}[u][[v]]}$ and $\widehat{\otimes}_{\mathbf{C}[[u,v]]}$, see Sections 1.3 and 4.4.)

Proposition 7.8. (a) *We have the inclusions*

$$A_+ \,\widehat{\otimes}_{\mathbf{C}[u][[v]]}\, A_+ \subset \widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{A}_+ \subset U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_+).$$

(b) *If $\Delta_{u,v}$ denotes the comultiplication of $U_{u,v}(\mathfrak{g}_+)$, then*

$$\Delta_{u,v}\big(A_+\big) \subset A_+ \,\widehat{\otimes}_{\mathbf{C}[u][[v]]}\, A_+$$

and

$$\Delta_{u,v}\big(\widehat{A}_+\big) \subset \widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{A}_+.$$

Proof. (a) The inclusion $\widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{A}_+ \subset U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_+)$ follows from Proposition 6.2 (a), Lemma 7.2, and Lemma 4.5 (b).

Let us consider the first inclusion. By Theorem 6.9 (a) and Lemma 7.5, it is enough to prove that the natural map

$$V_u(\mathfrak{g}_+)[[v]] \,\widehat{\otimes}_{\mathbf{C}[u][[v]]}\, V_u(\mathfrak{g}_+)[[v]] \to \widehat{V}_u(\mathfrak{g}_+)[[v]] \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{V}_u(\mathfrak{g}_+)[[v]] \tag{7.2}$$

induced by the inclusion $V_u(\mathfrak{g}_+)[[v]] \subset \widehat{V}_u(\mathfrak{g}_+)[[v]]$ is injective. By definition of $\widehat{\otimes}_{\mathbf{C}[u][[v]]}$, we see that

$$\begin{aligned}
&V_u(\mathfrak{g}_+)[[v]] \,\widehat{\otimes}_{\mathbf{C}[u][[v]]}\, V_u(\mathfrak{g}_+)[[v]] \\
&= \big(V_u(\mathfrak{g}_+) \otimes_{\mathbf{C}[u]} V_u(\mathfrak{g}_+)\big)[[v]] = V_u(\mathfrak{g}_+ \oplus \mathfrak{g}_+)[[v]].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\widehat{V}_u(\mathfrak{g}_+)[[v]] \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{V}_u(\mathfrak{g}_+)[[v]] \\
&= \varprojlim_n \Big(\widehat{V}_u(\mathfrak{g}_+)[[v]]/(u,v)^n \otimes_{\mathbf{C}[[u,v]]/(u,v)^n} \widehat{V}_u(\mathfrak{g}_+)[[v]]/(u,v)^n\Big) \\
&= \varprojlim_n \Big(V_u(\mathfrak{g}_+)[v]/(u,v)^n \otimes_{\mathbf{C}[u,v]/(u,v)^n} V_u(\mathfrak{g}_+)[v]/(u,v)^n\Big) \\
&= \varprojlim_n \Big(V_u(\mathfrak{g}_+) \otimes_{\mathbf{C}[u]} V_u(\mathfrak{g}_+)\Big)[v]/(u,v)^n \\
&= \varprojlim_n V_u(\mathfrak{g}_+ \oplus \mathfrak{g}_+)[v]/(u,v)^n \\
&= \varprojlim_n \widehat{V}_u(\mathfrak{g}_+ \oplus \mathfrak{g}_+)[[v]]/(u,v)^n \\
&= \widehat{V}_u(\mathfrak{g}_+ \oplus \mathfrak{g}_+)[[v]].
\end{aligned}$$

The last equality holds because $\widehat{V}_u(\mathfrak{g}_+ \oplus \mathfrak{g}_+)[[v]]$ is a topologically free $\mathbf{C}[[u,v]]$-module. The injectivity of (7.2) follows.

(b) In order to prove that the image of A_+ under $\Delta_{u,v}$ lies in the subalgebra $A_+ \widehat{\otimes}_{\mathbf{C}[u][[v]]} A_+$, it is enough to show that $\Delta_{u,v}\big(\psi_+(ux)\big)$ belongs to this subalgebra for all $x \in \mathfrak{g}_+$.

Let us consider the linear form $f_x \in U_h^*(\mathfrak{g}_-)$ of Section 5.5. Since $\rho_+ : U_h^*(\mathfrak{g}_-) \to U_h(\mathfrak{g}_+)$ is a morphism of coalgebras (see [**EK96**, Proposition 4.8]), we have $\Delta_h(\rho_+(f_x)) \in \mathrm{Im}\,\rho_+ \widehat{\otimes}_{\mathbf{C}[[h]]} \mathrm{Im}\,\rho_+$.

It follows from Lemma 5.6 that for any element $a \in U_h(\mathfrak{g}_+) \widehat{\otimes}_{\mathbf{C}[[h]]} U_h(\mathfrak{g}_+)$, there exists a unique family $\nu^{(n)}_{\underline{j},\underline{k}} \in \mathbf{C}$ indexed by a nonnegative integer n and two d-tuples $\underline{j}$ and $\underline{k}$ such that

$$a = \sum_{n\geq 0} \left(\sum_{|\underline{j}|+|\underline{k}|\leq c(n)} \nu^{(n)}_{\underline{j},\underline{k}}\, t_{\underline{j}} \otimes t_{\underline{k}} \right) h^n,$$

where $c(n)$ is an integer depending on a and n. If, in addition, $a \in \mathrm{Im}\,\rho_+ \widehat{\otimes}_{\mathbf{C}[[h]]} \mathrm{Im}\,\rho_+$, then $c(n) = n$, i.e., $\nu^{(n)}_{\underline{j},\underline{k}} = 0$ whenever $n < |\underline{j}| + |\underline{k}|$. Applying this to $a = \Delta_h(\rho_+(f_x))$, we obtain a family $\nu^{(n)}_{\underline{j},\underline{k}} \in \mathbf{C}$ as above such that

$$\begin{aligned}
&\Delta_h(\rho_+(f_x)) \\
&= \sum_{n\geq 0} \left(\sum_{|\underline{j}|+|\underline{k}|\leq n} \nu^{(n)}_{\underline{j},\underline{k}}\, t_{\underline{j}} \otimes t_{\underline{k}} \right) h^n \\
&= \sum_{\substack{n\geq 0, \underline{j}, \underline{k} \\ |\underline{j}|+|\underline{k}|\leq n}} \nu^{(n)}_{\underline{j},\underline{k}}\, \rho_+(f_{x_1})^{j_1} \dots \rho_+(f_{x_d})^{j_d} \otimes \rho_+(f_{x_1})^{k_1} \dots \rho_+(f_{x_d})^{k_d}\, h^{n-|\underline{j}|-|\underline{k}|},
\end{aligned}$$

where $\underline{j} = (j_1, \dots, j_d)$ and $\underline{k} = (k_1, \dots, k_d)$. Extending the scalars from $\mathbf{C}[[h]]$ to $\mathbf{C}[[u,v]]$ and using (6.3), we obtain

$$\begin{aligned}
&\Delta_{u,v}\big(\rho_+(\widetilde{f}_x)\big) \\
&= \sum_{\substack{n\geq 0; \underline{j}, \underline{k} \\ |\underline{j}|+|\underline{k}|\leq n}} \nu^{(n)}_{\underline{j},\underline{k}}\, \rho_+(\widetilde{f}_{x_1})^{j_1} \dots \rho_+(\widetilde{f}_{x_d})^{j_d} \otimes \rho_+(\widetilde{f}_{x_1})^{k_1} \dots \rho_+(\widetilde{f}_{x_d})^{k_d} (uv)^{n-|\underline{j}|-|\underline{k}|} \\
&= \sum_{n\geq 0; \underline{j}, \underline{k}, |\underline{j}|+|\underline{k}|\leq n} \nu^{(n)}_{\underline{j},\underline{k}}\, \psi_+(u^{|\underline{j}|} x_{\underline{j}}) \otimes \psi_+(u^{|\underline{k}|} x_{\underline{k}})\, u^{n-|\underline{j}|-|\underline{k}|}\, v^n \\
&= \sum_{n\geq 0} \left(\sum_{\underline{j}, \underline{k};\, |\underline{j}|+|\underline{k}|\leq n} \nu^{(n)}_{\underline{j},\underline{k}}\, u^{n-|\underline{j}|-|\underline{k}|}\, \psi_+(u^{|\underline{j}|} x_{\underline{j}}) \otimes \psi_+(u^{|\underline{k}|} x_{\underline{k}}) \right) v^n.
\end{aligned}$$

Therefore, $v\,\Delta_{u,v}\big(\psi_+(ux)\big) = \Delta_{u,v}\big(\rho_+(\widetilde{f}_x)\big)$ is a formal power series in v whose coefficients belong to the $\mathbf{C}[u]$-linear span of the elements $\psi_+(u^{|\underline{j}|}x_{\underline{j}})\otimes\psi_+(u^{|\underline{k}|}x_{\underline{k}})$. Hence, $v\,\Delta_{u,v}\big(\psi_+(ux)\big)$ belongs to $A_+\,\widehat{\otimes}_{\mathbf{C}[u][[v]]}\,A_+$.

The element $\Delta_{u,v}\big(\psi_+(ux)\big)\in U_{u,v}(\mathfrak{g}_+)\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\,U_{u,v}(\mathfrak{g}_+)$ can be expanded as

$$\Delta_{u,v}\big(\psi_+(ux)\big) = \sum_i a_i\otimes z_i,$$

where $(a_i)_i$ is a basis of the topologically free $\mathbf{C}[[u,v]]$-module $U_{u,v}(\mathfrak{g}_+)$ and $z_i\in U_{u,v}(\mathfrak{g}_+)$. Since

$$\sum_i a_i\otimes vz_i = v\,\Delta_{u,v}\big(\psi_+(ux)\big)\in U_{u,v}(\mathfrak{g}_+)\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\,\widehat{A}_+,$$

we have $vz_i\in\widehat{A}_+$ for all i. By Lemma 7.3 (b) it follows that $z_i\in\widehat{A}_+$ for all i. Now taking a basis $(b_j)_j$ of the topologically free $\mathbf{C}[[u,v]]$-module $\widehat{A}_+$, we can write

$$\Delta_{u,v}\big(\psi_+(ux)\big) = \sum_j z'_j\otimes b_j,$$

where $z'_j\in U_{u,v}(\mathfrak{g}_+)$. Since $\sum_j vz'_j\otimes b_j = v\,\Delta_{u,v}\big(\psi_+(ux)\big)\in\widehat{A}_+\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\,\widehat{A}_+$, we have $vz'_j\in\widehat{A}_+$, hence $z'_j\in\widehat{A}_+$ for all j. Therefore,

$$\Delta_{u,v}\big(\psi_+(ux)\big)\in\widehat{A}_+\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\,\widehat{A}_+.$$

The desired inclusion $\Delta_{u,v}\big(\psi_+(ux)\big)\in A_+\,\widehat{\otimes}_{\mathbf{C}[u][[v]]}\,A_+$ follows from

$$A_+^{\widehat{\otimes}2}\cap v\Big(\widehat{A}_+^{\widehat{\otimes}2}\Big) = v\Big(A_+^{\widehat{\otimes}2}\Big). \tag{7.3}$$

In view of Theorem 6.9 (a) and Lemma 7.5, Equality (7.3) is equivalent to

$$V_u(\mathfrak{g}_+)[[v]]^{\widehat{\otimes}2}\cap v\Big(\widehat{V}_u(\mathfrak{g}_+)[[v]]^{\widehat{\otimes}2}\Big) = v\Big(V_u(\mathfrak{g}_+)[[v]]^{\widehat{\otimes}2}\Big),$$

which is proved by using the identifications of the proof of Part (a). We have thus established that $\Delta_{u,v}\big(A_+\big)\subset A_+\,\widehat{\otimes}_{\mathbf{C}[u][[v]]}\,A_+$.

We now check that $\Delta_{u,v}\big(\widehat{A}_+\big)\subset\widehat{A}_+\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\,\widehat{A}_+$. By Lemma 7.5 any element of $\widehat{A}_+$ is of the form $\widehat{\psi}_+(a)$, where $a\in\widehat{V}_u(\mathfrak{g}_+)[[v]]$. For any $N>0$, there exists $b\in V_u(\mathfrak{g}_+)[[v]]$ such that $a-b=\sum_{n\geq 0}a_nv^n$ with $a_n\in\bigoplus_{p\geq N}U^p(\mathfrak{g}_+)u^p$. Now, $\widehat{\psi}_+(b)=\psi_+(b)\in A_+$, and $\widehat{\psi}_+(a-b)\in u^NU_{u,v}(\mathfrak{g}_+)$ by Lemma 6.8 (a). Therefore,

$$\Delta_{u,v}(\widehat{\psi}_+(a))\equiv\Delta_{u,v}(\psi_+(b))\mod u^N. \tag{7.4}$$

It follows from the considerations above that

$$\Delta_{u,v}(\psi_+(b))\in A_+\,\widehat{\otimes}_{\mathbf{C}[u][[v]]}\,A_+\subset\widehat{A}_+\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\,\widehat{A}_+.$$

The latter $\mathbf{C}[[u,v]]$-module being topologically free, Formula (7.4) for all $N > 0$ implies

$$\Delta_{u,v}(\widehat{\psi_+}(a)) \in \widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{A}_+ .$$

□

Corollary 7.9. *The algebras A_+ and $\widehat{A}_+$ are subbialgebras of $U_{u,v}(\mathfrak{g}_+)$.*

7.10. Remark. The bialgebra A_+ has the following alternative definition. Define the $\mathbf{C}[u][[v]]$-bialgebra

$$U'_{u,v}(\mathfrak{g}_+) = \varprojlim_n U_h(\mathfrak{g}_+) \otimes_{\mathbf{C}[[h]]/(h^n)} \mathbf{C}[u][[v]]/(v^n),$$

where $\mathbf{C}[u][[v]]$ is a $\mathbf{C}[[h]]$-module by the morphism ι of Section 4.6. One can check that $U'_{u,v}(\mathfrak{g}_+)$ embeds as a subbialgebra into the bialgebra $U_{u,v}(\mathfrak{g}_+)$ of Section 6.1, that the map $\widetilde{\alpha}_+$ of Section 6.6 sends the $\mathbf{C}[u][[v]]$-module $U'_{u,v}(\mathfrak{g}_+)$ isomorphically onto $U(\mathfrak{g}_+)[u][[v]]$, and that the bialgebra morphism p_v of Lemma 6.7 maps $U'_{u,v}(\mathfrak{g}_+)$ onto the bialgebra $U(\mathfrak{g}_+)[u]$ of polynomials with coefficients in $U(\mathfrak{g}_+)$.

Adapting the proofs of Sections 6–7, one can prove that A_+ is in $U'_{u,v}(\mathfrak{g}_+)$ and that

$$A_+ = \left\{ a \in U'_{u,v}(\mathfrak{g}_+) \;\middle|\; \delta^n(a) \in u^n U'_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n} \text{ for all } n \geq 1 \right\}.$$

8. Proofs of Theorems 2.3, 2.6, and 2.9 (I).

Let $A_{u,v}(\mathfrak{g}_+) = A_+$ be the bialgebra constructed in Sections 6–7. We first prove Theorem 2.6 and then determine A_+/uA_+ as an algebra (Part I of Theorem 2.9). The proof of Theorem 2.3 follows.

8.1. Proof of Theorem 2.6. It follows from Lemma 6.7, Lemma 6.8 (b), Theorem 6.9, and Corollary 7.9 applied to $\mathfrak{g}_+ = \mathfrak{g}$ that the morphism of bialgebras $p_v : U_{u,v}(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[u]]$ restricts to a surjective morphism of bialgebras $p_v : A_+ \to V_u(\mathfrak{g}_+)$ whose kernel is vA_+. Therefore, the induced map $A_+/vA_+ \to V_u(\mathfrak{g}_+)$ is an isomorphism of bialgebras. It remains to check that this isomorphism preserves the cobracket.

The bialgebra structure on A_+ induces on $V_u(\mathfrak{g}_+)$ a Poisson cobracket δ' given by (1.8), where $p = p_v$. We have to check that δ' coincides with the Poisson cobracket δ_u of $V_u(\mathfrak{g}_+)$ defined by (2.5). Since the algebra $V_u(\mathfrak{g}_+)$ is generated by the elements ux with $x \in \mathfrak{g}_+$, it suffices to show that $\delta'(ux) = \delta_u(ux)$ for all $x \in \mathfrak{g}_+$.

We identify the module $U_{u,v}(\mathfrak{g}_+)$ with $U(\mathfrak{g}_+)[[u,v]]$ via the isomorphism $\widetilde{\alpha}_+$ of Section 6.6. Let $a \in \widetilde{\alpha}_+^{-1}(ux) \subset U_{u,v}(\mathfrak{g}_+)$. We have $p_v(a) = ux$. Viewing $U_{u,v}(\mathfrak{g}_+)$ as a subbialgebra of $U_{u,v}(\mathfrak{d})$, we see by (5.3)–(5.4) that the comultiplication $\Delta_{u,v}$ of $U_{u,v}(\mathfrak{g}_+)$ satisfies

$$\Delta_{u,v}(a) \equiv \Delta(a) + uv \left[\Delta(a), \frac{r}{2}\right] \mod u^2v^2\, U_{u,v}(\mathfrak{d})^{\widehat{\otimes} 2},$$

where Δ is the standard comultiplication (2.4) on $U_{u,v}(\mathfrak{d}) = U(\mathfrak{d})[[u,v]]$. Therefore,

$$\frac{\Delta_{u,v}(a) - \Delta_{u,v}^{\mathrm{op}}(a)}{v} \equiv u \left[\Delta(a), \frac{r - r_{21}}{2}\right] \mod u^2 v\, U_{u,v}(\mathfrak{d})^{\widehat{\otimes} 2}.$$

It follows that

$$\begin{aligned}\delta'(ux) &= (p_v \otimes p_v)\left(\frac{\Delta_{u,v}(a) - \Delta_{u,v}^{\mathrm{op}}(a)}{v}\right)\\ &= u\left[\Delta(ux), \frac{r - r_{21}}{2}\right]\\ &= u^2\left[x\otimes 1 + 1\otimes x, \frac{r - r_{21}}{2}\right]\\ &= u^2\left([x\otimes 1 + 1\otimes x, r] - \frac{1}{2}[x\otimes 1 + 1\otimes x, r + r_{21}]\right)\\ &= u^2 [x\otimes 1 + 1\otimes x, r] = u^2 \delta(x) = \delta_u(ux).\end{aligned}$$

The vanishing of $[x\otimes 1 + 1\otimes x, r + r_{21}]$ is due to the invariance of the 2-tensor $r + r_{21}$. The identity $\delta(x) = [x\otimes 1 + 1\otimes x, r]$ follows from (5.2). □

8.2. Proof of Theorem 2.9. Part I. We prove here that $A_+/uA_+ = S(\mathfrak{g}_+)[[v]]$ as a $\mathbf{C}[[v]]$-algebra. We first observe that the algebra A_+/uA_+ is commutative. Indeed, $A_+/uA_+ \subset \widehat{A}_+/u\widehat{A}_+$ by the second equality of Corollary 7.6. By Proposition 3.5, the quotient algebra $\widehat{A}_+/u\widehat{A}_+$ is commutative; hence, so is A_+/uA_+.

Consider the $\mathbf{C}[u][[v]]$-linear isomorphism $\psi_+ : V_u(\mathfrak{g}_+)[[v]] \to A_+$ of Theorem 6.9 (a). It induces a $\mathbf{C}[[v]]$-linear isomorphism

$$\Psi_+ : S(\mathfrak{g}_+)[[v]] = V_u(\mathfrak{g}_+)[[v]]/uV_u(\mathfrak{g}_+)[[v]] \to A_+/uA_+.$$

By definition,

$$\Psi_+(x_1^{j_1} \dots x_d^{j_d}) = v^{-|\underline{j}|}\, \rho_+(\widetilde{f}_{x_1})^{j_1} \dots \rho_+(\widetilde{f}_{x_d})^{j_d} \mod u\, A_+ \tag{8.1}$$

for all d-tuples $\underline{j} = (j_1, \dots, j_d)$. (Recall that $(x_1, \dots, x_d)$ is a fixed basis of $\mathfrak{g}_+$.) Since A_+/uA_+ is commutative, Ψ_+ is an algebra morphism. □

8.3. Proof of Theorem 2.3. By Theorem 6.9 (a), the $\mathbf{C}[u][[v]]$-module A_+ is isomorphic to $V_u(\mathfrak{g}_+)[[v]]$, hence to $S(\mathfrak{g}_+)[u][[v]]$ (see Section 2.4 and Lemma 2.5). As a consequence of Theorem 2.6 and Section 8.2, the bialgebra A_+ is commutative modulo u and cocommutative modulo v. It follows from Theorem 2.6 and Lemma 2.5 that $A_+/(u,v) = S(\mathfrak{g})$ as bi-Poisson bialgebras. □

8.4. Remark. Since A_+ is a $\mathbf{C}[u][[v]]$-module, we may set $u = 1$. We claim that the quotient bialgebra $A_+/(u-1)$ is isomorphic to Etingof and Kazhdan's bialgebra $U_v(\mathfrak{g}_+)$ of Section 5.4 (with h replaced by v). Indeed, the bialgebra inclusion $A_+ \subset U'_{u,v}(\mathfrak{g}_+)$ of Remark 7.10 induces a bialgebra morphism $\xi : A_+/(u-1) \to U'_{u,v}(\mathfrak{g}_+)/(u-1) = U_v(\mathfrak{g}_+)$. It remains to show that ξ is an isomorphism. The isomorphism ψ_+ of Theorem 6.9 (a) induces a $\mathbf{C}[[v]]$-linear isomorphism $\overline{\psi}_+ : U(\mathfrak{g}_+)[[v]] = V_u(\mathfrak{g}_+)[[v]]/(u-1) \to A_+/(u-1)$. It now suffices to check that the composite map $\xi \circ \overline{\psi}_+$ is an isomorphism. By Sections 5.5, 6.4, and 6.6 the map $\xi \circ \overline{\psi}_+$ sends $x_{\underline{j}} = x_1^{j_1} \dots x_d^{j_d} \in U(\mathfrak{g}_+)[[v]]$ to $v^{-|\underline{j}|}\, \rho_+(f_{x_1})^{j_1} \dots \rho_+(f_{x_d})^{j_d}$ for all d-tuples $(j_1, \dots , j_d)$. In view of Lemma 5.6 (a) it follows that $\xi \circ \overline{\psi}_+$ is an isomorphism modulo v; hence, it is an isomorphism of topologically free $\mathbf{C}[[v]]$-modules.

9. A nondegenerate bialgebra pairing.

In this section, we construct a pairing between A_+ and a $\mathbf{C}[v][[u]]$-bialgebra A_-, using the element $R_{u,v} \in U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-)$ introduced in Section 6. We start by defining A_-, then we prove an important property of $R_{u,v}$. We resume the notation of Sections 5-8.

9.1. The Bialgebras A_- and $\widehat{A}_-$. They are defined by analogy with A_+ and $\widehat{A}_+$. Let us begin with the definition of A_-. Consider the $\mathbf{C}[[h]]$-linear isomorphism $\alpha_+ : U_h(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[h]]$ of Section 6.6. We have $\alpha_+(1) = 1$ and $\alpha_+ \equiv \mathrm{id}$ modulo h. Choose a $\mathbf{C}$-linear projection $\pi_+ : U(\mathfrak{g}_+) \to U^1(\mathfrak{g}_+) = \mathbf{C} \oplus \mathfrak{g}_+$ that is the identity on $U^1(\mathfrak{g}_+)$. For any $y \in \mathfrak{g}_-$ we define a $\mathbf{C}$-linear form $\langle -, y\rangle : U^1(\mathfrak{g}_+) \to \mathbf{C}$ extending the evaluation map $\langle -, y\rangle : \mathfrak{g}_+ \to \mathbf{C}$ and such that $\langle 1, y\rangle = 0$. We obtain a $\mathbf{C}[[h]]$-linear form $g_y : U_h(\mathfrak{g}_+) \to \mathbf{C}[[h]]$ by

$$g_y(a) = \langle \pi_+\alpha_+(a), y\rangle = \sum_{n\geq 0} \langle \pi_+(a_n), y\rangle\, h^n, \tag{9.1}$$

where $a \in U_h(\mathfrak{g}_+)$ and the elements $a_n \in U(\mathfrak{g}_+)$ are defined by $\alpha_+(a) = \sum_{n\geq 0} a_n h^n$. We have $g_y(1) = 0$.

By extension of scalars, we obtain a $\mathbf{C}[[u, v]]$-linear form $\widetilde{g}_y : U_{u,v}(\mathfrak{g}_+) \to \mathbf{C}[[u, v]]$ such that $\widetilde{g}_y(1) = 0$. We apply the map $\rho_- : U^*_{u,v}(\mathfrak{g}_+) \to U_{u,v}(\mathfrak{g}_-)$ of (6.2) to $\widetilde{g}_y$. By Lemma 6.5 adapted to this situation, $\rho_-(\widetilde{g}_y) \in U_{u,v}(\mathfrak{g}_-)$ is divisible by uv.

Let $V_v(\mathfrak{g}_-)$ be the $\mathbf{C}[v]$-bialgebra introduced in Section 2.4, where we have now replaced u by v. Let $(y_1, \dots , y_d)$ be the basis of $\mathfrak{g}_-$ dual to the fixed basis $(x_1, \dots , x_d)$ of $\mathfrak{g}_+$. The family $(v^{|\underline{k}|}\, y_{\underline{k}})$, where $\underline{k}$ runs over all d-tuples of nonnegative integers, is a $\mathbf{C}[v]$-basis of $V_v(\mathfrak{g}_-)$. We define a $\mathbf{C}[v]$-linear map $\psi_- : V_v(\mathfrak{g}_-) \to U_{u,v}(\mathfrak{g}_-)$ by $\psi_-(1) = 1$ and

$$\psi_-(v^{|\underline{k}|}\, y_{\underline{k}}) = u^{-|\underline{k}|}\, \rho_-(\widetilde{g}_{y_1})^{k_1} \dots \rho_-(\widetilde{g}_{y_d})^{k_d}, \tag{9.2}$$

where $\underline{k} = (k_1, \dots, k_d)$ is a d-tuple with $|\underline{k}| \geq 1$. This map extends uniquely to a $\mathbf{C}[v][[u]]$-linear map, still denoted ψ_-, from $V_v(\mathfrak{g}_-)[[u]]$ to $U_{u,v}(\mathfrak{g}_-)$ by

$$\psi_-\Big(\sum_{n\geq 0} w_n u^n\Big) = \sum_{n\geq 0} \psi_-(w_n) u^n,$$

where $w_0, w_1, w_2, \dots \in V_v(\mathfrak{g}_-)$. We then define the $\mathbf{C}[v][[u]]$-module A_- by

$$A_- = \psi_-(V_v(\mathfrak{g}_-)[[u]]) \subset U_{u,v}(\mathfrak{g}_-). \tag{9.3}$$

Recall the isomorphism $\alpha_- : U_h(\mathfrak{g}_-) \cong U(\mathfrak{g}_-)[[h]]$ of Section 5.5. It induces a $\mathbf{C}[[u,v]]$-linear isomorphism $\widetilde{\alpha}_- : U_{u,v}(\mathfrak{g}_-) \cong U(\mathfrak{g}_-)[[u,v]]$ such that $\widetilde{\alpha}_- \equiv \mathrm{id}$ modulo uv. Consider the composed map

$$p_u : U_{u,v}(\mathfrak{g}_-) \xrightarrow{\widetilde{\alpha}_-} U(\mathfrak{g}_-)[[u,v]] \to U(\mathfrak{g}_-)[[v]],$$

where the second map is the projection $u \mapsto 0$. The map p_u is a morphism of bialgebras when we equip $U(\mathfrak{g}_-)[[v]]$ with the power series multiplication and the comultiplication (2.4). Moreover, p_u sends A_- onto $V_v(\mathfrak{g}_-)$ and $p_u \circ \psi_- : V_v(\mathfrak{g}_-)[[u]] \to V_v(\mathfrak{g}_-)$ is the projection sending u to 0. This is proved as in Section 6.

By analogy with Section 7.1, we define a $\mathbf{C}[[u,v]]$-subalgebra $\widehat{A}_-$ of $U_{u,v}(\mathfrak{g}_-)$ by

$$\widehat{A}_- = \Big\{ a \in U_{u,v}(\mathfrak{g}_-) \;\big|\; \delta^n(a) \in v^n U_{u,v}(\mathfrak{g}_-)^{\widehat{\otimes} n} \text{ for all } n \geq 1 \Big\}. \tag{9.4}$$

It is clear that the results of Sections 6–8 apply to A_- and $\widehat{A}_-$, namely we have the following properties.

(i) The map $\psi_- : V_v(\mathfrak{g}_-)[[u]] \to A_-$ is an isomorphism of $\mathbf{C}[v][[u]]$-modules. It extends to an isomorphism of $\mathbf{C}[[u,v]]$-modules $\widehat{\psi}_- : \widehat{V}_v(\mathfrak{g}_-)[[u]] \to \widehat{A}_-$.

(ii) $A_- \subset \widehat{A}_-$ are subalgebras of $U_{u,v}(\mathfrak{g}_-)$.

(iii) A_- is independent of the choices of the isomorphism $\alpha_+ : U_h(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[h]]$, of the projection $\pi_+ : U(\mathfrak{g}_+) \to U^1(\mathfrak{g}_+)$, and of the basis of $\mathfrak{g}_-$.

(iv) A_- and $\widehat{A}_-$ are topological bialgebras for the u-adic topology and the (u,v)-adic topology, respectively.

(v) A_- and $\widehat{A}_-$ are commutative modulo v and cocommutative modulo u. There are isomorphisms of co-Poisson bialgebras

$$A_-/uA_- = V_v(\mathfrak{g}_-), \tag{9.5}$$

isomorphisms of bi-Poisson bialgebras

$$A_-/(u,v)A_- = S(\mathfrak{g}_-), \tag{9.6}$$

and isomorphisms of algebras

$$A_-/vA_- = S(\mathfrak{g}_-)[[u]]. \tag{9.7}$$

Recall the two-variable universal R-matrix

$$R_{u,v} \in U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-)$$

of Section 6. We now give a stronger version of Lemma 6.3 (b).

Lemma 9.2. *The element* $R_{u,v} - 1\otimes 1$ *belongs to the submodules*

$$v\,\widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-) \quad \textit{and} \quad U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, u\,\widehat{A}_-$$

of $U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-)$.

Proof. Recall the element $R' \in U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-)$ of Lemma 6.3 (b). It is enough to show that

$$uR' \in \widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-) \quad \text{and} \quad vR' \in U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{A}_-.$$

We shall prove the first inclusion. The second one has a similar proof.

Let $(b_j)_j$ be a basis over $\mathbf{C}[[u,v]]$ of the (topologically free) $\mathbf{C}[[u,v]]$-module $U_{u,v}(\mathfrak{g}_-)$. We can expand R' as $R' = \sum_j z_j \otimes b_j$, where z_j are elements of $U_{u,v}(\mathfrak{g}_+)$. The proof of Lemma 7.4 shows that $(\delta^n \otimes \mathrm{id})(uvR')$ is divisible by u^n for any $n \geq 1$. Hence,

$$(\delta^n \otimes \mathrm{id})(R') = \sum_j \delta^n(z_j) \otimes b_j$$

is divisible by u^{n-1}. The elements b_j being linearly independent, it follows that $\delta^n(z_j)$ is divisible by u^{n-1} for all $n \geq 1$ and all j. Therefore, $uz_j \in \widehat{A}_+$ for all j and $uR' \in \widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-)$. ☐

Corollary 9.3. *The element* $R_{u,v}$ *belongs to the submodules*

$$\widehat{A}_+ \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-) \quad \textit{and} \quad U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, \widehat{A}_-.$$

We consider the dual $\mathbf{C}[[u,v]]$-modules $\widehat{A}_+^* = \mathrm{Hom}_{\mathbf{C}[[u,v]]}(\widehat{A}_+, \mathbf{C}[[u,v]])$ and $\widehat{A}_-^* = \mathrm{Hom}_{\mathbf{C}[[u,v]]}(\widehat{A}_-, \mathbf{C}[[u,v]])$. In view of Corollary 9.3, Formulas (6.2) now define $\mathbf{C}[[u,v]]$-linear maps $\widehat{A}_-^* \to U_{u,v}(\mathfrak{g}_+)$ and $\widehat{A}_+^* \to U_{u,v}(\mathfrak{g}_-)$, which we still denote by ρ_+ and ρ_-, respectively. The comultiplications of $\widehat{A}_+$ and of $\widehat{A}_-$ induce algebra structures on $\widehat{A}_+^*$ and $\widehat{A}_-^*$. As in Section 6, the map ρ_+ is an antimorphism of algebras and ρ_- is a morphism of algebras.

Lemma 9.4. *We have*

$$A_+ \subset \rho_+(\widehat{A}_-^*) \subset \widehat{A}_+ \quad \textit{and} \quad A_- \subset \rho_-(\widehat{A}_+^*) \subset \widehat{A}_-.$$

Proof. Let us prove the first two inclusions. The other two inclusions have similar proofs.

(a) We use the notation of Sections 6.4 and 6.6. We first show that, for any $x \in \mathfrak{g}_+$, the element $v^{-1}\,\rho_+(\widetilde{f}_x) \in A_+$ sits in $\rho_+\big(\widehat{A}_-^*\big)$. Indeed, if $b \in \widehat{A}_-$, then $\delta^1(b) = b - \varepsilon(b)1$ is divisible by v in $U_{u,v}(\mathfrak{g}_-)$. Hence,

$\widetilde{f}_x(b) = \widetilde{f}_x(b) - \varepsilon(b)\widetilde{f}_x(1) \in \mathbf{C}[[u,v]]$ is divisible by v. We then define $\widehat{f}_x \in \widehat{A}_-^*$ by

$$\widehat{f}_x(b) = v^{-1}\,\widetilde{f}_x(b) \in \mathbf{C}[[u,v]] \tag{9.8}$$

for any $b \in \widehat{A}_-$. It follows that the restriction of $\widetilde{f}_x$ to $\widehat{A}_-$ equals $v\,\widehat{f}_x$. Therefore, $v^{-1}\,\rho_+(\widetilde{f}_x) = \rho_+(\widehat{f}_x) \in \rho_+(\widehat{A}_-^*)$.

By Section 6.6, any element $a \in A_+$ is of the form

$$a = \sum_{n\geq 0} v^n \left(\sum_{\underline{j}} P_{\underline{j}}(u)\, v^{-|\underline{j}|}\, \rho_+(\widetilde{f}_{x_1})^{j_1} \ldots \rho_+(\widetilde{f}_{x_d})^{j_d} \right),$$

where the sums inside the brackets are finite and $P_{\underline{j}}(u) \in \mathbf{C}[u]$. The formal power series

$$\sum_{n\geq 0} v^n \left(\sum_{\underline{j}} P_{\underline{j}}(u)\, \widehat{f}_{x_d}^{j_d} \ldots \widehat{f}_{x_1}^{j_1} \right)$$

converges to an element f in the topologically free $\mathbf{C}[[u,v]]$-module $\widehat{A}_-^*$. Since $\rho_+ : \widehat{A}_-^* \to U_{u,v}(\mathfrak{g}_+)$ is an antimorphism of algebras, we have $\rho_+(f) = a$. This implies that $A_+ \subset \rho_+(\widehat{A}_-^*)$.

(b) Let us prove that $\rho_+(\widehat{A}_-^*) \subset \widehat{A}_+$. Given $f \in \widehat{A}_-^*$, we have to check that $\delta^n(\rho_+(f))$ is divisible by u^n for all $n \geq 1$. By Lemma 9.2, $vR' \in U_{u,v}(\mathfrak{g}_+)\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\,\widehat{A}_-$, hence

$$v^n R'_{1,n+1} R'_{2,n+1} \cdots R'_{n-1,n+1} R'_{n,n+1} \in U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\,\widehat{A}_-.$$

This allows us to apply $\mathrm{id}\otimes f$ to $v^n R'_{1,n+1} R'_{2,n+1} \cdots R'_{n-1,n+1} R'_{n,n+1}$. A computation similar to the one in the proof of Lemma 7.4 yields

$$\begin{aligned}&\delta^n(\rho_+(f))\\ &= u^n(\mathrm{id}\otimes f)(v^n R'_{1,n+1} R'_{2,n+1} \cdots R'_{n-1,n+1} R'_{n,n+1}) \in u^n U_{u,v}(\mathfrak{g}_+)^{\widehat{\otimes} n}.\end{aligned}$$

□

Lemma 9.5. *For $a \in A_+$ and $b \in A_-$, the formulas*

$$(a,b)_{u,v} = \big(\rho_+^{-1}(a)\big)(b) = \big(\rho_-^{-1}(b)\big)(a), \tag{9.9}$$

yield a well-defined bialgebra pairing $A_+ \times A_-^{\mathrm{cop}} \to \mathbf{C}[[u,v]]$.

Here A_-^{cop} denotes the bialgebra A_- with the opposite comultiplication. The pairing $(\ ,\ )_{u,v}$ is in the sense of Section 2.10 with $K_1 = \mathbf{C}[u][[v]]$, $K_2 = \mathbf{C}[v][[u]]$, and $K = \mathbf{C}[[u,v]]$.

Proof. Let us prove that the expression $\big(\rho_-^{-1}(b)\big)(a)$ is well defined. It suffices to check that, if $g \in \widehat{A}_+^*$ satisfies $\rho_-(g) = 0$, then $g(a) = 0$. Suppose first that $a = \psi_+(u^{|\underline{j}|}x_{\underline{j}})$ for some d-tuple $\underline{j}$. By (6.3), $v^{|\underline{j}|}\,a = \rho_+(f)$, where

$f = \widetilde{f}_{x_d}^{j_d} \dots \widetilde{f}_{x_1}^{j_1} \in U_{u,v}^*(\mathfrak{g}_-)$. Applying $g \otimes f$ to $R_{u,v} \in \widehat{A}_+ \widehat{\otimes}_{\mathbf{C}[[u,v]]} U_{u,v}(\mathfrak{g}_-)$, we obtain

$$v^{|\underline{j}|}\, g(a) = g(\rho_+(f)) = (g \otimes f)(R_{u,v}) = f(\rho_-(g)) = 0.$$

Since $\mathbf{C}[[u,v]]$ is v-torsion-free, we obtain $g(a) = 0$. By $\mathbf{C}[[u,v]]$-linearity, $g(a) = 0$ for all $a \in \widehat{A}_+$.

A similar argument proves that $\big(\rho_+^{-1}(a)\big)(b)$ is well defined. Let us show that

$$\big(\rho_+^{-1}(a)\big)(b) = \big(\rho_-^{-1}(b)\big)(a). \tag{9.10}$$

By linearity, it suffices to consider the case $a = \psi_+(u^{|\underline{j}|} x_{\underline{j}})$ as above. We have $v^{|\underline{j}|}\, a = \rho_+(f)$ with $f \in U_{u,v}^*(\mathfrak{g}_-)$. Let $g \in \rho_-^{-1}(b) \subset \widehat{A}_+^*$. Then

$$\begin{aligned} v^{|\underline{j}|}\, \big(\rho_+^{-1}(a)\big)(b) &= f(b) = f(\rho_-(g)) = (g \otimes f)(R_{u,v}) \\ &= g(\rho_+(f)) = v^{|\underline{j}|}\, g(a) = v^{|\underline{j}|}\, \big(\rho_-^{-1}(b)\big)(a). \end{aligned}$$

Hence, (9.10) holds.

That $(\ ,\ )_{u,v}$ is a bialgebra pairing follows directly from the fact that ρ_+ is an antimorphism of algebras and ρ_- is a morphism of algebras. □

9.6. Remark. Proceeding as in the proof of Lemma 9.5, we can show that the maps $\rho_+ : \widehat{A}_-^* \to \widehat{A}_+$ and $\rho_- : \widehat{A}_+^* \to \widehat{A}_-$ are injective.

9.7. Induced Bialgebra Pairings. Passing to the quotient modulo u, the pairing $(\ ,\ )_{u,v}$ induces a bialgebra pairing

$$(\ ,\ )_v : A_+/uA_+ \times A_-/uA_- \to \mathbf{C}[[v]]. \tag{9.11}$$

(The bialgebra A_-/uA_- is cocommutative by (9.5), so that $(A_-/uA_-)^{\mathrm{cop}} = A_-/uA_-$.) Recall the isomorphism of algebras $\Psi_+ : S(\mathfrak{g}_+)[[v]] \to A_+/uA_+$ defined by (8.1). On the other hand, the composition of $\psi_- : V_v(\mathfrak{g}_-) \to A_-$ defined by (9.2) and the projection $A_- \to A_-/uA_-$ is an isomorphism of $\mathbf{C}[v]$-bialgebras $\Psi'_- : V_v(\mathfrak{g}_-) \to A_-/uA_-$, which is defined on the $\mathbf{C}[v]$-basis $(v^{|\underline{k}|}\, y_{\underline{k}})_{\underline{k}}$ of $V_v(\mathfrak{g}_-)$ by

$$\begin{aligned} \Psi'_-(v^{|\underline{k}|}\, y_{\underline{k}}) &= \psi_-(v^{|\underline{k}|}\, y_{\underline{k}}) \mod uA_- \\ &= u^{-|\underline{k}|}\, \rho_-(\widetilde{g}_{y_1})^{k_1} \cdots \rho_-(\widetilde{g}_{y_d})^{k_d} \mod uA_-, \end{aligned} \tag{9.12}$$

where $\underline{k} = (k_1, \dots, k_d)$ and the maps $\widetilde{g}_{y_i}$ were introduced in Section 9.1.

Lemma 9.8. *If $\underline{j} = (j_1, \dots, j_d)$ and $\underline{k} = (k_1, \dots, k_d)$ are d-tuples of non-negative integers, then*

$$(\Psi_+(x_{\underline{j}}), \Psi'_-(v^{|\underline{k}|}\, y_{\underline{k}}))_v = \begin{cases} 0 & \text{if } |\underline{j}| > |\underline{k}|, \\ \delta_{j_1,k_1} \dots \delta_{j_d,k_d}\, j_1! \dots j_d! & \text{if } |\underline{j}| = |\underline{k}|, \\ \in v^{|\underline{k}|-|\underline{j}|}\, \mathbf{C}[[v]] & \text{if } |\underline{j}| < |\underline{k}|. \end{cases}$$

Proof. We first claim that for any $x \in \mathfrak{g}_+$ and any d-tuple $\underline{k} = (k_1, \dots, k_d)$,

$$(\Psi_+(x), \Psi'_-(v^{|\underline{k}|}\, y_{\underline{k}}))_v = \begin{cases} 0 & \text{if } |\underline{k}| = 0, \\ v^{|\underline{k}|-1}\, \langle x, \pi_-(y_{\underline{k}})\rangle & \text{if } |\underline{k}| \geq 1. \end{cases} \tag{9.13}$$

Indeed, consider the diagram

$$\begin{array}{ccccc} U_{u,v}(\mathfrak{g}_-) & \xrightarrow{\widetilde{\alpha}_-} & U(\mathfrak{g}_-)[[u,v]] & \xrightarrow{\langle x, \pi_-(-)\rangle} & \mathbf{C}[[u,v]] \\ \downarrow{\scriptstyle p_u} & & \downarrow & & \downarrow \\ U(\mathfrak{g}_-)[[v]] & \xrightarrow{\mathrm{id}} & U(\mathfrak{g}_-)[[v]] & \xrightarrow{\langle x, \pi_-(-)\rangle} & \mathbf{C}[[v]] \end{array}$$

where the unmarked vertical maps are the projections sending u to 0. The left-hand and the right-hand squares commute by definition of p_u and by linearity, respectively. It follows that, for any $b \in U_{u,v}(\mathfrak{g}_-)$,

$$\widetilde{f}_x(b) \mod u\mathbf{C}[[u,v]] = \langle x, \pi_-(p_u(b))\rangle. \tag{9.14}$$

Since $\Psi_+(x) = v^{-1}\, \rho_+(\widetilde{f}_x) \mod uA_+$ and $\Psi'_-(v^{|\underline{k}|}\, y_{\underline{k}}) = \psi_-(v^{|\underline{k}|}\, y_{\underline{k}}) \mod uA_-$, we have

$$\begin{aligned} (\Psi_+(x), \Psi'_-(v^{|\underline{k}|}\, y_{\underline{k}}))_v &= v^{-1}\, \widetilde{f}_x(\psi_-(v^{|\underline{k}|}\, y_{\underline{k}})) \mod u\mathbf{C}[[u,v]] \\ &= v^{-1}\, \langle x, \pi_-(p_u(\psi_-(v^{|\underline{k}|}\, y_{\underline{k}})))\rangle \\ &= v^{-1}\, \langle x, v^{|\underline{k}|}\, \pi_-(y_{\underline{k}})\rangle = v^{|\underline{k}|-1}\, \langle x, \pi_-(y_{\underline{k}})\rangle \end{aligned}$$

for all $\underline{k}$. If $|\underline{k}| = 0$, then $v^{|\underline{k}|}\, y_{\underline{k}} = 1$, on which $\langle x, -\rangle$ vanishes. This proves (9.13).

Formula (9.13) implies that Lemma 9.8 holds for any $\underline{j}$ and $\underline{k}$ such that $|\underline{j}| = 1$. For the general case, observe that

$$\begin{aligned} &(\Psi_+(x_{\underline{j}}), \Psi'_-(v^{|\underline{k}|}\, y_{\underline{k}}))_v \\ &= (\Psi_+(x_1)^{j_1} \dots \Psi_+(x_d)^{j_d}, \Psi'_-(v^{|\underline{k}|}\, y_{\underline{k}}))_v \\ &= (\Psi_+(x_1)^{\otimes j_1} \otimes \dots \otimes \Psi_+(x_d)^{\otimes j_d}, \Delta^{|\underline{j}|}_{u,v}(\Psi'_-(v^{|\underline{k}|}\, y_{\underline{k}})))_v \\ &= (\Psi_+(x_1)^{\otimes j_1} \otimes \dots \otimes \Psi_+(x_d)^{\otimes j_d}, (\Psi'_-)^{\otimes |\underline{j}|}(\Delta^{|\underline{j}|}(v^{|\underline{k}|}\, y_{\underline{k}})))_v \end{aligned} \tag{9.15}$$

in view of Lemma 9.5, and the fact that Ψ_+ preserves the multiplication and Ψ'_- preserves the comultiplication. Here Δ is given by (2.4). Then the formulas of Lemma 9.8 for a general $\underline{j}$ follow from (2.4), (9.15), and the formulas for $\underline{j}$ such that $|\underline{j}| = 1$. $\square$

Passing to the quotients modulo v and modulo (u, v), the pairing $(\ ,\ )_{u,v}$ induces bialgebra pairings

$$(\ ,\ )_u : A_+/vA_+ \times (A_-/vA_-)^{\mathrm{cop}} \to \mathbf{C}[[u]] \tag{9.16}$$

and

$$A_+/(u,v) \times A_-/(u,v) \to \mathbf{C}. \tag{9.17}$$

The latter can also be obtained from the pairing $(\ ,\ )_v$ of (9.11) by setting $v = 0$.

The isomorphism $\Psi_+ : S(\mathfrak{g}_+)[[v]] \to A_+/uA_+$ defined by (8.1) induces a canonical isomorphism of bialgebras $S(\mathfrak{g}_+) = A_+/(u,v)$. The isomorphism $\Psi'_- : V_v(\mathfrak{g}_-) \to A_-/uA_-$ defined above induces a canonical isomorphism of bialgebras $S(\mathfrak{g}_-) = A_-/(u,v)$. We denote by $(\ ,\ )_0$ the bialgebra pairing $S(\mathfrak{g}_+) \times S(\mathfrak{g}_-) \to \mathbf{C}$ obtained from (9.17) under these identifications. Lemma 9.8 implies that

$$(x_{\underline{j}}, y_{\underline{k}})_0 = \begin{cases} 0 & \text{if } |\underline{j}| \neq |\underline{k}|, \\ \delta_{j_1,k_1} \dots \delta_{j_d,k_d}\, j_1! \dots j_d! & \text{if } |\underline{j}| = |\underline{k}| \end{cases} \tag{9.18}$$

for all d-tuples $\underline{j} = (j_1, \dots, j_d)$ and $\underline{k} = (k_1, \dots, k_d)$.

Corollary 9.9. *The pairings*

$$(\ ,\ )_{u,v} : A_+ \times A_-^{\mathrm{cop}} \to \mathbf{C}[[v]], \quad (\ ,\ )_v : A_+/uA_+ \times A_-/uA_- \to \mathbf{C}[[v]],$$

$$(\ ,\ )_u : A_+/vA_+ \times (A_-/vA_-)^{\mathrm{cop}} \to \mathbf{C}[[u]], \quad \textit{and} \quad (\ ,\ )_0 : S(\mathfrak{g}_+) \times S(\mathfrak{g}_-) \to \mathbf{C}$$

are nondegenerate.

Proof. It follows from (9.18) that $(\ ,\ )_0$ is nondegenerate. (Actually, $(\ ,\ )_0$ is the standard pairing between $S(\mathfrak{g}_+)$ and $S(\mathfrak{g}_-)$.)

We check that $(\ ,\ )_v$ is nondegenerate. Let $a \in A_+/uA_+$ such that $(a, -)_v = 0$. If $\bar{a}$ denotes the image of a under the projection $A_+/uA_+ \to S(\mathfrak{g}_+)$, then $(\bar{a}, -)_0 = 0$. It follows from the nondegeneracy of $(\ ,\ )_0$ that $\bar{a} = 0$, which implies that $a \in vA_+/uA_+$. Let $a_1 \in A_+/uA_+$ be such that $a = va_1$. We now have $(a_1, -)_v = 0$. A similar argument shows that a_1 is divisible by v, hence a is divisible by v^2 in A_+/uA_+. Proceeding in the same way, we see that a is divisible by any power of v, which is possible only if $a = 0$. A similar argument shows that $(-, b)_v = 0$ implies $b = 0$.

The nondegeneracy of $(\ ,\ )_{u,v}$ and $(\ ,\ )_u$ is proved in a similar fashion. □

10. Completion of the proof of Theorem 2.9.

Before proceeding to prove Theorem 2.9, we establish a few facts about a topological dual of the $\mathbf{C}[v]$-bialgebra

$$V_v(\mathfrak{g}_-) = \left\{ \sum_{n \geq 0} b_n v^n \in U(\mathfrak{g}_-)[v] \;\middle|\; b_n \in U^n(\mathfrak{g}_-) \text{ for all } n \geq 0 \right\}.$$

10.1. A Topological Dual. Inside the dual

$$V_v^*(\mathfrak{g}_-) = \mathrm{Hom}_{\mathbf{C}[v]}(V_v(\mathfrak{g}_-), \mathbf{C}[[v]])$$

of $V_v(\mathfrak{g}_-)$ there is a $\mathbf{C}[[v]]$-submodule $V_v^{\mathrm{o}}(\mathfrak{g}_-)$ consisting of all $f \in V_v^*(\mathfrak{g}_-)$ satisfying the following condition: For every $m \geq 0$ there exists $N \geq 0$ such that

$$f(U^p(\mathfrak{g}_-)\, v^p) \subset v^m \mathbf{C}[[v]] \tag{10.1}$$

for all $p \geq N$. In other words, $V_v^{\mathrm{o}}(\mathfrak{g}_-)$ consists of all $\mathbf{C}[v]$-linear forms that are continuous when we equip $\mathbf{C}[[v]]$ with the v-adic topology and $V_v(\mathfrak{g}_-)$ with the I-adic topology, where I is the two-sided ideal

$$I = \bigoplus_{p \geq 1} U^p(\mathfrak{g}_-)\, v^p \subset V_v(\mathfrak{g}_-).$$

Lemma 10.2. *The $\mathbf{C}[[v]]$-module $V_v^{\mathrm{o}}(\mathfrak{g}_-)$ is topologically free and*

$$V_v^{\mathrm{o}}(\mathfrak{g}_-) \bigcap v\, V_v^*(\mathfrak{g}_-) = v\, V_v^{\mathrm{o}}(\mathfrak{g}_-).$$

Proof. For the first statement, it is enough to check that, if $(f_n)_{n \geq 0}$ is a family of elements of $V_v^{\mathrm{o}}(\mathfrak{g}_-)$ such that $f_n \equiv f_{n+1} \bmod v^n$ for all $n \geq 0$, then there exists a unique $f \in V_v^{\mathrm{o}}(\mathfrak{g}_-)$ such that $f \equiv f_n \bmod v^n$ for all $n \geq 0$.

Indeed, since the linear forms f_n are with values in $\mathbf{C}[[v]]$, there exists a unique $f \in V_v^*(\mathfrak{g}_-)$ such that $f \equiv f_n \bmod v^n$ for all $n \geq 0$. Let us show that f belongs to $V_v^{\mathrm{o}}(\mathfrak{g}_-)$. Fix $m \geq 0$. By definition of $V_v^{\mathrm{o}}(\mathfrak{g}_-)$, there exists $N \geq 0$ such that $f_m(U^p(\mathfrak{g}_-)\, v^p) \subset v^m \mathbf{C}[[v]]$ for all $p \geq N$. Since $f \equiv f_m \bmod v^m$, we have $f(a) \equiv f_m(a) \bmod v^m$ for all $a \in V_v(\mathfrak{g}_-)$, hence

$$f(U^p(\mathfrak{g}_-)\, v^p) \equiv f_m(U^p(\mathfrak{g}_-)\, v^p) \equiv 0 \mod v^m$$

for all p. Therefore, $f(U^p(\mathfrak{g}_-)\, v^p) \subset v^m \mathbf{C}[[v]]$ for all $p \geq N$.

The second statement is an easy exercise left to the reader. □

We now relate $V_v^{\mathrm{o}}(\mathfrak{g}_-)$ to $S(\mathfrak{g}_+)[[v]]$. As before, we fix a basis $(x_1, \dots, x_d)$ of $\mathfrak{g}_+$ and the dual basis $(y_1, \dots, y_d)$ of $\mathfrak{g}_-$. The family of elements $x_{\underline{j}} = x_1^{j_1} \dots x_d^{j_d}$ indexed by all d-tuples $\underline{j} = (j_1, \dots, j_d)$ of nonnegative integers is a $\mathbf{C}$-basis of $S(\mathfrak{g}_+)$; the family of elements $(v^{|\underline{k}|}\, y_{\underline{k}})$ indexed by all d-tuples $\underline{k}$ of nonnegative integers is a $\mathbf{C}[v]$-basis of $V_v(\mathfrak{g}_-)$.

Suppose there exists a pairing $(\ ,\ ) : S(\mathfrak{g}_+)[[v]] \times V_v(\mathfrak{g}_-) \to \mathbf{C}[[v]]$ (in the sense of Section 2.10 with $K = K_1 = \mathbf{C}[[v]] \supset K_2 = \mathbf{C}[v]$) such that for all $\underline{j} = (j_1, \dots, j_d)$ and $\underline{k} = (k_1, \dots, k_d)$ we have

$$(x_{\underline{j}}, v^{|\underline{k}|}\, y_{\underline{k}}) = \begin{cases} 0 & \text{if } |\underline{j}| > |\underline{k}|, \\ \delta_{j_1,k_1} \dots \delta_{j_d,k_d}\, j_1! \dots j_d! & \text{if } |\underline{j}| = |\underline{k}|, \\ \in v^{|\underline{k}|-|\underline{j}|}\, \mathbf{C}[[v]] & \text{if } |\underline{j}| < |\underline{k}|. \end{cases} \tag{10.2}$$

The pairing $(\ ,\ )$ induces a $\mathbf{C}[[v]]$-linear map $\varphi : S(\mathfrak{g}_+)[[v]] \to V_v^*(\mathfrak{g}_-)$ defined for $a \in S(\mathfrak{g}_+)[[v]]$ by $\varphi(a) = (a, -)$.

Proposition 10.3. *Under Condition* (10.2) *the map* φ *sends* $S(\mathfrak{g}_+)[[v]]$ *isomorphically onto* $V_v^{\rm o}(\mathfrak{g}_-)$.

Proof. The same argument as in the proof of Corollary 9.9 shows that the pairing (,) is nondegenerate. This implies the injectivity of φ.

Let us prove that φ sends $S(\mathfrak{g}_+)[[v]]$ into $V_v^{\rm o}(\mathfrak{g}_-)$. Any element of $S(\mathfrak{g}_+)[[v]]$ is of the form

$$a = \sum_{n\geq 0;\, \underline{j}} \mu_{\underline{j},n}\, x_{\underline{j}}\, v^n,$$

where $(\mu_{\underline{j},n})_{n\geq 0;\, \underline{j}}$ is a family of scalars indexed by a nonnegative integer n and a d-tuple $\underline{j}$ of nonnegative integers, such that for all n there exists an integer N_n with $\mu_{\underline{j},n} = 0$ whenever $|\underline{j}| \geq N_n$.

In order to check that $\varphi(a)$ lies in $V_v^{\rm o}(\mathfrak{g}_-)$, we have to prove that, given $m \geq 0$, there exists N such that for all $p \geq N$ we have

$$\varphi(a)(U^p(\mathfrak{g}_+)\, v^p) \subset v^m \mathbf{C}[[v]].$$

Let N'_m be any integer such that $N'_m \geq N_n$ for all $n = 0, \ldots, m-1$. It is clear that $\mu_{\underline{j},n} = 0$ when $|\underline{j}| \geq N'_m$ and $0 \leq n \leq m-1$. For any $p \geq 1$, the family $(v^p\, y_{\underline{k}})$ with $|\underline{k}| \leq p$ is a basis of $U^p(\mathfrak{g}_-)\, v^p$. Let us compute $\varphi(a)(v^p\, y_{\underline{k}})$ when $|\underline{k}| \leq p$. Using (10.2), we get

$$\begin{aligned}
\varphi(a)(v^p\, y_{\underline{k}}) &= (a, v^p\, y_{\underline{k}}) \\
&= \sum_{n\geq 0;\, \underline{j}} \mu_{\underline{j},n}\, (x_{\underline{j}}, v^p\, y_{\underline{k}})\, v^n \\
&= \sum_{n\geq 0;\, \underline{j}} \mu_{\underline{j},n}\, (x_{\underline{j}}, v^{|\underline{k}|}\, y_{\underline{k}})\, v^{n+p-|\underline{k}|} \\
&= \sum_{\substack{\underline{j} \\ |\underline{j}|\leq|\underline{k}|}} P_{\underline{j}}(v),
\end{aligned}$$

where $P_{\underline{j}}(v) = \Big(\sum_{n\geq 0} \mu_{\underline{j},n}\, v^n\Big)(x_{\underline{j}}, v^{|\underline{k}|}\, y_{\underline{k}})\, v^{p-|\underline{k}|}$. If $|\underline{j}| \geq N'_m$, then

$$\sum_{n\geq 0} \mu_{\underline{j},n}\, v^n = \sum_{n\geq m} \mu_{\underline{j},n}\, v^n$$

is divisible by v^m. Hence $P_{\underline{j}}(v)$ is divisible by v^m. If $|\underline{j}| < N'_m$ and $|\underline{j}| \leq |\underline{k}|$, then by (10.2) $(x_{\underline{j}}, v^{|\underline{k}|}\, y_{\underline{k}})$ is divisible by $v^{|\underline{k}|-|\underline{j}|}$. Therefore, $P_{\underline{j}}(v)$ is divisible by $v^{p-|\underline{j}|}$, hence by $v^{p-N'_m+1}$. If $|\underline{j}| < N'_m$ and $|\underline{j}| > |\underline{k}|$, then $p - |\underline{k}| \geq p - N'_m + 1$. Therefore, $P_{\underline{j}}(v)$ is divisible by $v^{p-N'_m+1}$. Summing up, we see that $\varphi(a)(U^p(\mathfrak{g}_+)\, v^p) \subset v^m\mathbf{C}[[v]]$ for all $p \geq m + N'_m - 1$. Hence, $\varphi(a) \in V_v^{\rm o}(\mathfrak{g}_-)$.

It remains to show that $V_v^{\rm o}(\mathfrak{g}_-) \subset \varphi(S(\mathfrak{g}_+)[[v]])$. Since $(v^{|\underline{j}|}\, y_{\underline{j}})_{\underline{j}}$ is a $\mathbf{C}[v]$-basis of $V_v(\mathfrak{g}_-)$, a $\mathbf{C}[v]$-linear form $f \in V_v^*(\mathfrak{g}_-)$ is uniquely determined by the family $(\nu_{\underline{j}}(v))_{\underline{j}}$ of formal power series defined by

$$\nu_{\underline{j}}(v) = f(v^{|\underline{j}|}\, y_{\underline{j}}) \in \mathbf{C}[[v]].$$

Suppose that $f \in V_v^{\rm o}(\mathfrak{g}_-)$. Then for every m there exists N such that for all $\underline{j}$ with $|\underline{j}| \geq N$ the formal power series $\nu_{\underline{j}}(v)$ is divisible by v^m. Consider the formal sum

$$a_0 = \sum_{\underline{j}} \frac{\nu_{\underline{j}}(v)}{\underline{j}!}\, x_{\underline{j}},$$

where $\underline{j}! = j_1! \dots j_d!$ if $\underline{j} = (j_1, \dots , j_d)$. By the divisibility property of $\nu_{\underline{j}}(v)$ obtained above, a_0 is a well-defined element of $S(\mathfrak{g}_+)[[v]]$. Let us compute $\varphi(a_0) \in V_v^{\rm o}(\mathfrak{g}_-)$.

Given a d-tuple $\underline{k} = (k_1, \dots , k_d)$, we have

$$\begin{aligned}\varphi(a_0)(v^{|\underline{k}|}\, y_{\underline{k}}) &= (a_0, v^{|\underline{k}|}\, y_{\underline{k}})\\ &= \sum_{\underline{j}} \frac{\nu_{\underline{j}}(v)}{\underline{j}!}\, (x_{\underline{j}}, v^{|\underline{k}|}\, y_{\underline{k}})\\ &= \sum_{\underline{j};\, |\underline{j}|=|\underline{k}|} \frac{\nu_{\underline{j}}(v)}{\underline{j}!}\, (x_{\underline{j}}, v^{|\underline{k}|}\, y_{\underline{k}}) + \sum_{\underline{j};\, |\underline{j}|<|\underline{k}|} \frac{\nu_{\underline{j}}(v)}{\underline{j}!}\, (x_{\underline{j}}, v^{|\underline{k}|}\, y_{\underline{k}}).\end{aligned}$$

From (10.2) we derive

$$\sum_{\underline{j};\, |\underline{j}|=|\underline{k}|} \frac{\nu_{\underline{j}}(v)}{\underline{j}!}\, (x_{\underline{j}}, v^{|\underline{k}|}\, y_{\underline{k}}) = \sum_{\underline{j};\, |\underline{j}|=|\underline{k}|} \frac{\nu_{\underline{j}}(v)}{\underline{j}!}\, \delta_{\underline{j},\underline{k}}\, \underline{k}! = \nu_{\underline{k}}(v),$$

where $\delta_{\underline{j},\underline{k}} = \delta_{j_1,k_1} \dots \delta_{j_d,k_d}$. On the other hand, by (10.2), $(x_{\underline{j}}, v^{|\underline{k}|}\, y_{\underline{k}})$ is divisible by v if $|\underline{j}| < |\underline{k}|$. It follows that, for all $\underline{k}$,

$$\varphi(a_0)(y_{\underline{k}} v^{|\underline{k}|}) = \nu_{\underline{k}}(v) + v\, \mathbf{C}[[v]] = f(y_{\underline{k}} v^{|\underline{k}|}) + v\, \mathbf{C}[[v]].$$

Therefore, $f = \varphi(a_0) + vf_1$, where f_1 is a linear form on $V_v(\mathfrak{g}_-)$ such that vf_1 belongs to the subspace $V_v^{\rm o}(\mathfrak{g}_-)$. By Lemma 10.2, this implies that $f_1 \in V_v^{\rm o}(\mathfrak{g}_-)$. Starting all over again, we get an element $f_2 \in V_v^{\rm o}(\mathfrak{g}_-)$ and an element $a_1 \in S(\mathfrak{g}_+)[[v]]$ such that $f_1 = \varphi(a_1) + vf_2$. Hence, $f = \varphi(a_0 + va_1) + v^2 f_2$. Proceeding in this way, we see that for all $n \geq 0$

$$V_v^{\rm o}(\mathfrak{g}_-) = \varphi(S(\mathfrak{g}_+)[[v]]) + v^n V_v^{\rm o}(\mathfrak{g}_-).$$

Together with the topological freeness of $V_v^{\rm o}(\mathfrak{g}_-)$ proved in Lemma 10.2, this implies that $V_v^{\rm o}(\mathfrak{g}_-)$ sits inside the image of φ. □

Recall the nondegenerate bialgebra pairing (9.11)

$$(\, , \,)_v : A_+/uA_+ \times A_-/uA_- \to \mathbf{C}[[v]]$$

and the bialgebra isomorphism $\Psi'_v : V_v(\mathfrak{g}_-) \to A_-/uA_-$ of Section 9.7. They give rise to a $\mathbf{C}[[v]]$-linear morphism of algebras $\varphi : A_+/uA_+ \to V_v^*(\mathfrak{g}_-)$ defined for $a \in A_+/uA_+$ and $b \in V_v(\mathfrak{g}_-)$ by

$$\varphi(a)(b) = (a, \Psi'_-(b))_v. \tag{10.3}$$

Corollary 10.4. *$V_v^\circ(\mathfrak{g}_-)$ is a subalgebra of $V_v^*(\mathfrak{g}_-)$ and $\varphi : A_+/uA_+ \to V_v^*(\mathfrak{g}_-)$ is an injective morphism of algebras whose image is $V_v^\circ(\mathfrak{g}_-)$.*

Proof. By Lemma 9.8 the pairing

$$(-,-) = (\Psi_+(-), \Psi'_-(-))_v : S(\mathfrak{g}_+)[[v]] \times V_v(\mathfrak{g}_-) \to \mathbf{C}[[v]]$$

satisfies Condition (10.2). By Proposition 10.3 the map $\varphi \circ \Psi_+$ is injective with image $V_v^\circ(\mathfrak{g}_-)$. Since $\varphi \circ \Psi_+$ is an algebra morphism, its image $V_v^\circ(\mathfrak{g}_-)$ is necessarily a subalgebra of $V_v^*(\mathfrak{g}_-)$. One concludes by recalling that $\Psi_+ : S(\mathfrak{g}_+)[[v]] \to A_+/uA_+$ is an algebra isomorphism. □

Consider the Poisson $\mathbf{C}[[v]]$-bialgebra $E_v(\mathfrak{g}_+)$ of Section 2.7. As an algebra, $E_v(\mathfrak{g}_+) = S(\mathfrak{g}_+)[[v]]$. By (2.8) its comultiplication Δ' fulfills the following condition: For all $x \in \mathfrak{g}_+ \subset E_v(\mathfrak{g}_+)$,

$$\Delta'(x) = x \otimes 1 + 1 \otimes x + \sum_{k \geq 1} X_k v^k, \tag{10.4}$$

where $X_k \in \bigoplus_{p+q=k+1} S^p(\mathfrak{g}_+) \otimes S^q(\mathfrak{g}_+)$ for all $k \geq 1$. The Poisson bracket $\{ \, , \, \}$ of $E_v(\mathfrak{g}_+)$ is uniquely determined by Condition (2.9).

In [**Tur91**, Section 12] a bialgebra pairing $(\, , \,)'_v : E_v(\mathfrak{g}_+) \times V_v(\mathfrak{g}_-) \to \mathbf{C}[[v]]$ was constructed such that

$$(x, vy)'_v = \langle x, y \rangle \in \mathbf{C} \tag{10.5}$$

for all $x \in \mathfrak{g}_+ \subset S(\mathfrak{g}_+)[[v]] = E_v(\mathfrak{g}_+)$ and $vy \in v\mathfrak{g}_- \subset V_v(\mathfrak{g}_-)$, where $\langle \, , \, \rangle : \mathfrak{g}_+ \times \mathfrak{g}_- \to \mathbf{C}$ is the evaluation pairing. The pairing $(\, , \,)'_v$ has the following properties.

Lemma 10.5. *Let $X_1, \dots, X_m \in \mathfrak{g}_+$ and $Y_1, \dots, Y_n \in \mathfrak{g}_-$. If $m > n$, then*

$$(X_1 \cdots X_m, v^n Y_1 \cdots Y_n)'_v = 0. \tag{10.6}$$

If $m = n$, then

$$(X_1 \cdots X_m, v^n Y_1 \cdots Y_n)'_v = \sum_\sigma \langle X_{\sigma(1)}, Y_1 \rangle \cdots \langle X_{\sigma(m)}, Y_m \rangle, \tag{10.7}$$

where σ runs over all permutations of $\{1, \dots, n\}$.

If $m < n$, then

$$(X_1 \cdots X_m, v^n Y_1 \cdots Y_n)'_v \subset v^{n-m} \, \mathbf{C}[[v]]. \tag{10.8}$$

Proof. (i) We prove (10.6) and (10.7) by induction on n, using (2.11) and (10.5). The case $m = n = 1$ follows from (10.5). If $m > n = 1$, then by (2.4) and (2.11)

$$\begin{aligned}&(X_1 \cdots X_m, v\, Y_1)'_v\\&= (X_1 \otimes X_2 \cdots X_m, \Delta(v\, Y_1))'_v = (X_1 \otimes X_2 \cdots X_m, vY_1 \otimes 1 + 1 \otimes vY_1)'_v\\&= (X_1, vY_1)'_v\, (X_2 \cdots X_m, 1)'_v + (X_1, 1)'_v\, (X_2 \cdots X_m, vY_1)'_v = 0.\end{aligned}$$

Suppose we have proved (10.6) and (10.7) for $1, \ldots, n-1$. By (2.4),

$$\begin{aligned}&\Delta(Y_1 \cdots Y_n)\\&= 1 \otimes Y_1 \cdots Y_n + \sum_{p=1}^{n-1} \sum_{\sigma} Y_{\sigma(1)} \cdots Y_{\sigma(p)} \otimes Y_{\sigma(p+1)} \cdots Y_{\sigma(n)} + Y_1 \cdots Y_n \otimes 1,\end{aligned}$$

where σ runs over all $(p, n-p)$-shuffles, i.e., all permutations of $\{1, \ldots, n\}$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(n)$. Therefore,

$$\begin{aligned}&(X_1 \cdots X_m, v^n\, Y_1 \cdots Y_n)'_v\\&= (X_1 \otimes X_2 \cdots X_m, \Delta(v^n\, Y_1 \cdots Y_n))'_v\\&= (X_1, 1)'_v\, (X_2 \cdots X_m, Y_1 \cdots Y_n)'_v\\&\quad + \sum_{p=1}^{n-1} \sum_{\sigma} (X_1, v^p\, Y_{\sigma(1)} \cdots Y_{\sigma(p)})'_v\, (X_2 \cdots X_m, v^{n-p}\, Y_{\sigma(p+1)} \cdots Y_{\sigma(n)})'_v\\&\quad + (X_1, v^n\, Y_1 \cdots Y_n)'_v\, (X_2 \cdots X_m, 1)'_v,\end{aligned}$$

where σ runs over the same set of permutations as above. The first and last terms vanish by (2.11). If $m > n$, the middle sum is zero by the induction hypothesis on (10.6). If $m = n$, by (10.6), the only nonzero term is for $p = 1$, so that

$$\begin{aligned}&(X_1 \cdots X_m, v^n\, Y_1 \cdots Y_n)'_v\\&= \sum_{\sigma} (X_1, v\, Y_{\sigma(1)})'_v\, (X_2 \cdots X_m, v^{n-1}\, Y_{\sigma(2)} \cdots Y_{\sigma(n)})'_v,\end{aligned}$$

where σ runs over all permutations of $\{1, \ldots, n\}$ such that $\sigma(2) < \cdots < \sigma(n)$. Therefore,

$$\begin{aligned}&(X_1 \cdots X_m, v^n\, Y_1 \cdots Y_n)'_v\\&= \sum_{i=1}^{n} (X_1, vY_i)'_v\, (X_2 \cdots X_m, v^{n-1}\, Y_1 \cdots \widehat{Y_i} \cdots Y_n)'_v,\end{aligned}$$

where the hat on Y_i means that it is omitted from the product. We conclude with (10.5) and the induction hypothesis on (10.7).

We also prove (10.8) by induction on n. If $n = 1$, then necessarily $m = 0$ and the claim follows from (2.11). For the inductive step, observe that (10.4) implies that, for $X_1, \dots, X_m \in \mathfrak{g}_+$,

$$\Delta'(X_1 \dots X_m) = \sum_{k \geq 0} X'_k \otimes X''_k \, v^k,$$

where $X'_k \otimes X''_k \in \bigoplus_{p+q=k+m} S^p(\mathfrak{g}_+) \otimes S^q(\mathfrak{g}_+)$. By (2.11), we obtain

$$\begin{aligned}(X_1 \cdots X_m, v^n \, Y_1 \cdots Y_n)'_v &= (\Delta'(X_1 \cdots X_m), vY_1 \otimes v^{n-1} Y_2 \cdots Y_n)'_v \\ &= \sum_{k \geq 0} v^k (X'_k, vY_1)'_v \, (X''_k, v^{n-1} \, Y_2 \cdots Y_n)'_v.\end{aligned}$$

By (10.6) the only case where $(X'_k, vY_1)'_v$ may be nonzero is when $X'_k \in S^1(\mathfrak{g}_+)$, therefore when $X''_k \in S^{k+m-1}(\mathfrak{g}_+)$. If $k + m - 1 \leq n - 1$, we use (10.7) and the induction hypothesis on (10.8). Thus, $(X''_k, v^{n-1} \, Y_2 \cdots Y_n)'_v$ is divisible by v^{n-m-k}. If $k + m - 1 > n - 1$, then $(X''_k, v^{n-1} \, Y_2 \cdots Y_n)'_v = 0$ by (10.6). Therefore, $(X''_k, v^{n-1} \, Y_2 \cdots Y_n)'_v$ is divisible by v^{n-m-k} in all cases. Hence, $(X_1 \cdots X_m, v^n \, Y_1 \cdots Y_n)'_v$ is divisible by v^{n-m}. □

From the bialgebra pairing $(\, , \,)'_v$ we get a morphism of algebras $\varphi' : E_v(\mathfrak{g}_+) \to V_v^*(\mathfrak{g}_-)$ defined by $\varphi'(a) = (a, -)'_v$ for $a \in E_v(\mathfrak{g}_+)$.

Corollary 10.6. *The bialgebra pairing $(\, , \,)'_v$ is nondegenerate and the morphism of algebras φ' induces an isomorphism*

$$\varphi' : E_v(\mathfrak{g}_+) \to V_v^{\circ}(\mathfrak{g}_-) \subset V_v^*(\mathfrak{g}_-).$$

Proof. By Proposition 10.3 it is enough to check that the pairing $(\, , \,)'_v$ satisfies Condition (10.2). An easy computation shows that (10.2) is equivalent to (10.6–10.8). □

10.7. Proof of Theorem 2.9. Part II. By Corollaries 10.4 and 10.6 we have two algebra isomorphisms $\varphi : A_+/uA_+ \to V_v^{\circ}(\mathfrak{g}_-)$ and $\varphi' : E_v(\mathfrak{g}_+) \to V_v^{\circ}(\mathfrak{g}_-)$. Composing φ with the inverse of φ', we obtain an algebra isomorphism

$$\chi = \varphi'^{-1} \varphi : A_+/uA_+ \to E_v(\mathfrak{g}_+).$$

Let us check that χ is a morphism of coalgebras. By definition of φ, φ' and χ,

$$(a, \Psi'_-(b))_v = \varphi(a)(b) = \varphi'\big(\chi(a)\big)(b) = (\chi(a), b)'_v \tag{10.9}$$

for all $a \in A_+/uA_+$ and $b \in V_v(\mathfrak{g}_-)$. (For the definition of Ψ'_-, see Section 9.7.) Using (2.11) and (10.9), we obtain

$$\begin{aligned}(\Delta'(\chi(a)), b_1 \otimes b_2)'_v &= (\chi(a), b_1 b_2)'_v \\ &= (a, \Psi'_-(b_1 b_2))_v \\ &= (a, \Psi'_-(b_1) \Psi'_-(b_2))_v\end{aligned}$$

$$= (\Delta(a), \Psi'_-(b_1)\otimes\Psi'_-(b_2))_v$$
$$= \big((\chi\otimes\chi)(\Delta(a)), b_1\otimes b_2\big)'_v$$

for all $a \in A_+/uA_+$ and $b_1, b_2 \in V_v(\mathfrak{g}_-)$. Here Δ' is the comultiplication in $E_v(\mathfrak{g}_+)$ and Δ is the comultiplication in A_+/uA_+ induced by $\Delta_{u,v}$. Since the pairing $(\ ,\ )'_v$ is nondegenerate, $\Delta'\chi = (\chi\otimes\chi)\Delta$.

The bialgebra A_+/uA_+ is a (commutative) Poisson bialgebra with Poisson bracket $\{\ ,\ \}_v$ defined for $a_1, a_2 \in A_+$ by

$$\{p(a_1), p(a_2)\}_v = p\left(\frac{a_1a_2 - a_2a_1}{u}\right), \tag{10.10}$$

where $p : A_+ \to A_+/uA_+$ is the projection. The bialgebra isomorphism $\chi : A_+/uA_+ \to E_v(\mathfrak{g}_+)$ transfers this Poisson bracket to a Poisson bracket $\{\ ,\ \}'$ on $E_v(\mathfrak{g}_+)$. In order to show that χ is a morphism of Poisson bialgebras, we have to prove that $\{\ ,\ \}' = \{\ ,\ \}$. It suffices to check that $\{\ ,\ \}'$ satisfies Condition (2.9).

The pairing of Lemma 9.5 pairs the bialgebras A_+ and A_-^{cop}. Consequently,

$$(a_1a_2 - a_2a_1, b)_{u,v} = \big(a_1\otimes a_2, \Delta^{\mathrm{op}}_{u,v}(b) - \Delta_{u,v}(b)\big)_{u,v}$$

for all $a_1, a_2 \in A_+$ and $b \in A_-$. The bialgebra A_- being cocommutative modulo u (see Section 9.1), it follows that $\Delta^{\mathrm{op}}_{u,v}(b) - \Delta_{u,v}(b)$ is divisible by u; hence,

$$\left(\frac{a_1a_2 - a_2a_1}{u}, b\right)_{u,v} = \left(a_1\otimes a_2, \frac{\Delta^{\mathrm{op}}_{u,v}(b) - \Delta_{u,v}(b)}{u}\right)_{u,v}. \tag{10.11}$$

By Section 8.1 applied to A_- and by (9.5), the isomorphism $\psi_- : V_v(\mathfrak{g}_-)[[u]] \to A_-$ of Section 9.1 induces the isomorphism $\Psi'_- : V_v(\mathfrak{g}_-) \to A_-/uA_-$ of co-Poisson bialgebras. Therefore,

$$(\Psi'_-\otimes\Psi'_-)(\delta_v(vy)) = \frac{\Delta_{u,v}(b) - \Delta^{\mathrm{op}}_{u,v}(b)}{u} \mod u\, A_- \,\widehat{\otimes}_{\mathbf{C}[v][[u]]}\, A_- \tag{10.12}$$

for $vy \in v\mathfrak{g}_- \subset V_v(\mathfrak{g}_-)$ and $b \in A_-$ mapped onto $\Psi'_-(vy)$ under the projection $A_- \to A_-/uA_-$. Here, $\delta_v : V_v(\mathfrak{g}_-) \to V_v(\mathfrak{g}_-)\otimes_{\mathbf{C}[v]} V_v(\mathfrak{g}_-)$ is the Poisson cobracket defined by (2.5), where we have replaced u by v, and the Lie cobracket δ of $\mathfrak{g}$ by the Lie cobracket δ_- of $\mathfrak{g}_-$. By definition of $\mathfrak{g}_- = (\mathfrak{g}_+^{\mathrm{op}})^*$,

$$\langle x_1\otimes x_2, \delta_-(y)\rangle = -\langle [x_1\otimes x_2], y\rangle \tag{10.13}$$

for all $x_1, x_2 \in \mathfrak{g}_+$ and $y \in \mathfrak{g}_-$.

Combining (10.10)-(10.12), we obtain

$$(\{p(a_1), p(a_2)\}_v, \Psi'_-(vy))_v = -\big(p(a_1)\otimes p(a_2), (\Psi'_-\otimes\Psi'_-)(\delta_v(vy))\big)_v \tag{10.14}$$

for all $a_1, a_2 \in A_+$ and $y \in \mathfrak{g}_-$. It follows from (2.5), (10.9), (10.13), and (10.14) that

$$(\{x_1, x_2\}', vy)'_v = \big(\chi^{-1}(\{x_1, x_2\}'), \Psi'_-(vy)\big)_v \tag{10.15}$$

$$
\begin{aligned}
&= \big(\{\chi^{-1}(x_1), \chi^{-1}(x_2)\}_v, \Psi'_-(vy)\big)_v \\
&= -\big(\chi^{-1}(x_1)\otimes\chi^{-1}(x_2), (\Psi'_-\otimes\Psi'_-)(\delta_v(vy))\big)_v \\
&= -\big(x_1\otimes x_2, \delta_v(vy)\big)'_v \\
&= -\big(x_1\otimes x_2, v^2\,\delta_-(y)\big)'_v \\
&= \big([x_1, x_2], vy\big)'_v
\end{aligned}
$$

for all x_1, $x_2 \in \mathfrak{g}_+$ and $y \in \mathfrak{g}_-$.

On the other hand, the Poisson bracket $\{\ ,\ \}'$ induces the Poisson bracket (2.3) on $E_v(\mathfrak{g}_+)/vE_v(\mathfrak{g}_+) = S(\mathfrak{g}_+)$. Consequently, for all $x_1, x_2 \in \mathfrak{g}_+$,

$$\{x_1, x_2\}' = [x_1, x_2] + \sum_{m\geq 1} X_m\, v^m, \tag{10.16}$$

where $X_m \in S(\mathfrak{g}_+)$. Let $X_m^{(p)}$ be the component of X_m in $S^p(\mathfrak{g}_+)$. In order to check Condition (2.9) for $\{\ ,\ \}'$, it is enough to show that $X_m^{(p)} = 0$ for all $p = 0, 1$ and $m \geq 1$.

For the case $p = 0$, we use the counits ε of the bialgebras involved. Since ε vanishes on commutators in A_+, we have $\varepsilon(\{a_1, a_2\}_v) = 0$ in the quotient bialgebra A_+/uA_+. The map χ being also a morphism of bialgebras, $\varepsilon(\{x_1, x_2\}') = 0$ for all x_1, $x_2 \in \mathfrak{g}_+$. The map ε vanishing on $S^p(\mathfrak{g}_+)$ for $p \geq 1$ and being the identity on $S^0(\mathfrak{g}_+)$, Formula (10.16) implies

$$0 = \varepsilon(\{x_1, x_2\}') = \varepsilon([x_1, x_2]) + \sum_{m\geq 1} \varepsilon(X_m)\, v^m = \sum_{m\geq 1} X_m^{(0)}\, v^m.$$

Hence, $X_m^{(0)} = 0$ for all $m \geq 1$.

For $p = 1$, we use Lemma 10.5, (10.2), (10.15) and (10.16) in the following computation holding for all x_1, $x_2 \in \mathfrak{g}_+$ and $y \in \mathfrak{g}_-$:

$$
\begin{aligned}
0 &= (\{x_1, x_2\}' - [x_1, x_2], vy)'_v \\
&= \sum_{m\geq 1} (X_m^{(1)}, vy)'_v\, v^m + \sum_{m\geq 1;\, p\geq 2} (X_m^{(p)}, vy)'_v\, v^m \\
&= \sum_{m\geq 1} \langle X_m^{(1)}, y\rangle\, v^m.
\end{aligned}
$$

Hence, $\langle X_m^{(1)}, y\rangle = 0$ for all $y \in \mathfrak{g}_-$ and all $m \geq 1$. Therefore, $X_m^{(1)} = 0$ for all $m \geq 1$. □

10.8. Remark. Our definition of the Poisson bracket $\{\ ,\ \}'$ gives a construction of a Poisson bracket on $E_v(\mathfrak{g}_+)$ that is independent of [**Tur91**, Theorem 11.4]. We have also proved that the topological dual $V_v^o(\mathfrak{g}_-)$ has a natural structure of a Poisson $\mathbf{C}[[v]]$-bialgebra.

10.9. Remark. There are similar versions of Theorems 2.3, 2.6, and 2.9 for the bialgebra $\widehat{A}_+$ of Section 7.1. To state them, we need the bi-Poisson bialgebra $\widehat{S}(\mathfrak{g}_+)$. As an algebra, it is the completion of $S(\mathfrak{g})$ with respect to its augmentation ideal $I_0 = \bigoplus_{m\geq 1} S^m(\mathfrak{g}_+)$:

$$\widehat{S}(\mathfrak{g}_+) = \prod_{n\geq 0} S^n(\mathfrak{g}_+).$$

The bi-Poisson bialgebra structure on $S(\mathfrak{g}_+)$ defined in Section 2.2 extends to a topological bi-Poisson bialgebra structure on $\widehat{S}(\mathfrak{g}_+)$, where the comultiplication and the Poisson cobracket take values in the completed tensor product

$$\widehat{S}(\mathfrak{g})\,\widehat{\otimes}_{\mathbf{C}}\,\widehat{S}(\mathfrak{g}) = \varprojlim_n \Big(S(\mathfrak{g})/I_0^n \otimes_{\mathbf{C}} S(\mathfrak{g})/I_0^n\Big).$$

The natural projection $q_u : V_u(\mathfrak{g}_+) \to S(\mathfrak{g}_+)$ of Section 2.4 extends to a bialgebra morphism $\widehat{V}_u(\mathfrak{g}_+) \to \widehat{S}(\mathfrak{g}_+)$ that induces a canonical isomorphism of bi-Poisson bialgebras

$$\widehat{V}_u(\mathfrak{g}_+)/u\widehat{V}_u(\mathfrak{g}_+) = \widehat{S}(\mathfrak{g}_+).$$

Similarly, the Poisson $\mathbf{C}[[v]]$-bialgebra structure on $E_v(\mathfrak{g}_+) = S(\mathfrak{g}_+)[[v]]$ extends uniquely to a topological Poisson $\mathbf{C}[[v]]$-bialgebra structure on $\widehat{E}_v(\mathfrak{g}_+) = \widehat{S}(\mathfrak{g}_+)[[v]]$. The projection $\widehat{E}_v(\mathfrak{g}_+) \to \widehat{S}(\mathfrak{g}_+)$ sending v to 0 induces a canonical isomorphism of bi-Poisson bialgebras

$$\widehat{E}_v(\mathfrak{g}_+)/v\widehat{E}_v(\mathfrak{g}_+) \to \widehat{S}(\mathfrak{g}_+).$$

Proceeding for $\widehat{A}_+$ as we did for A_+ in Sections 8–10, we can prove that there is an isomorphism of co-Poisson bialgebras $\widehat{A}_+/v\widehat{A}_+ = \widehat{V}_u(\mathfrak{g}_+)$, an isomorphism of Poisson bialgebras $\widehat{A}_+/u\widehat{A}_+ \cong \widehat{E}_v(\mathfrak{g}_+)$, and an isomorphism of bi-Poisson bialgebras $\widehat{A}_+/(u,v) = \widehat{S}(\mathfrak{g}_+)$.

11. Exchanging $\mathfrak{g}_+$ and $\mathfrak{g}_-$.

Consider the Lie bialgebra $\mathfrak{g}'_+ = \mathfrak{g}_-$ and its double $\mathfrak{d}'$. By definition of the double, $\mathfrak{d}'$ contains $\mathfrak{g}'_- = (\mathfrak{g}'^*_+)^{\mathrm{cop}}$ as a Lie subbialgebra. Following Sections 5.3-5.4 for $\mathfrak{g}'_+$, we obtain three $\mathbf{C}[[h]]$-bialgebras $U_h(\mathfrak{g}'_+) \hookrightarrow U_h(\mathfrak{d}') \hookleftarrow U_h(\mathfrak{g}'_-)$. The aim of this section is to prove the following addition to [**EK96**], [**EK97**]. Here, for a bialgebra A, we denote by A^{cop} the bialgebra A obtained by replacing the comultiplication by the opposite comultiplication.

Theorem 11.1. *There is an isomorphism of* $\mathbf{C}[[h]]$*-bialgebras*

$$U_h(\mathfrak{d}') \cong U_h(\mathfrak{d})^{\mathrm{cop}}$$

sending $U_h(\mathfrak{g}'_+)$ *onto* $U_h(\mathfrak{g}_-)^{\mathrm{cop}}$ *and* $U_h(\mathfrak{g}'_-)$ *onto* $U_h(\mathfrak{g}_+)^{\mathrm{cop}}$.

Theorem 11.1 does not follow directly from the functoriality of Etingof and Kazhdan's quantization because in general there is no isomorphism between the triples $(\mathfrak{g}_+, \mathfrak{d}, \mathfrak{g}_-)$ and $(\mathfrak{g}'_+, \mathfrak{d}', \mathfrak{g}'_-)$ (nevertheless, see the proof of Theorem 1.18 in [**EK98**]). We have chosen to give a proof of this theorem using the original definitions of the bialgebras $U_h(\mathfrak{d})$, $U_h(\mathfrak{g}_+)$, $U_h(\mathfrak{g}_-)$ as given in [**EK96**]. These definitions will be recalled in Sections 11.2-11.4 below.

11.2. A Braided Monoidal Category. Consider the double Lie bialgebra $\mathfrak{d} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ of $\mathfrak{g}_+$ and let $\mathcal{S}$ be the category of $U(\mathfrak{d})$-modules. This is a symmetric monoidal category: The tensor product of two $U(\mathfrak{d})$-modules is given by $M \otimes N = M \otimes_{\mathbf{C}} N$ on which $U(\mathfrak{d})$ acts through its comultiplication, and the symmetry $\sigma_{M,N} : M \otimes N \to N \otimes M$ by the standard transposition $m \otimes n \mapsto n \otimes m$. The category $\mathcal{S}$ has an infinitesimal braiding $t_{M,N} : M \otimes N \to M \otimes N$ in the sense of Cartier [**Car93**] (see also [**Kas95**, Definition XX.4.1]). The morphism $t_{M,N}$ is given by the action of the two-tensor $t = r + r_{21} = \sum_{i=1}^{d} (x_i \otimes y_i + y_i \otimes x_i)$ of Section 5.3.

We now fix a Drinfeld associator Φ, as defined, e.g., in [**Dri89**], [**Dri90**], [**Kas95**, Section XIX.8], [**KT98**, Section 4.6]. This is a series $\Phi(A, B)$ in two non-commuting variables A and B with coefficients in $\mathbf{C}$ and constant term 1, subject to a certain set of equations (for details see the references above). Such a Φ exists by [**Dri90**] and can be assumed to be the exponential of a Lie series in A and B.

From $\mathcal{S}$ and Φ we construct a braided monoidal category $\mathcal{C}$ as follows: The objects of $\mathcal{C}$ are the same as the objects of $\mathcal{S}$. A morphism from M to N in $\mathcal{C}$ is a formal power series $\sum_{n \geq 0} f_n h^n$, where $f_n \in \mathrm{Hom}_{\mathcal{S}}(M, N) = \mathrm{Hom}_{U(\mathfrak{d})}(M, N)$ for all n. The composition in $\mathcal{C}$ is defined using the composition in $\mathcal{S}$ and the standard multiplication of formal power series. The identity morphism of an object M in $\mathcal{C}$ is the constant formal power series $\sum_{n \geq 0} f_n h^n$, where $f_0 = \mathrm{id}_M$ and $f_n = 0$ when $n > 0$. The category $\mathcal{C}$ has a tensor product: On objects it is the same as on the objects of $\mathcal{S}$; on morphisms it is obtained by extending $\mathbf{C}[[h]]$-linearly the tensor product of morphisms of $\mathcal{S}$. The unit object is the same as in $\mathcal{S}$, namely the trivial module $\mathbf{C}$ on which $U(\mathfrak{g})$ acts by the counit.

For any triple (L, M, N) of objects in $\mathcal{C}$ we define an associativity isomorphism $a_{L,M,N}$ and a braiding $c_{M,N}$ by

$$a_{L,M,N} = \Phi(h\, t_{L,M} \otimes \mathrm{id}_N, h\, \mathrm{id}_L \otimes t_{M,N}) : (L \otimes M) \otimes N \xrightarrow{\cong} L \otimes (M \otimes N) \tag{11.1}$$

and

$$c_{M,N} = \sigma_{M,N} \exp\Big(\frac{h}{2}\, t_{M,N}\Big) : M \otimes N \xrightarrow{\cong} N \otimes M, \tag{11.2}$$

where $\sigma_{M,N}$ is the transposition. For details, see [**Kas95**, XX.6].

The construction of $\mathcal{C}$, Formulas (11.1-11.2), and $\Phi(0,0) = 1$ imply that the braided monoidal category obtained as the quotient of $\mathcal{C}$ by the subclass of morphisms whose constant term as a formal power series in h is 0 is nothing else than the category $\mathcal{S}$ we started from. In this sense, $\mathcal{C}$ is a quantization of $\mathcal{S}$.

11.3. Definition of J_h. Following [**EK96**, Section 2.3], we first define $U(\mathfrak{d})$-modules $M_\pm = U(\mathfrak{d}) \otimes_{U(\mathfrak{g}_\pm)} \mathbf{C}$, where $\mathbf{C}$ is the trivial $U(\mathfrak{g}_\pm)$-module. The Verma module $M_\pm$ is a free $U(\mathfrak{g}_\mp)$-module on a generator $1_\pm$ such that $a \cdot 1_\pm = \varepsilon(a) 1_\pm$ for all $a \in U(\mathfrak{g}_\pm)$, where ε is the counit of $U(\mathfrak{g}_\pm)$. There is an isomorphism $\varphi : U(\mathfrak{d}) \to M_+ \otimes M_-$ of $U(\mathfrak{d})$-modules such that

$$\varphi(1) = 1_+ \otimes 1_- . \tag{11.3}$$

There are also $U(\mathfrak{d})$-linear maps $i_\pm : M_\pm \to M_\pm \otimes M_\pm$ defined by $i_\pm(1_\pm) = 1_\pm \otimes 1_\pm$.

In the braided monoidal category $\mathcal{C}$ of Section 11.2 consider the isomorphism

$$\chi = \beta^{-1} \circ (\mathrm{id}_+ \otimes c_{M_+,M_-} \otimes \mathrm{id}_-) \circ \alpha : (M_+ \otimes M_+) \otimes (M_- \otimes M_-) \\ \to (M_+ \otimes M_-) \otimes (M_+ \otimes M_-),$$

where $\mathrm{id}_\pm$ is the identity morphism of $M_\pm$, $c_{M_+,M_-} : M_+ \otimes M_- \to M_- \otimes M_+$ is the braiding, α is the composition of the associativity isomorphisms

$$\begin{array}{ccc} (M_+ \otimes M_+) \otimes (M_- \otimes M_-) & \xrightarrow{a^{-1}_{M_+ \otimes M_+, M_-, M_-}} & \big((M_+ \otimes M_+) \otimes M_-\big) \otimes M_- \\ & & \Big\downarrow a_{M_+,M_+,M_-} \otimes \mathrm{id}_- \\ & & \big(M_+ \otimes (M_+ \otimes M_-)\big) \otimes M_- \end{array}$$

and β is the composition of the isomorphisms

$$\begin{array}{ccc} (M_+ \otimes M_-) \otimes (M_+ \otimes M_-) & \xrightarrow{a^{-1}_{M_+ \otimes M_-, M_+, M_-}} & \big((M_+ \otimes M_-) \otimes M_+\big) \otimes M_- \\ & & \Big\downarrow a_{M_+,M_-,M_+} \otimes \mathrm{id}_- \\ & & \big(M_+ \otimes (M_- \otimes M_+)\big) \otimes M_- . \end{array}$$

Then, by [**EK96**, Formula (3.1)], the element $J_h \in (U(\mathfrak{d}) \otimes U(\mathfrak{d}))[[h]]$ determining the comultiplication of $U_h(\mathfrak{d})$ in (5.3) is defined by

$$(\varphi \otimes \varphi)(J_h) = \chi(1_+ \otimes 1_+ \otimes 1_- \otimes 1_-) = \chi(i_+ \otimes i_-)(\varphi(1)). \tag{11.4}$$

11.4. Definition of $U_h(\mathfrak{g}_\pm)$. For any $f \in \mathrm{Hom}_{\mathcal{C}}(M_+ \otimes M_-, M_-)$ consider the endomorphism $\mu_+(f) \in \mathrm{End}_{\mathcal{C}}(M_+ \otimes M_-)$ defined as the following composition of morphisms in the monoidal category $\mathcal{C}$ of Section 11.2:

(11.5)

$$M_+ \otimes M_- \xrightarrow{i_+ \otimes \mathrm{id}_-} (M_+ \otimes M_+) \otimes M_- \xrightarrow{a} M_+ \otimes (M_+ \otimes M_-) \xrightarrow{\mathrm{id}_+ \otimes f} M_+ \otimes M_-,$$

where $a = a_{M_+,M_+,M_-}$ is the associativity isomorphism defined by (11.1). Conjugating by the isomorphism φ of (11.3), we obtain the endomorphism $\varphi^{-1}\mu_+(f)\varphi \in \mathrm{End}_{\mathcal{C}}(U(\mathfrak{d}))$. Applying this endomorphism to the unit element in $U(\mathfrak{d})[[h]]$, we get the formal power series

$$f^+ = \big(\varphi^{-1}\mu_+(f)\varphi\big)(1) \in U(\mathfrak{d})[[h]].$$

By [**EK96**, Section 4.1], $U_h(\mathfrak{g}_+)$ is the image of the map $f \mapsto f^+$ from $\mathrm{Hom}_{\mathcal{C}}(M_+ \otimes M_-, M_-)$ to $U_h(\mathfrak{d}) = U(\mathfrak{d})[[h]]$.

There is a similar definition for $U_h(\mathfrak{g}_-)$. For any $g \in \mathrm{Hom}_{\mathcal{C}}(M_+ \otimes M_-, M_+)$ define $\mu_-(g) \in \mathrm{End}_{\mathcal{C}}(M_+ \otimes M_-)$ as the following composition of morphisms in $\mathcal{C}$:

(11.6)
$$M_+ \otimes M_- \xrightarrow{\mathrm{id}_+ \otimes i_-} M_+ \otimes (M_- \otimes M_-) \xrightarrow{a^{-1}} (M_+ \otimes M_-) \otimes M_- \xrightarrow{g \otimes \mathrm{id}_-} M_+ \otimes M_-.$$

Applying the endomorphism $\varphi^{-1}\mu_+(f)\varphi \in \mathrm{End}_{\mathcal{C}}(U(\mathfrak{d}))$ to $1 \in U(\mathfrak{d})[[h]]$, we obtain

$$g^- = \big(\varphi^{-1}\mu_-(g)\varphi\big)(1) \in U(\mathfrak{d})[[h]].$$

By [**EK96**, Section 4.1], $U_h(\mathfrak{g}_-)$ is the image of the map $g \mapsto g^-$ from $\mathrm{Hom}_{\mathcal{C}}(M_+ \otimes M_-, M_+)$ to $U_h(\mathfrak{d}) = U(\mathfrak{d})[[h]]$.

11.5. Proof of Theorem 11.1. By Section 2.1,

$$\mathfrak{g}'_- = ({\mathfrak{g}'_+}^*)^{\mathrm{cop}} = (\mathfrak{g}_-^*)^{\mathrm{cop}} = (\mathfrak{g}_+^{\mathrm{op}})^{\mathrm{cop}}$$

is isomorphic to $\mathfrak{g}_+$ via the map $-\mathrm{id}_{\mathfrak{g}_+}$. Let $\mathfrak{d}' = \mathfrak{g}'_+ \oplus \mathfrak{g}'_-$ be the double Lie bialgebra of $\mathfrak{g}'_+$. We have $\mathfrak{d}' = \mathfrak{g}_- \oplus \mathfrak{g}_+ = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as vector spaces. The following lemma is easily checked.

Lemma 11.6. *The endomorphism σ of $\mathfrak{g}_+ \oplus \mathfrak{g}_-$ that is the identity on $\mathfrak{g}_-$ and the opposite of the identity on $\mathfrak{g}_+$ is an isomorphism of Lie bialgebras $\sigma : \mathfrak{d} \to \mathfrak{d}'$ which fits in the following commutative diagram of Lie bialgebras, where the horizontal morphisms are the natural injections:*

$$\begin{array}{ccccc}
\mathfrak{g}_- & \hookrightarrow & \mathfrak{d} & \hookleftarrow & \mathfrak{g}_+ \\
{\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \sigma}\downarrow & & {\scriptstyle -\mathrm{id}}\downarrow \\
\mathfrak{g}'_+ = \mathfrak{g}_- & \hookrightarrow & \mathfrak{d}' & \hookleftarrow & \mathfrak{g}'_- = (\mathfrak{g}_+^{\mathrm{op}})^{\mathrm{cop}}.
\end{array}$$

The morphism σ sends the 2-tensor $r = \sum_{i=1}^d x_i \otimes y_i \in \mathfrak{d} \otimes \mathfrak{d}$ to

$$\sigma(r) = \sum_{i=1}^d (-x_i) \otimes y_i = -r \in \mathfrak{d}' \otimes \mathfrak{d}'.$$

Consequently, for the symmetric 2-tensor $t = r + r_{21}$, we have $\sigma(t) = -t$.

The Lie bialgebra isomorphism $\sigma : \mathfrak{d} \to \mathfrak{d}'$ induces a bialgebra isomorphism $\sigma : U(\mathfrak{d}) \to U(\mathfrak{d}')$, hence an algebra isomorphism between their quantizations (cf. Section 5.3):

$$\sigma : U_h(\mathfrak{d}) = U(\mathfrak{d})[[h]] \to U(\mathfrak{d}')[[h]] = U_h(\mathfrak{d}').$$

For the definition of the comultiplication Δ'_h of $U_h(\mathfrak{d}')$ we follow Section 11.2 and construct a braided monoidal category $\mathcal{C}'$, using now the double Lie bialgebra $\mathfrak{d}' = \sigma(\mathfrak{d})$, the same Drinfeld associator Φ as above, and the two-tensor $t' = \sigma(t)$. The morphism σ induces a canonical isomorphism $\mathcal{C} = \mathcal{C}'$ of braided monoidal categories.

We also need Verma modules for $\mathfrak{d}'$. Following Section 11.4, they are defined by $M'_\pm = U(\mathfrak{d}') \otimes_{U(\mathfrak{g}'_\pm)} \mathbf{C}$. As a $U(\mathfrak{g}'_\mp)$-module, $M'_\pm$ is free on a generator $1'_\pm$. There is an isomorphism $\varphi' : U(\mathfrak{d}') \to M'_+ \otimes M'_-$ defined by $\varphi'(1) = 1'_+ \otimes 1'_-$. The homomorphism $\sigma : \mathfrak{d} \to \mathfrak{d}'$ induces canonical algebra isomorphisms $U(\mathfrak{g}_\pm) = U(\mathfrak{g}'_\mp)$, hence canonical isomorphisms

$$M_\pm = U(\mathfrak{d}) \otimes_{U(\mathfrak{g}_\pm)} \mathbf{C} = U(\mathfrak{d}') \otimes_{U(\mathfrak{g}'_\mp)} \mathbf{C} = M'_\mp.$$

Using these isomorphisms, we henceforth identify $\mathfrak{d}'$ with $\mathfrak{d}$, M'_+ with M_-, M'_- with M_+, $\varphi' : U(\mathfrak{d}') \to M'_+ \otimes M'_-$ with the isomorphism of $U(\mathfrak{d})$-modules $\varphi' : U(\mathfrak{d}) \to M_- \otimes M_+$ determined by

$$\varphi'(1) = 1_- \otimes 1_+. \tag{11.7}$$

By (5.3) the comultiplication Δ'_h of the bialgebra $U_h(\mathfrak{d}') = U_h(\mathfrak{d})$ is given for $a \in U(\mathfrak{d})[[h]]$ by

$$\Delta'_h(a) = (J'_h)^{-1} \Delta(a) J'_h,$$

where Δ is the standard comultiplication and J'_h is the element in $(U(\mathfrak{d}') \otimes U(\mathfrak{d}'))[[h]] = (U(\mathfrak{d}) \otimes U(\mathfrak{d}))[[h]]$ defined, according to (11.4) and using the above identifications, by

$$(\varphi' \otimes \varphi')(J'_h) = \chi'(1_- \otimes 1_- \otimes 1_+ \otimes 1_+) = \chi'(i_- \otimes i_+)(\varphi'(1)) \tag{11.8}$$

where χ' is obtained from the morphism χ of Section 11.3 by exchanging M_+ and M_-.

Consider the $U(\mathfrak{d})$-linear automorphism ν of $U(\mathfrak{d})$ defined by

$$\nu = (\varphi')^{-1} c_{M_+,M_-} \varphi, \tag{11.9}$$

where $c_{M_+,M_-} : M_+ \otimes M_- \to M_- \otimes M_+$ is the braiding. The morphism ν is the right multiplication by the invertible element $\omega = \nu(1) \in U(\mathfrak{d})[[h]]$:

$$\nu(a) = a\omega \tag{11.10}$$

for all $a \in U(\mathfrak{d})[[h]]$.

Lemma 11.7. *We have $\omega \equiv 1$ mod h and*

$$J'_h = \Delta(\omega)^{-1} \exp(ht/2)(J_h)_{21}(\omega \otimes \omega).$$

Proof. By (11.2), (11.3), (11.7), and (11.9) we have

$$\begin{aligned}\omega &= (\varphi')^{-1}\big(\exp(ht/2)(1_+ \otimes 1_-)\big)_{21} \\ &\equiv (\varphi')^{-1}(1_- \otimes 1_+) \equiv 1 \mod h.\end{aligned}$$

Let us compute J'_h. Below we shall prove that

(11.11)
$$c_{M_-\otimes M_+,M_-\otimes M_+}(c_{M_+,M_-}\otimes c_{M_+,M_-})\chi(i_+\otimes i_-) = \chi'(i_-\otimes i_+)c_{M_+,M_-},$$

where $c_{M_-\otimes M_+,M_-\otimes M_+} : (M_-\otimes M_+)\otimes(M_-\otimes M_+) \to (M_-\otimes M_+)\otimes(M_-\otimes M_+)$ is the braiding. Then, (11.9), (11.11) and the naturality of the braiding imply

$$\begin{aligned}(11.12)\quad & ((\varphi')^{-1}\otimes(\varphi')^{-1})\chi'(i_-\otimes i_+)\varphi'\nu \\ &= ((\varphi')^{-1}\otimes(\varphi')^{-1})\chi'(i_-\otimes i_+)c_{M_+,M_-}\varphi \\ &= ((\varphi')^{-1}\otimes(\varphi')^{-1})c_{M_-\otimes M_+,M_-\otimes M_+}(c_{M_+,M_-}\otimes c_{M_+,M_-})\chi(i_+\otimes i_-)\varphi \\ &= c_{U(\mathfrak{d}),U(\mathfrak{d})}((\varphi')^{-1}\otimes(\varphi')^{-1})(c_{M_+,M_-}\otimes c_{M_+,M_-})\chi(i_+\otimes i_-)\varphi \\ &= c_{U(\mathfrak{d}),U(\mathfrak{d})}(\nu\otimes\nu)(\varphi^{-1}\otimes\varphi^{-1})\chi(i_+\otimes i_-)\varphi.\end{aligned}$$

Let us apply both sides of (11.12) to the unit in $U(\mathfrak{d})[[h]]$. By (11.8) and (11.10), we obtain for the left-hand side

$$\begin{aligned}&\Big(((\varphi')^{-1}\otimes(\varphi')^{-1})\chi'(i_-\otimes i_+)\varphi'\nu\Big)(1) \\ &= \big(((\varphi')^{-1}\otimes(\varphi')^{-1})\chi'(i_-\otimes i_+)\varphi'\big)(\omega) \\ &= \Delta(\omega)\big(((\varphi')^{-1}\otimes(\varphi')^{-1})\chi'(i_-\otimes i_+)\varphi'\big)(1) \\ &= \Delta(\omega)J'_h.\end{aligned}$$

For the right-hand side, using (11.2), (11.4), (11.10), and the symmetry of t, we obtain

$$\begin{aligned}\big(c_{U(\mathfrak{d}),U(\mathfrak{d})}(\nu\otimes\nu)(\varphi^{-1}\otimes\varphi^{-1})\chi(i_+\otimes i_-)\varphi\big)(1) &= c_{U(\mathfrak{d}),U(\mathfrak{d})}\big((\nu\otimes\nu)(J_h)\big) \\ &= \big(\exp(ht/2)J_h(\omega\otimes\omega)\big)_{21} \\ &= \exp(ht/2)(J_h)_{21}(\omega\otimes\omega).\end{aligned}$$

Putting both computations together, we obtain the desired formula for J'_h.

Let us prove (11.11). By a well-known result of Mac Lane's, any braided monoidal category is equivalent to a strict braided monoidal category. It is therefore licit to omit the associativity isomorphisms in the computations. To simplify notation, we replace in the braidings the subscripts $M_\pm$ by $\pm$ and we omit the tensor product signs. With these conventions, $\chi = \mathrm{id}_+\otimes c_{+,-}\otimes\mathrm{id}_-$ and $\chi' = \mathrm{id}_-\otimes c_{-,+}\otimes\mathrm{id}_+$. In $\mathcal{C}$ we have the following sequence of equalities implying (11.11) and justified below:

$$\begin{aligned}(11.13)\qquad & c_{-+,-+}(c_{+,-}\otimes c_{+,-})\chi(i_+\otimes i_-) \\ &= c_{-+,-+}(c_{+,-}\otimes c_{+,-})(\mathrm{id}_+\otimes c_{+,-}\otimes\mathrm{id}_-)(i_+\otimes i_-) \\ &= (\mathrm{id}_-\otimes c_{-,+}\otimes\mathrm{id}_+)c_{++,--}(c_{+,+}\otimes c_{-,-})(i_+\otimes i_-) \\ &= (\mathrm{id}_-\otimes c_{-,+}\otimes\mathrm{id}_+)c_{++,--}(i_+\otimes i_-) \\ &= (\mathrm{id}_-\otimes c_{-,+}\otimes\mathrm{id}_+)(i_-\otimes i_+)c_{+,-}\end{aligned}$$

$$= \chi'(i_-\otimes i_+)c_{+,-}$$

Here, the first and the last equalities hold by definition of χ and χ'. The second one is a consequence of the equality

$$c_{-+,-+}(c_{+,-}\otimes c_{+,-})(\mathrm{id}_+\otimes c_{+,-}\otimes \mathrm{id}_-) = (\mathrm{id}_-\otimes c_{-,+}\otimes \mathrm{id}_+)c_{++,--}(c_{+,+}\otimes c_{-,-}), \tag{11.14}$$

which holds in any braided monoidal category. This equality follows from the identity

$$\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2 = \sigma_2^2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3, \tag{11.15}$$

which holds in Artin's braid group on four strands B_4, where $\sigma_1, \sigma_2, \sigma_3$ are the standard generators of B_4.

The third equality in (11.13) is a consequence of

$$c_{\pm,\pm} = \mathrm{id}_\pm\otimes \mathrm{id}_\pm. \tag{11.16}$$

Since both sides of (11.16) are $U(\mathfrak{d})$-linear, it suffices to check this equality on the generator $1_\pm\otimes 1_\pm$ of $M_\pm\otimes M_\pm$. Now, by (11.2) and the vanishing of $t(1_\pm\otimes 1_\pm)$, we have

$$c_{\pm,\pm}(1_\pm\otimes 1_\pm) = \big(\exp(ht/2)(1_\pm\otimes 1_\pm)\big)_{21} = (1_\pm\otimes 1_\pm)_{21} = 1_\pm\otimes 1_\pm.$$

This proves (11.16). The fourth equality in (11.13) holds by naturality of the braiding. □

Corollary 11.8. *Let $\sigma_\omega : U_h(\mathfrak{d}) \to U_h(\mathfrak{d}')$ be the algebra isomorphism defined by $\sigma_\omega(a) = \sigma(\omega^{-1}a\omega)$ for all $a \in U_h(\mathfrak{d})$. Then σ_ω is a bialgebra isomorphism $U_h(\mathfrak{d})^{\mathrm{cop}} \cong U_h(\mathfrak{d}')$.*

Proof. We have to check that

$$\Delta'_h\sigma_\omega = (\sigma_\omega\otimes\sigma_\omega)\Delta_h^{\mathrm{op}}. \tag{11.17}$$

It follows from Lemma 11.7 that, for all $a \in U(\mathfrak{d})[[h]]$,

$$\begin{aligned}
&(\omega^{-1}\otimes\omega^{-1})\Delta_h^{\mathrm{op}}(a)(\omega\otimes\omega)\\
&= (\omega^{-1}\otimes\omega^{-1})(J_h^{-1})_{21}\Delta(a)(J_h)_{21}(\omega\otimes\omega)\\
&= (J'_h)^{-1}\Delta(\omega)^{-1}\exp(ht/2)\Delta(a)\exp(-ht/2)\Delta(\omega)J'_h.
\end{aligned}$$

The 2-tensor t being invariant, $\Delta(a)t = t\Delta(a)$, hence $\Delta(a)\exp(ht/2) = \exp(ht/2)\Delta(a)$. Therefore,

$$\begin{aligned}
(\omega^{-1}\otimes\omega^{-1})\Delta_h^{\mathrm{op}}(a)(\omega\otimes\omega) &= (J'_h)^{-1}\Delta(\omega)^{-1}\Delta(a)\Delta(\omega)J'_h\\
&= (J'_h)^{-1}\Delta(\omega^{-1}a\omega)J'_h\\
&= \Delta'_h(\omega^{-1}a\omega).
\end{aligned}$$

This implies (11.17). □

We now complete the proof of Theorem 11.1 by establishing that the bialgebra isomorphism $\sigma_\omega : U_h(\mathfrak{d})^{\mathrm{cop}} \to U_h(\mathfrak{d}')$ sends $U_h(\mathfrak{g}_\mp)$ onto $U_h(\mathfrak{g}'_\pm)$. We give the proof only for $\mathfrak{g}'_+$. The proof for $\mathfrak{g}'_-$ is similar.

For $f' \in \mathrm{Hom}_{\mathcal{C}'}(M'_+ \otimes M'_-, M'_-)$ consider the endomorphism $\mu'_+(f') \in \mathrm{End}_{\mathcal{C}}(M'_+ \otimes M'_-)$ defined as the following composition of morphisms in $\mathcal{C}'$:

(11.18)
$$M'_+ \otimes M'_- \xrightarrow{i'_+ \otimes \mathrm{id}'_-} (M'_+ \otimes M'_+) \otimes M'_- \xrightarrow{a'} M'_+ \otimes (M'_+ \otimes M'_-) \xrightarrow{\mathrm{id}'_+ \otimes f'} M'_+ \otimes M'_-.$$

Here $\mathrm{id}'_\pm$ is the identity morphism of $M'_\pm$, $i'_+ : M'_+ \to M'_+ \otimes M'_+$ is the analogue of $i_+ : M_+ \to M_+ \otimes M_+$, and a' is the corresponding associativity isomorphism. Conjugating by the isomorphism $\varphi' : U(\mathfrak{d}') \to M'_+ \otimes M'_-$, we obtain the endomorphism $(\varphi')^{-1} \mu'_+(f') \varphi' \in \mathrm{End}_{\mathcal{C}'}(U(\mathfrak{d}'))$, hence the formal power series
$$(f')^+ = \big((\varphi')^{-1} \mu'_+(f') \varphi'\big)(1) \in U(\mathfrak{d}')[[h]].$$
By definition, $U_h(\mathfrak{g}'_+)$ is the image of the map $f' \mapsto (f')^+$ from $\mathrm{Hom}_{\mathcal{C}}(M'_+ \otimes M'_-, M'_-)$ to $U_h(\mathfrak{d}') = U(\mathfrak{d}')[[h]]$. Under the above identifications, the morphism (11.18) in $\mathcal{C}'$ becomes for $f \in \mathrm{Hom}_{\mathcal{C}}(M_- \otimes M_+, M_+)$ the composition of morphisms in $\mathcal{C}$

$$\begin{aligned}(11.19)\quad \mu(f) : M_- \otimes M_+ &\xrightarrow{i_- \otimes \mathrm{id}_+} (M_- \otimes M_-) \otimes M_+ \\ &\xrightarrow{a} M_- \otimes (M_- \otimes M_+) \xrightarrow{\mathrm{id}_- \otimes f} M_- \otimes M_+.\end{aligned}$$

Therefore, the submodule $\sigma^{-1}(U_h(\mathfrak{g}'_+))$ of $U_h(\mathfrak{d})$ is the image of the map
$$f \mapsto f_- = \big((\varphi')^{-1} \mu(f) \varphi'\big)(1)$$
from $\mathrm{Hom}_{\mathcal{C}}(M_- \otimes M_+, M_+)$ to $U_h(\mathfrak{d}) = U(\mathfrak{d})[[h]]$, where $\varphi' : U(\mathfrak{d}) \to M_- \otimes M_+$ is defined by (11.7).

Let us compare the map $f \mapsto f_-$ with the map $g \mapsto g^-$ of Section 11.4. We shall prove below that

$$(11.20)\qquad c_{M_+,M_-} \mu_-(g) = \mu(g c^{-1}_{M_+,M_-}) c_{M_+,M_-}$$

for all $g \in \mathrm{Hom}_{\mathcal{C}}(M_+ \otimes M_-, M_+)$. It follows from (11.9), (11.10), (11.20), and from the definitions of g^- and of f_- that

$$\begin{aligned}(g c^{-1}_{M_+,M_-})_- &= \big((\varphi')^{-1} \mu(g c^{-1}_{M_+,M_-}) \varphi'\big)(1) \\ &= \big(\nu \varphi^{-1} c^{-1}_{M_+,M_-} \mu(g c^{-1}_{M_+,M_-}) c_{M_+,M_-} \varphi \nu^{-1}\big)(1) \\ &= \big(\nu \varphi^{-1} \mu_-(g) \varphi \nu^{-1}\big)(1) \\ &= \big(\nu \varphi^{-1} \mu_-(g) \varphi\big)(\omega^{-1}) \\ &= \nu\big(\omega^{-1} (\varphi^{-1} \mu_-(g) \varphi)(1)\big) \\ &= \nu(\omega^{-1} g^-) = \omega^{-1} g^- \omega = \sigma_\omega(g^-).\end{aligned}$$

Consequently, $\sigma_\omega(U_h(\mathfrak{g}_-)) = U_h(\mathfrak{g}'_+)$.

It remains to prove (11.20). We use the simplified notation introduced in the proof of (11.11). By functoriality of the braiding in $\mathcal{C}$, we have
(11.21)
$$(i_-\otimes \mathrm{id}_+)c_{+,-} = c_{+,--}(\mathrm{id}_+\otimes i_-) \quad \text{and} \quad c_{+,-}(g\otimes \mathrm{id}_-) = (\mathrm{id}_-\otimes g)c_{+-,-}$$

for $g \in \mathrm{Hom}_{\mathcal{C}}(M_+\otimes M_-, M_+)$. Therefore, by definition of μ,

$$\begin{aligned} c_{+,-}^{-1}\mu(gc_{+,-}^{-1})c_{+,-} &= c_{+,-}^{-1}(\mathrm{id}_-\otimes(gc_{+,-}^{-1}))a(i_-\otimes \mathrm{id}_+)c_{+,-} \\ &= c_{+,-}^{-1}(\mathrm{id}_-\otimes g)(\mathrm{id}_-\otimes c_{+,-}^{-1})a(i_-\otimes \mathrm{id}_+)c_{+,-} \\ &= (g\otimes \mathrm{id}_-)c_{+-,-}^{-1}(\mathrm{id}_-\otimes c_{+,-}^{-1})ac_{+,--}(\mathrm{id}_+\otimes i_-). \end{aligned}$$

Since $\mu_-(g) = (g\otimes \mathrm{id}_-)a^{-1}(\mathrm{id}_+\otimes i_-)$, it suffices to observe that by the general properties of braided categories and (11.16),

$$ac_{+,--} = ac_{+,--}(\mathrm{id}_+\otimes c_{-,-}) = (\mathrm{id}_-\otimes c_{+,-})c_{+-,-}\, a^{-1}. \tag{11.22}$$

This completes the proof of (11.20) and Theorem 11.1. □

We end this section by computing the universal R-matrix R'_h of $U_h(\mathfrak{d}')$ in terms of the universal R-matrix R_h of $U_h(\mathfrak{d})$ and the invertible element $\omega \in U_h(\mathfrak{d})$.

Lemma 11.9. *We have $R'_h = (\sigma_\omega\otimes\sigma_\omega)(R_h)_{21}$.*

Proof. By (5.6) and Lemma 11.7 we have

$$\begin{aligned} R'_h &= (J'_h)_{21}^{-1}\,\exp(ht/2)\,J'_h \\ &= (\omega^{-1}\otimes\omega^{-1})J_h^{-1}\exp(-ht/2)\Delta(\omega) \\ &\qquad \cdot \exp(ht/2)\,\Delta(\omega)^{-1}\exp(ht/2)(J_h)_{21}(\omega\otimes\omega). \end{aligned}$$

As observed in the proof of Corollary 11.8, $\Delta(a)$ commutes with $\exp(ht/2)$ for any $a \in U_h(\mathfrak{d})$. Hence,

$$R'_h = (\omega^{-1}\otimes\omega^{-1})J_h^{-1}\exp(ht/2)(J_h)_{21}(\omega\otimes\omega) = (\omega^{-1}\otimes\omega^{-1})(R_h)_{21}(\omega\otimes\omega).$$

□

12. Proof of Theorem 2.11.

The aim of this section is to identify the bialgebra A_- of Section 9. As an application, we prove Theorem 2.11.

Let us apply the constructions of Sections 6–7 to the Lie bialgebra $\mathfrak{g}'_+ = \mathfrak{g}_-$ of Sections 5.2 and 11. We obtain a $\mathbf{C}[[u,v]]$-bialgebra $U_{u,v}(\mathfrak{g}'_+)$ containing a $\mathbf{C}[u][[v]]$-bialgebra $A_{u,v}(\mathfrak{g}'_+)$.

12.1. Exchanging u and v. Any $\mathbf{C}[[u,v]]$-module M gives rise to a $\mathbf{C}[[u,v]]$-module $\tau(M)$ defined as follows. As a vector space $\tau(M)=M$, but the action of u, v is different: The new action of u is defined as the multiplication by v and the new action of v is defined as the multiplication by u. Clearly, $\tau(\tau(M))=M$. Similarly, exchanging the actions of u and v, we transform any $\mathbf{C}[u][[v]]$-module M into a $\mathbf{C}[v][[u]]$-module $\tau(M)$.

For the Lie bialgebra $\mathfrak{g}'_+=\mathfrak{g}_-$, we obtain a $\mathbf{C}[v][[u]]$-bialgebra $A_{v,u}(\mathfrak{g}'_+)$ and a $\mathbf{C}[[u,v]]$-bialgebra $U_{v,u}(\mathfrak{g}'_+)$ by

$$A_{v,u}(\mathfrak{g}'_+)=\tau\big(A_{u,v}(\mathfrak{g}'_+)\big) \quad \text{and} \quad U_{v,u}(\mathfrak{g}'_+)=\tau\big(U_{u,v}(\mathfrak{g}'_+)\big). \tag{12.1}$$

It is clear that $A_{v,u}(\mathfrak{g}'_+)\subset U_{v,u}(\mathfrak{g}'_+)$.

Theorem 12.2. *There is an isomorphism of* $\mathbf{C}[[u,v]]$*-bialgebras*

$$\sigma_{\widetilde{\omega}}:U_{u,v}(\mathfrak{g}_-)^{\mathrm{cop}}\to U_{v,u}(\mathfrak{g}'_+)$$

sending A_-^{cop} *onto* $A_{v,u}(\mathfrak{g}'_+)$.

Proof. After extending the scalars from $\mathbf{C}[[h]]$ to $\mathbf{C}[[u,v]]$ and exchanging u and v, the $\mathbf{C}[[h]]$-bialgebra isomorphism $\sigma_\omega:U_h(\mathfrak{d})^{\mathrm{cop}}\cong U_h(\mathfrak{d}')$ of Theorem 11.1 gives rise to a $\mathbf{C}[[u,v]]$-bialgebra isomorphism

$$\sigma_{\widetilde{\omega}}:U_{u,v}(\mathfrak{d})^{\mathrm{cop}}\to U_{v,u}(\mathfrak{d}') \tag{12.2}$$

sending $U_{u,v}(\mathfrak{g}_-)^{\mathrm{cop}}$ onto $U_{v,u}(\mathfrak{g}'_+)$ and $U_{u,v}(\mathfrak{g}_+)^{\mathrm{cop}}$ onto $U_{v,u}(\mathfrak{g}'_-)$. The isomorphism $\sigma_{\widetilde{\omega}}$ is given by $a\mapsto\widetilde{\sigma}(\widetilde{\omega}^{-1}a\widetilde{\omega})$, where $\widetilde{\sigma}:U_{u,v}(\mathfrak{g}_-)\cong U_{v,u}(\mathfrak{g}'_+)$ is the algebra isomorphism induced by extension of scalars from the algebra isomorphism $\sigma:U_h(\mathfrak{d})\cong U_h(\mathfrak{d}')$ of Section 11.5, and where $\widetilde{\omega}$ is the invertible element of $U_{u,v}(\mathfrak{d})=U(\mathfrak{d})[[u,v]]$ coming from the element $\omega\in U_h(\mathfrak{d})=U(\mathfrak{d})[[h]]$, cf. Section 4.6. As a consequence of Lemma 11.7, we have

$$\widetilde{\omega}\equiv 1 \mod uv. \tag{12.3}$$

The bialgebra $U_{u,v}(\mathfrak{d}')$ contains a universal R-matrix

$$R'_{u,v}\in U_{u,v}(\mathfrak{d}')\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\,U_{u,v}(\mathfrak{d}')$$

defined in the same way as the element $R_{u,v}\in U_{u,v}(\mathfrak{d})\,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\,U_{u,v}(\mathfrak{d})$ in Section 6. As an immediate corollary of Lemma 11.9,

$$R'_{u,v}=(\sigma_{\widetilde{\omega}}\otimes\sigma_{\widetilde{\omega}})(R_{u,v})_{21}. \tag{12.4}$$

We have to show that $\sigma_{\widetilde{\omega}}$ maps A_- onto $A_{v,u}(\mathfrak{g}'_+)$. We first describe $A_{u,v}(\mathfrak{g}'_+)$ following Sections 5.5 and 6.6. To begin with, we need a $\mathbf{C}[[h]]$-linear isomorphism $\alpha'_-:U_h(\mathfrak{g}'_-)\to U(\mathfrak{g}'_-)[[h]]$ such that $\alpha'_-(1)=1$ and $\alpha'_-\equiv\mathrm{id}$ modulo h, and a $\mathbf{C}$-linear projection $\pi'_-:U(\mathfrak{g}'_-)\to U^1(\mathfrak{g}'_-)=\mathbf{C}\oplus\mathfrak{g}'_-$

that is the identity on $U^1(\mathfrak{g}'_-)$. We choose them in such a way that the following squares commute:

$$\begin{array}{ccccccc} U_h(\mathfrak{g}_+) & \xrightarrow{\alpha_+} & U(\mathfrak{g}_+)[[h]] & & U(\mathfrak{g}_+) & \xrightarrow{\pi_+} & U^1(\mathfrak{g}_+) \\ \sigma_\omega \downarrow & & \sigma \downarrow & & \sigma \downarrow & & \sigma \downarrow \\ U_h(\mathfrak{g}'_-) & \xrightarrow{\alpha'_-} & U(\mathfrak{g}'_-)[[h]], & & U(\mathfrak{g}'_-) & \xrightarrow{\pi'_-} & U^1(\mathfrak{g}'_-) \end{array} \tag{12.5}$$

where $\alpha_+ : U_h(\mathfrak{g}_+) \to U(\mathfrak{g}_+)[[h]]$ has been chosen in Section 6.6 and $\pi_+ : U(\mathfrak{g}_+) \to U^1(\mathfrak{g}_+)$ in Section 9.1.

For any $y \in \mathfrak{g}_-$, let $\sigma(y)$ be the corresponding element in $\mathfrak{g}'_+$ and $\langle \sigma(y), - \rangle'$: $U^1(\mathfrak{g}'_-) \to \mathbf{C}$ be the $\mathbf{C}$-linear form extending the standard evaluation map $\langle \sigma(y), - \rangle' : \mathfrak{g}'_- \to \mathbf{C}$ and such that $\langle \sigma(y), 1 \rangle' = 0$. Following Section 5.5, given $y \in \mathfrak{g}_-$, we define a $\mathbf{C}[[h]]$-linear form $f'_{\sigma(y)} : U_h(\mathfrak{g}'_-) \to \mathbf{C}[[h]]$ for $a \in U_h(\mathfrak{g}'_-)$ by

$$f'_{\sigma(y)}(a) = \langle \sigma(y), \pi'_- \alpha'_-(a) \rangle'. \tag{12.6}$$

By extension of scalars, we obtain a $\mathbf{C}[[u,v]]$-linear form $\widetilde{f}'_{\sigma(y)} : U_{u,v}(\mathfrak{g}_-) \to \mathbf{C}[[u,v]]$. By Lemma 6.5, the element

$$\rho'_+(\widetilde{f}'_{\sigma(y)}) = (\mathrm{id} \otimes \widetilde{f}'_{\sigma(y)})(R'_{u,v}) \in U_{u,v}(\mathfrak{g}'_+) \tag{12.7}$$

is divisible by uv. Let $(y_1, \ldots, y_d)$ be the basis of $\mathfrak{g}_-$ dual to the basis $(x_1, \ldots, x_d)$ of $\mathfrak{g}_+$. In view of Section 6.6, $A_{u,v}(\mathfrak{g}'_+)$ is the $\mathbf{C}[u][[v]]$-submodule of $U_{u,v}(\mathfrak{g}'_+)$ generated by the elements

$$v^{-|\underline{k}|}\, \rho'_+(\widetilde{f}'_{\sigma(y_1)})^{k_1} \ldots \rho'_+(\widetilde{f}'_{\sigma(y_d)})^{k_d},$$

where $\underline{k}$ runs over all finite sequences of nonnegative integers.

Therefore, $A_{v,u}(\mathfrak{g}'_+) = \tau(A_{u,v}(\mathfrak{g}'_+))$ is the $\mathbf{C}[v][[u]]$-submodule of $U_{v,u}(\mathfrak{g}'_+)$ generated by the elements

$$u^{-|\underline{k}|}\, \rho'_+(\widetilde{f}'_{\sigma(y_1)})^{k_1} \ldots \rho'_+(\widetilde{f}'_{\sigma(y_d)})^{k_d}, \tag{12.8}$$

where $\underline{k}$ runs over all finite sequences of nonnegative integers.

In view of the definition of A_- (see Section 9.1), in order to prove that $\sigma_{\widetilde{\omega}}(A_-) = A_{v,u}(\mathfrak{g}'_+)$, it suffices to check that for all $y \in \mathfrak{g}_-$

$$\sigma_{\widetilde{\omega}}(\rho_-(\widetilde{g}_y)) = -\rho'_+(\widetilde{f}'_{\sigma(y)}), \tag{12.9}$$

where ρ_- is defined by (6.2) and $\widetilde{g}_y : U_{u,v}(\mathfrak{g}_+) \to \mathbf{C}[[u,v]]$ is the $\mathbf{C}[[u,v]]$-linear form extended from the linear form $g_y : U_h(\mathfrak{g}_+) \to \mathbf{C}[[h]]$ defined by (9.1).

Let us prove (12.9). First observe that, since $\sigma = -\mathrm{id}$ on $\mathfrak{g}_+$ and $\sigma = \mathrm{id}$ on $\mathfrak{g}_-$, we have

$$\langle \sigma(y), \sigma(x) \rangle' = -\langle x, y \rangle \tag{12.10}$$

for all $x \in \mathfrak{g}_+$ and $y \in \mathfrak{g}_-$. It follows from (9.1), (12.5), (12.6), and (12.10) that

$$\begin{aligned} f'_{\sigma(y)}(\sigma_\omega(a)) &= \langle \sigma(y), \pi'_-\alpha'_-(\sigma_\omega(a))\rangle' \\ &= \langle \sigma(y), \sigma\pi_+\alpha_+(a)\rangle' \\ &= -\langle \pi_+\alpha_+(a), y\rangle = -g_y(a) \end{aligned}$$

for all $y \in \mathfrak{g}_-$ and $a \in U_h(\mathfrak{g}_+)$. By extension of scalars, we obtain

$$\widetilde{f}'_{\sigma(y)}(\sigma_{\widetilde{\omega}}(a)) = -\widetilde{g}_y(a) \tag{12.11}$$

for all $y \in \mathfrak{g}_-$ and $a \in U_{u,v}(\mathfrak{g}_+)$.

As a consequence of (6.2), (12.4), (12.7), and (12.11),

$$\begin{aligned} \rho'_+(\widetilde{f}'_{\sigma(y)}) &= (\mathrm{id}\otimes \widetilde{f}'_{\sigma(y)})(R'_{u,v}) \\ &= (\mathrm{id}\otimes \widetilde{f}'_{\sigma(y)})\Big((\sigma_{\widetilde{\omega}}\otimes\sigma_{\widetilde{\omega}})(R_{u,v})_{21}\Big) \\ &= (\widetilde{f}'_{\sigma(y)}\otimes \mathrm{id})(\sigma_{\widetilde{\omega}}\otimes\sigma_{\widetilde{\omega}})(R_{u,v}) \\ &= \sigma_{\widetilde{\omega}}\Big((\widetilde{f}'_{\sigma(y)}\sigma_{\widetilde{\omega}}\otimes \mathrm{id})(R_{u,v})\Big) \\ &= -\sigma_{\widetilde{\omega}}\Big((\widetilde{g}_y\otimes \mathrm{id})(R_{u,v})\Big) \\ &= -\sigma_{\widetilde{\omega}}(\rho_-(\widetilde{g}_y)). \end{aligned}$$

This proves (12.9) and completes the proof of Theorem 12.2. □

12.3. Proof of Theorem 2.11. Under the bialgebra isomorphism $A_-^{\mathrm{cop}} \cong A_{v,u}(\mathfrak{g}'_+)$ of Theorem 12.2, the nondegenerate bialgebra pairing $(\ ,\ )_{u,v}$ of Lemma 9.5 and Corollary 9.9 gives rise to a nondegenerate bialgebra pairing $A_{u,v}(\mathfrak{g}_+) \times A_{v,u}(\mathfrak{g}'_+) \to \mathbf{C}[[u,v]]$. The second assertion in Theorem 2.11 follows from (9.18) and (12.3). □

Appendix A. Biquantization of the trivial bialgebra.

Let $\mathfrak{g}_+$ be a d-dimensional Lie bialgebra with basis $(x_1, \dots, x_d)$ and with dual basis $(y_1, \dots, y_d)$. Assume throughout the appendix that $\mathfrak{g}_+$ is the trivial Lie bialgebra, i.e., with zero Lie bracket and cobracket:

$$[x_i, x_j] = 0 \quad \text{and} \quad \delta(x_i) = 0 \tag{A.1}$$

for all i and $j = 1, \dots, d$. We now give a complete description of the biquantization $A_{u,v}(\mathfrak{g}_+)$ and of the pairing (9.9) under the hypothesis (A.1).

The dual Lie bialgebra $\mathfrak{g}_- = (\mathfrak{g}_+^*)^{\mathrm{cop}}$ is also trivial, whereas the double Lie bialgebra $\mathfrak{d} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ is not: It follows from (5.1) and (5.2) that the Lie bracket of $\mathfrak{d}$ is equal to zero, but not its Lie cobracket, which is given by $\delta(u) = [u\otimes 1 + 1\otimes u, r]$, where $r = \sum_{i=1}^d x_i\otimes y_i$.

We first determine the bialgebras $U_h(\mathfrak{d})$ and $U_h(\mathfrak{g}_\pm)$ of Section 5. Since $\mathfrak{d}$ is a trivial Lie algebra, we have

$$U_h(\mathfrak{d}) = U(\mathfrak{d})[[h]] = S(\mathfrak{d})[[h]]. \tag{A.2}$$

This is not only an isomorphism of algebras, but also of bialgebras. Indeed, since $U_h(\mathfrak{d})$ is commutative, it follows from (5.3) that its comultiplication is the standard one: $\Delta_h = \Delta$.

In order to determine the subbialgebras $U_h(\mathfrak{g}_\pm)$ of $U_h(\mathfrak{d})$, we need Sections 11.2–11.4, whose notation we use freely. Consider the braided monoidal category $\mathcal{C}$ of Section 11.2. We claim that the associativity isomorphims are trivial:

$$a_{L,M,N} = \mathrm{id}_{L\otimes M\otimes N} \tag{A.3}$$

for any triple (L, M, N) of objects in $\mathcal{C}$. Indeed, since the Lie algebra $\mathfrak{d}$ is abelian, the morphisms $t_{L,M} \otimes \mathrm{id}_N$ and $\mathrm{id}_L \otimes t_{M,N}$ coming up in (11.1) commute with one another. Now, the Drinfeld associator $\Phi(A, B)$, being the exponential of a Lie series in the variables A and B, is equal to 1 if A and B commute. This proves (A.3).

On the Verma modules $M_\pm$, the braiding c_{M_+,M_-} is given by

$$c_{M_+,M_-}(1_+\otimes 1_-) = \exp(ht/2)(1_-\otimes 1_+)$$

in view of (11.2) and the symmetry of t. Since $\mathfrak{d}$ is abelian, we have

$$\exp(ht/2) = \prod_{i=1}^{d} \exp(h(x_i\otimes y_i)/2) \exp(hr_{21}/2).$$

Now, $r_{21}(1_-\otimes 1_+) = \sum_{i=1}^{d} y_i 1_-\otimes x_i 1_+ = 0$. Therefore

$$c_{M_+,M_-}(1_+\otimes 1_-) = \prod_{i=1}^{d} \exp(h(x_i\otimes y_i)/2)(1_-\otimes 1_+). \tag{A.4}$$

Let us give a formula for the isomorphism $\varphi : U(\mathfrak{d}) \to M_+\otimes M_-$ of (11.3). Since $\mathfrak{d} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as Lie algebras, any element of $U(\mathfrak{d}) = S(\mathfrak{d})$ is a linear combination of elements of the form ab, where $a \in S(\mathfrak{g}_+) \subset S(\mathfrak{d})$ and $b \in S(\mathfrak{g}_-) \subset S(\mathfrak{d})$. We have

$$\varphi(ab) = b1_+\otimes a1_-. \tag{A.5}$$

Indeed, using Sweedler's notation, the definition of $M_\pm$ as modules, and the commutativity of $U(\mathfrak{d}) = S(\mathfrak{d})$, we have

$$\begin{aligned}\varphi(ab) &= \Delta(ab)(1_+ \otimes 1_-)\\ &= \sum_{(a)(b)} a_{(1)}b_{(1)}1_+\otimes a_{(2)}b_{(2)}1_-\\ &= \sum_{(a)(b)} b_{(1)}a_{(1)}1_+\otimes a_{(2)}b_{(2)}1_-\end{aligned}$$

$$= \sum_{(a)(b)} b_{(1)}\varepsilon(a_{(1)})1_+ \otimes a_{(2)}\varepsilon(b_{(2)})1_-$$
$$= \left(\sum_{(b)} b_{(1)}\varepsilon(b_{(2)})1_+\right) \otimes \left(\sum_{(a)} \varepsilon(a_{(1)})a_{(2)}1_-\right)$$
$$= b1_+ \otimes a1_-.$$

It follows that, for $a \in S(\mathfrak{g}_+)$ and $b \in S(\mathfrak{g}_-)$,

$$\varphi(\exp(ab)) = \exp(b\otimes a)(1_+\otimes 1_-). \tag{A.6}$$

Proposition A.1. $U_h(\mathfrak{g}_\pm) = S(\mathfrak{g}_\pm)[[h]]$ *as bialgebras.*

Proof. We prove this for $U_h(\mathfrak{g}_+)$. There is a similar proof for $U_h(\mathfrak{g}_-)$.

By Section 11.4, $U_h(\mathfrak{g}_+)$ is the image of the map $f \mapsto f^+$ from $\mathrm{Hom}_{\mathcal{C}}(M_+\otimes M_-, M_-)$ to $U_h(\mathfrak{d}) = U(\mathfrak{d})[[h]]$. We claim that this image is exactly the submodule $U(\mathfrak{g}_+)[[h]]$ of $U(\mathfrak{d})[[h]]$ consisting of the formal power series with coefficients in $U(\mathfrak{g}_+) \subset U(\mathfrak{d})$. Indeed, an element $f \in \mathrm{Hom}_{\mathcal{C}}(M_+\otimes M_-, M_-)$ is of the form $f = \sum_{i\geq 0} f_i\, h^i$ where the maps $f_i : M_+\otimes M_- \to M_-$ are $U(\mathfrak{d})$-linear. Since $M_+\otimes M_-$ is a rank-one free module generated by $1_+\otimes 1_-$, the map f_i is determined by the element $a_i\, 1_- = f_i(1_+\otimes 1_-) \in M_-$, where a_i is a well-defined element of $U(\mathfrak{g}_+)$. The claim will be proved if we show that $f^+ = \sum_{i\geq 0} a_i\, h^i$.

By (11.5), (A.3) and (A.5) we have

$$\begin{aligned}
f^+ &= \big(\varphi^{-1}\mu_+(f)\varphi\big)(1)\\
&= \sum_{i\geq 0} \big(\varphi^{-1}(\mathrm{id}_+\otimes f_i)\, a\, (i_+\otimes \mathrm{id}_-)\varphi\big)(1)\, h^i\\
&= \sum_{i\geq 0} \big(\varphi^{-1}(\mathrm{id}_+\otimes f_i)(i_+\otimes \mathrm{id}_-)\varphi\big)(1)\, h^i\\
&= \sum_{i\geq 0} \big(\varphi^{-1}(\mathrm{id}_+\otimes f_i)(i_+\otimes \mathrm{id}_-)\big)(1_+\otimes 1_-)\, h^i\\
&= \sum_{i\geq 0} \big(\varphi^{-1}(\mathrm{id}_+\otimes f_i)\big)(1_+\otimes 1_+\otimes 1_-)\, h^i\\
&= \sum_{i\geq 0} \varphi^{-1}(1_+\otimes a_i\, 1_-)\, h^i = \sum_{i\geq 0} a_i\, h^i.
\end{aligned}$$

The fact that $U_h(\mathfrak{g}_\pm) = U(\mathfrak{g}_\pm)[[h]]$ is a subbialgebra of $U(\mathfrak{d})[[h]]$, hence has the standard product and coproduct, follows from the obvious fact that $U(\mathfrak{g}_\pm)$ is a subbialgebra of $U(\mathfrak{d})$. The Lie algebras $\mathfrak{d}$ and $\mathfrak{g}_\pm$ being abelian, we have $U(\mathfrak{g}_\pm) = S(\mathfrak{g}_\pm)$. Consequently, $U_h(\mathfrak{g}_\pm) = S(\mathfrak{g}_\pm)[[h]]$ as bialgebras. □

Corollary A.2. *The bialgebra $\widehat{A}_+$ is the subbialgebra of $S(\mathfrak{g}_+)[[u,v]]$ consisting of the formal power series $\sum_{m,n\geq 0} a_{m,n}\, u^m v^n$ such that $a_{m,n} \in \bigoplus_{k=0}^{m} S^k(\mathfrak{g}_+)$ for all $m \geq 0$.*

Proof. By (6.1), Proposition A.1 and Lemma 4.7, we have $U_{u,v}(\mathfrak{g}_\pm) = S(\mathfrak{g}_\pm)[[u,v]]$. We conclude in view of (7.1) and of Proposition 3.8. □

Similarly, the bialgebra $\widehat{A}_-$ of Section 9.1 is the subbialgebra of $S(\mathfrak{g}_-)[[u,v]]$ consisting of the formal power series $\sum_{m,n\geq 0} b_{m,n}\, u^m v^n$ such that $b_{m,n} \in \bigoplus_{k=0}^{n} S^k(\mathfrak{g}_-)$ for all $n \geq 0$.

In order to determine the subalgebras $A_{u,v}(\mathfrak{g}_+)$ and A_- defined in Sections 6.6 and 9.1, we have to make explicit the element

$$R_{u,v} \in U_{u,v}(\mathfrak{g}_+) \,\widehat{\otimes}_{\mathbf{C}[[u,v]]}\, U_{u,v}(\mathfrak{g}_-)$$

of Section 6. Let J_h and R_h be the elements of $(U(\mathfrak{d})\otimes_{\mathbf{C}} U(\mathfrak{d}))[[h]]$ given by (11.4) and (5.6), respectively.

Lemma A.3. *We have $J_h = \exp(hr/2)$ and $R_h = \exp(hr)$.*

Proof. By (11.4), (A.4) and (A.5), we have

$$\begin{aligned}
(\varphi\otimes\varphi)(J_h) &= \chi(1_+\otimes 1_+\otimes 1_-\otimes 1_-)\\
&= \exp(ht_{23}/2)\cdot(1_+\otimes 1_-\otimes 1_+\otimes 1_-)\\
&= \exp(hr_{23}/2)\cdot(1_+\otimes 1_-\otimes 1_+\otimes 1_-)\\
&= \sum_{n\geq 0} \frac{h^n}{2^n n!}\left(\sum_{i=1}^{d} 1\otimes x_i\otimes y_i\otimes 1\right)^n (1_+\otimes 1_-\otimes 1_+\otimes 1_-)\\
&= 1_+\otimes\left(\sum_{n\geq 0}\frac{h^n}{2^n n!}\sum_{i_1,\dots,i_n=1}^{d} x_{i_1}\cdots x_{i_n}1_-\otimes y_{i_1}\cdots y_{i_n}1_+\right)\otimes 1_-\\
&= (\varphi\otimes\varphi)\left(\sum_{n\geq 0}\frac{h^n}{2^n n!}\sum_{i_1,\dots,i_n=1}^{d} x_{i_1}\cdots x_{i_n}\otimes y_{i_1}\cdots y_{i_n}\right)\\
&= (\varphi\otimes\varphi)\big(\exp(hr/2)\big).
\end{aligned}$$

Formula (5.6) implies

$$R_h = (J_h^{-1})_{21}\, \exp\left(\frac{ht}{2}\right)\, J_h = \exp\left((-r_{21}+r+r_{21}+r)\frac{h}{2}\right) = \exp(hr).$$

□

Corollary A.4. *We have*

$$R_{u,v} = \exp(uvr) = \sum_{n\geq 0}\frac{u^n v^n}{n!}\sum_{i_1,\dots,i_n=1}^{d} x_{i_1}\cdots x_{i_n}\otimes y_{i_1}\cdots y_{i_n}.$$

From $R_{u,v}$ we get maps $\rho_+ : U^*_{u,v}(\mathfrak{g}_-) \to U_{u,v}(\mathfrak{g}_+)$ and $\rho_- : U^*_{u,v}(\mathfrak{g}_+) \to U_{u,v}(\mathfrak{g}_-)$ as in Section 6. Formula (5.10) defines a $\mathbf{C}[[h]]$-linear form $f_x : U_h(\mathfrak{g}_-) = S(\mathfrak{g}_-)[[h]] \to \mathbf{C}[[h]]$, where we may take $\alpha_- = \mathrm{id}$ and $\pi_- : U(\mathfrak{g}_-) = S(\mathfrak{g}_-) \to U^1(\mathfrak{g}_-) = \mathbf{C} \oplus \mathfrak{g}_-$ the natural projection. It follows that the map $\widetilde{f}_x : U_{u,v}(\mathfrak{g}_-) = S(\mathfrak{g}_-)[[u,v]] \to \mathbf{C}[[u,v]]$ of Section 6.4 is given for $b = \sum_{m,n\geq 0} b_{m,n}\, u^m v^n \in S(\mathfrak{g}_-)[[u,v]]$ by

$$\widetilde{f}_x(b) = \sum_{n\geq 0} \langle x, \pi(b_{m,n})\rangle\, u^m v^n. \tag{A.7}$$

Lemma A.5. *We have* $v^{-1}\,\rho_+(\widetilde{f}_x) = ux$ *for all* $x \in \mathfrak{g}_+$.

Proof. By (6.2), (A.7) and Corollary A.4 we get

$$\begin{aligned}
\rho_+(\widetilde{f}_x) &= (\mathrm{id}\otimes \widetilde{f}_x)(R_{u,v}) \\
&= \sum_{n\geq 0} \frac{u^n v^n}{n!} \sum_{i_1,\dots,i_n=1}^{d} \widetilde{f}_x(y_{i_1}\cdots y_{i_n})\, x_{i_1}\cdots x_{i_n} \\
&= \sum_{n\geq 0} \frac{u^n v^n}{n!} \sum_{i_1,\dots,i_n=1}^{d} \langle x, \pi(y_{i_1}\cdots y_{i_n})\rangle\, x_{i_1}\cdots x_{i_n} \\
&= uv \sum_{i=1}^{d} \langle x, \pi(y_i)\rangle\, x_i = uv \sum_{i=1}^{d} \langle x, y_i\rangle\, x_i = uv\, x.
\end{aligned}$$

□

Corollary A.6. $A_{u,v}(\mathfrak{g}_+)$ *consists of the formal power series* $\sum_{m,n\geq 0} a_{m,n}\, u^m v^n$ *such that* $a_{m,n} \in \bigoplus_{k=0}^{m} S^k(\mathfrak{g}_+)$ *for all* $m \geq 0$, *and for all* $n \geq 0$ *there exists* N *with* $a_{m,n} = 0$ *for all* $m > N$.

Similarly, the bialgebra A_- consists of the formal power series $\sum_{m,n\geq 0} b_{m,n}\, u^m v^n$ such that $b_{m,n} \in \bigoplus_{k=0}^{n} S^k(\mathfrak{g}_-)$ for all $n \geq 0$, and for all $m \geq 0$ there exists M with $b_{m,n} = 0$ for all $n > M$. Together with Corollary A.6, this implies that

$$A_- = A_{v,u}(\mathfrak{g}_-).$$

Let us describe the bialgebra pairing $(\ ,\ )_{u,v} : A_{u,v}(\mathfrak{g}_+) \times A_-^{\mathrm{cop}} \to \mathbf{C}[[u,v]]$ defined by (9.9). By (2.11) and Corollary A.6, it suffices to compute $(ux, vy)_{u,v}$ when $x \in \mathfrak{g}_+$ and $y \in \mathfrak{g}_-$. The following result shows that the pairing $(\ ,\ )_{u,v}$ is the standard one.

Lemma A.7. *We have* $(ux, vy)_{u,v} = \langle x, y\rangle$ *for all* $x \in \mathfrak{g}_+$ *and* $y \in \mathfrak{g}_-$.

Proof. By (9.9), (A.7), and Lemma A.5 we have

$$(ux, vy)_{u,v} = \big(\rho_+^{-1}(ux)\big)(vy) = v^{-1}\, \widetilde{f}_x(vy) = v^{-1}v\,\langle x, \pi(y)\rangle = \langle x, y\rangle.$$

□

A.8. Remark. The reader may check, using (A.4) and (A.6), that the invertible element $\omega \in U_h(\mathfrak{d}) = S(\mathfrak{d})[[h]]$ defined by (11.10) is given by

$$\omega = \exp\left(\frac{h}{2}\sum_{i=1}^{d} x_i y_i\right).$$

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