Nonabelian Algebraic Topology:

Higher homotopy groupoids of filtered spaces

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Preface

This book aims to present the work on crossed complexes and related higher homotopy groupoids carried out mainly by the first two authors from 1974 to 2005, resulting in 12 joint papers.

This account also elucidates fully, as did [BH81a], a paragraph near the end of the Introduction to [Bro67] which referred speculatively to an n-dimensional version of the van Kampen theorem. The intuition behind that speculation drove the subsequent research, and it was interesting to see how well it all worked.

The aim became to develop the theories of groupoids and higher groupoids in a similar spirit to that of combinatorial group theory.

The contribution of Philip Higgins' imagination, algebraic insight and expository skills to this research is seen throughout this book, and so he is rightly a joint author. However, this presentation has been carried out by Brown and Sivera who carry full responsibility for the final result and in particular for any errors.

Obtaining these results depended on other fortunate collaborations, particularly initial work on double groupoids and crossed modules with Chris Spencer in 1971-3. Collaboration at Bangor over the years with Tim Porter and Chris Wensley has been especially important. Other collaborators on joint papers relevant to the 'groupoid project' were: Lew Hardy, Jean-Louis Loday, Sid Morris, Phil Heath, Peter Booth, Johannes Huebschmann, Graham Ellis, Heiner Kamps, Nick Gilbert, Tim Porter, David Johnson, Edmund Robertson, Hans Baues, Razak Salleh, Kirill Mackenzie, Marek Golasinski, Mohammed Aof, Rafael Sivera, Osman Mucuk, George Janelidze, Ilhan Icen, James Glazebrook; research students at Bangor (with date of completion, supervised by Brown unless marked P for Porter or W for Wensley): Lew Hardy (1974), Tony Seda (1974), A. Razak Salleh (1975), Keith Dakin (1976), Nick Ashley (1978), David Jones (1984), Graham Ellis (1984), Fahmi Korkes (1985, P), Ghafar H. Mosa (1987), Mohammed Aof (1988), Fahd Al-Agl (1988), Osman Mucuk (1993), Andy Tonks (1993), Ilhan Icen (1996), Phil Ehlers (1994, P), J. Shrimpton (1990, with W), Zaki Arvasi (1995, P), Murat Alp (1997, W), Ali Mutlu (1998, W), Anne Heyworth (1998 with W), Emma Moore (2001, with W).

It is a pleasure to acknowledge also:

(i) the influence of the work of Henry Whitehead, who was Brown's supervisor until 1960, when Henry died suddenly in Princeton at the age of 55. It was then that Michael Barratt guided Brown's thesis towards the homotopy type of function spaces; Michael's splendid example of how to go about mathematical research is gratefully acknowledged. In writing the first edition of the book which has now become 'Topology and Groupoids', Brown turned to Whitehead's work on 'Combinatorial Homotopy'; this title shows the influence on Whitehead of the analogy between ideas in homotopy theory and in combinatorial group theory.

(ii) the further contributions of research students at Bangor, and of other colleagues, who all contributed key ideas to the whole programme.

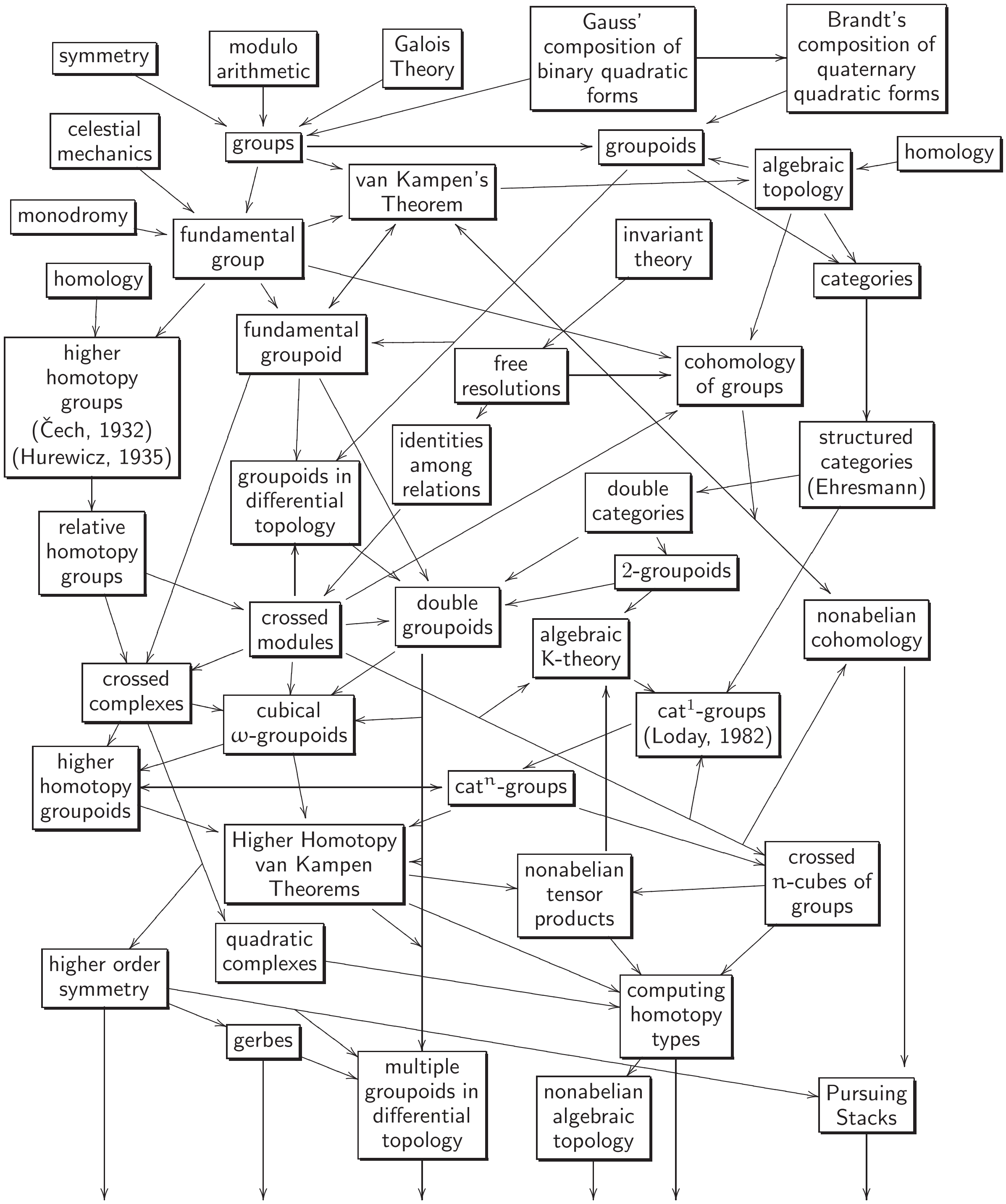
(iii) the stimulus of a correspondence with Alexander Grothendieck in the years 1982-1991. This correspondence is to be published by the Société Mathématique de France, as an Appendix to the visionary manuscript by Grothendieck, 'Pursuing Stacks', which was distributed from Bangor in 1983.

(iv) the support of the Leverhulme Trust through an Emeritus Fellowship for Brown in 2002-2004, and to my chosen referees for that, Professors A. Bak (Bielefeld) and J.P. May (Chicago). This Fellowship provided the support for: the revision and publication of 'Topology and Groupoids'; meetings of Brown and Sivera; for LaTeX work, well done by Genevieve Tan and Peilang Wu; for other travel and visitors; and also a moral impetus to complete this project.

(v) Support by the University of Wales and by the SERC for the collaboration with Higgins in the years 1974-1985.

We would like to make a number of acknowledgements. We must thank Chris Spencer, whose work with one of us [BS76a] in 1972 gave the crucial algebraic impetus to work in dimension 2. We must thank Keith Dakin, whose invention of T-complexes [Dak83] made it clear that an acceptable generalisation of double groupoids to all dimensions did exist. We thank the Science Research Council for support. Finally, we thank many friends whose sceptical interest has often been a stimulus and challenge.

Some Historical Context for Non Abelian Algebraic Topology



This diagram aims to give a sketch of some influences and interactions leading to the development of nonabelian algebraic topology, and higher dimensional algebra, so that this exposition is seen as part of a continuing development.

Introduction

Aims

Our aim for this text is to give in one place a complete, consistent and we hope readable account of basic algebraic topology on the border between homology and homotopy, using where sensible the tools of groupoids, crossed modules, crossed complexes and a higher homotopy groupoid. The principal research for this work was done by the first two authors in the years 1974–1995.

The major tool is a classical functor defined using relative homotopy groups and the fundamental groupoid on a set of base points

$$\Pi : \mathsf{FTop} \to \mathsf{Crs} \tag{0.0.1}$$

from the category of filtered spaces to the category crossed complexes. Here a *filtered space*

$$X_* : X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty$$

is a topological space X_∞ with an increasing sequence of subspaces. For such a filtered space X_* one obtains the fundamental groupoid $\pi_1(X_1, X_0)$ and the relative homotopy groups $\pi_n(X_n, X_{n-1}, x), x \in X_0, n \geqslant 2$. They are defined in terms of homotopy classes of maps $I^n \to X_n$, where I^n is the unit n-cube, which map to x the set J^{n-1} of all $(n-1)$-faces of I^n except the $(-,1)$-th face, map the remaining face to X_{n-1}, and all homotopies keep J^{n-1} fixed. These groups are abelian for $n \geqslant 3$, and admit an operation of $\pi_1(X_1, X_0)$ as well as boundary maps from dimension n to dimension $n-1$. The structure all these satisfy is called a *crossed complex*.

We can calculate this functor directly in many useful cases because of the first of our major results, a *Higher Homotopy van Kampen Theorem*, (HHvKT), that the functor Π preserves certain colimits: this mean that in some cases, Π of a filtered space X_* can be calculated from the way X_* is built up by gluing together smaller filtered spaces. The applications and proof of this result are main themes of the theory.

One major application is a *homotopical excision theorem* (Theorem 8.3.7): this says that if A, B are open, $A, B, A \cap B$ are connected, and the pair $(A, A \cap B)$ is $(n-1)$-connected, then the pair $(A \cup B, B)$ is $(n-1)$-connected and the excision morphism

$$\varepsilon : \pi_n(A, A \cap B, x) \to \pi_n(A \cup B, B, x) \tag{0.0.2}$$

determines the right hand *module (crossed if* $n = 2$*)* as induced from the left hand (crossed) module by the morphism $\lambda : \pi_1(A \cap B, x) \to \pi_1(B, x)$. In dimension 2, this is a nonabelian result (cf. Theorem 5.4.1) which has not been otherwise obtained. Thus the theory copes happily and in a uniform fashion with the action of the fundamental group, or indeed groupoid, whereas actions of a fundamental group are usually treated via covering spaces.

As a simple example of the approach to modules, consider the map of spaces

$$f : S^n \vee [0,1] \to S^n \vee S^1 \tag{0.0.3}$$

where the n-sphere ($n \geqslant 2$) is attached to the unit interval at 0, and f identifies $0, 1$. We see the nth homotopy of the first space as the free $\mathcal{I}$–module on one generator of dimension n at 0, where $\mathcal{I}$ is the 'unit interval groupoid', and of the second space as the free module on one generator over the infinite cyclic group C_∞. Now to say an object is free means it satisfies a universal property. It is desirable aesthetically, and is good practice, to prove these result by direct verifications of the universal property. We set up the machinery to do exactly this. Of course this particular result could be proved using covering spaces and homology; but we do not use homology theory!

Corollaries of this homotopical excision result include:

(i) the Brouwer degree theorem (the n-sphere S^n is $(n-1)$-connected and the homotopy classes of maps of S^n to itself are classified by an integer called the *degree* of the map);

(ii) the relative Hurewicz theorem, which usually relates relative homotopy and homology groups, but seen here as describing the morphism $\pi_n(X, A, x) \to \pi_n(X \cup CA, x)$ when (X, A) is $(n-1)$-connected;

(iii) Whitehead's theorem that $\pi_2(A \cup \{e^2_\lambda\}, A, x)$ is a free crossed $\pi_1(A, x)$-module; and

(iv) computations of the second homotopy group, and indeed of the homotopy 2-type, of the mapping cone of the map $Bf : BG \to BH$ of classifying spaces of groups induced by a morphism $f : G \to H$ of groups.

The last two results involve the nonabelian structure of crossed modules.

We have used the word 'induced' several times in the above comments. This is put in the general context of fibred categories in Appendix A.7.

A second major theme is to develop the techniques to define the *classifying topological space* BC of a crossed complex C and to prove that if X_* is the skeletal filtration of a *CW*-complex X, then there is a bijection of sets of homotopy classes

$$[X, BC] \cong [\Pi X_*, C], \tag{0.0.4}$$

where on the left we have homotopy classes of continuous maps and on the right homotopy classes in the algebraic category of crossed complexes. This result, Theorem 10.4.17, is shown in Chapter 12 to allow some specific calculation, not possible by other means, of homotopy classes of maps of spaces.

The routes to these results do not involve traditional tools of algebraic topology such as homology or simplicial approximation, but are not direct. They involve less well known methods of a cubically defined *higher homotopy groupoid* ρX_*. One advantage of cubical methods is an easy and intuitive description of *algebraic inverses to subdivision*; this means algebraic control not only of cutting something up, but also putting it back together again. This combination is especially suitable for *local-to-global problems*.

A second advantage which we exploit is the use of tensor products derived from the standard formula for cubes $I^m \times I^n \cong I^{m+n}$.

The complete and intricate story has its main facts summarised in the following diagram and comments:

$$
\begin{tikzcd}
 & & \text{filtered spaces} \arrow[dll, bend right, "C_* = \nabla \circ \Pi"'] \arrow[dl, shift right, "\Pi"'] \arrow[dr, shift left, "\rho"] & \text{filtered cubical sets} \arrow[l, "| \, |"'] \\
\text{operator chain complexes} \arrow[r, shift right, "\Theta"'] & \text{crossed complexes} \arrow[l, shift right, "\nabla"'] \arrow[ur, shift right, "\mathcal{B}"'] \arrow[rr, shift left, "\lambda"] & & \text{cubical } \omega\text{-groupoids with connections} \arrow[ll, shift left, "\gamma"] \arrow[ul, shift left, "| \, | \circ U_*"] \arrow[u, "U_*"']
\end{tikzcd}
\qquad \text{(Main Diagram)}
$$

in which

MD 1) the categories FTop of filtered spaces, Crs of crossed complexes and ω-Gpds of ω-groupoids, are monoidal closed, and have a notion of homotopy using $\otimes$ and unit interval objects;

MD 2) ρ, Π are homotopical functors (that is they are defined in terms of homotopy classes of certain maps), and preserve homotopies;

MD 3) λ, γ are inverse adjoint equivalences of monoidal closed categories, and λ is a kind of 'nerve' functor;

MD 4) there is a natural equivalence $\gamma\rho \simeq \Pi$, so that either ρ or Π can be used as appropriate;

MD 5) ρ preserves certain colimits and certain tensor products, and hence so also does Π;

MD 6) the category Chn of chain complexes with a groupoid of operators is monoidal closed, and ∇ is a monoidal functor which has a right adjoint Θ;

MD 7) by definition, the *cubical filtered classifying space* is $\mathcal{B} = | \, | \circ U_* \circ \lambda$ where U_* is the forgetful functor to filtered cubical sets using the filtration of an ω-groupoid by skeleta, and $| \, |$ is geometric realisation of a cubical set;

MD 8) there is a natural equivalence $\Pi \circ \mathcal{B} \simeq 1$;

MD 9) if C is a crossed complex and its cubical classifying space is defined as $BC = (\mathcal{B}C)_\infty$, then for a CW-complex X, and using homotopy as in MD1) for crossed complexes, there is a natural bijection of sets of homotopy classes as in equation (0.0.4).

Why crossed complexes?

• They generalise groupoids and crossed modules to all dimensions, and the functor Π is classical, involving relative homotopy groups.

• They are good for modelling CW-complexes.

• Free crossed resolutions enable calculations with small CW-models of $K(G, 1)$s *and their maps* (Whitehead, Wall, Baues).

• Crossed complexes give a kind of 'linear model' of homotopy types which includes all 2-types. Thus although they are not the most general model by any means (they do no contain quadratic information such as Whitehead products), this simplicity makes them easier to handle and to relate to classical tools. The new methods and results obtained for crossed complexes can be used as a model for more complicated situations. This is how a general n-adic Hurewicz Theorem was found [BL87a].

• They are convenient for some *calculations* generalising methods of computational group theory, e.g. trees in Cayley graphs. We explain some results of this kind in Chapter 11.

• They are close to the traditional chain complexes with a group(oid) of operators, as shown in MD6), and are related to some classical homological algebra (e.g. *identities among relations for groups*). Further, if SX is the simplicial singular complex of a space, with its skeletal filtration, then the crossed complex $\Pi(SX)$ can be considered as a slightly non commutative version of the singular chains of a space. However crossed complexes have better realisation properties than the related chain complexes.

• The monoidal structure is suggestive of further developments (e.g. *crossed differential algebras*).

• They have a good homotopy theory, with a *cylinder object, and homotopy colimits*. The homotopy classification result (0.0.4) generalises a classical theorem of Eilenberg-Mac Lane.

• They have an interesting relation with the Moore complex of simplicial groups and of simplicial groupoids, [Ash88, NT89, EP97].

Why cubical ω-groupoids with connections?

The definition of these objects is more difficult to give. Here we explain why we need to introduce such new structures.

• The functor ρ gives a form of *higher homotopy groupoid*, thus confirming the visions of topologists of the early 20th century of higher dimensional nonabelian forms of the fundamental group.

• They are equivalent to crossed complexes, and this equivalence is a kind of cubical and nonabelian form of the Dold-Kan theorem, relating chain complexes with simplicial abelian groups.

• They have a clear *monoidal closed structure*, and notion of homotopy, from which one can deduce that on crossed complexes, using the equivalence of categories.

• It is easy to relate the functor ρ to tensor products, but quite difficult to do this for Π.

• Cubical methods, unlike globular or simplicial methods, allow for a simple *algebraic inverse to subdivision*, involving multiple compositions in many directions, Remark 13.1.11, which is crucial for the proof of our HHvKT in Chapter 14.

• The additional structure of 'connections', and the equivalence with crossed complexes, allows the notion of *thin cube*, Section 13.7, which subsumes the idea of commutative cube, and the proof that *multiple compositions of thin cubes are thin*. The last fact is another key component of the proof of the HHvKT, see Theorem 14.2.9.

• They yield a construction of a *(cubical) classifying space* $BC = (\mathcal{B}C)_\infty$ of a crossed complex C, which generalises (cubical) versions of Eilenberg-Mac Lane spaces, including the local coefficient case.

• There is a current *resurgence of the use of cubes* in for example combinatorics, algebraic topology, and concurrency.

Structure of the book

Because of the complications set out above in the Main Diagram, and in order to communicate the basic intuitions, we divide our account into three parts, each with an Introduction.

In Part I we give some history of work on the fundamental group and groupoid, in particular explaining how the van Kampen theorem with a set of base points gives a method of computation of fundamental groups. It was the extension of this classical theorem from groups to groupoids that led to the question of the putative uses of groupoids in higher homotopy theory.

We are then mainly concerned with the extension of nonabelian work to dimension 2, using the key concept, due to J.H.C. Whitehead in 1946, of *crossed module*. This is a morphism

$$\mu : M \to P$$

of groups together with an action of the group P on the right of the group M, written $(m, p) \mapsto m^p$, satisfying the two rules:

CM1) $\mu(m^p) = p^{-1}(\mu m)p$;

CM2) $m^{-1}nm = n^{\mu m}$,

for all $p \in P,\ m, n \in M$. Algebraic examples of crossed modules include normal subgroups M of P; P-modules; the inner automorphism crossed module $M \to \mathrm{Aut}\, M$; and many others. There is the beginnings of a combinatorial and also computational crossed module theory.

The standard geometric example of crossed module is the boundary morphism of the second relative homotopy group

$$\partial : \pi_2(X, X_1, x) \to \pi_1(X_1, x)$$

where X_1 is a subspace of the topological space X and $x \in X_1$. This relative homotopy group is defined in terms of certain homotopy classes of maps $I^2 \to X$. For this reason, and because they are a good model of 2-dimensional pointed homotopy theory, crossed modules are commonly seen as good candidates for 2-*dimensional groups*.

The remarkable fact is that we can calculate with these 2-dimensional structures and apply these calculations to topology using a 2-dimensional version of the van Kampen theorem for the fundamental group.

We give a substantial account of this 2-dimensional theory because the step from dimension 1 to dimension 2 involves a number of new ideas for which the reader's intuition needs to be developed. In particular, calculation with crossed modules requires some extensions of combinatorial group theory, for example to induced crossed modules. Finally in this Part, the proof of the van Kampen type theorem for crossed modules, involves a notion of *homotopy double groupoid*, based on composing squares with common edges. The intuition for this construction was the start of the theory of this book.

The aim of Part II is to give a kind of handbook of applications of *crossed complexes*, assuming some major properties which are proved in Part III. The theory extends many basic results in homotopy theory, such as the relative Hurewicz theorem. Among results found in no other text on algebraic topology is the homotopy classification theorem referred to above.

In Part III we define the *cubical ω-groupoids with connections* whose properties are the power behind the applications of crossed complexes. In principle, and this would be the logical order, Part III can be read independently of the previous parts, referring back for some basic definitions.

At the end, Part IV, we give a 'Conclusion', in which we try to evaluate what has been done, to direct the reader to some other current directions, and to indicate some of the many things yet to be done with these tools.

Prerequisites: The aim is for large parts of this book to be readable by a graduate student acquainted with general topology, the fundamental group, notions of homotopy, and some basic methods of category theory. Many of these areas, including the concept of groupoid and its uses, are covered in Brown's text 'Topology and Groupoids', [Bro06]. The only theory we have to assume for the homotopy classification theorem in Chapter 10 is some results on the geometric realisation of cubical sets.

Some aspects of category theory perhaps less familiar to a graduate student are summarised in an Appendix, particularly the notion of representable functor, the notion of dense subcategory, and the preservation of colimits by a left adjoint functor. This last fact is a basic tool of algebraic computation for those algebraic structures which are built up in several levels, since it can often show that a colimit of such a structure can be built up level by level.

We make no use of classical tools such as simplicial approximation, but some knowledge of homology and homotopy of chain complexes could be useful at a few points, to help motivate some definitions.

We feel it is important for readers to understand how this theory derives from the basic intuitions and history of algebraic topology, and so we start Part I with some history. After that, historical comments are given in Notes at the end of each chapter.

Part I

1 and 2-dimensional results

Introduction to Part I

Part I develops in dimensions 1 and 2 that aspect of nonabelian algebraic topology related to the van Kampen Theorem (vKT).

We start by giving a Historical background, and outline the proof of the van Kampen theorem in dimension 1. It was an analysis of this proof which suggested the higher dimensional possibilities.

We then explain the functor

$$\Pi_2 : (\text{pointed pairs of spaces}) \to (\text{crossed modules})$$

in terms of second relative homotopy groups, state a Higher Homotopy van Kampen Theorem (HHvKT) for this, and give applications. These applications involve the algebra of crossed modules, and two important constructions for calculations with crossed modules, namely coproducts of crossed modules on a fixed base group (Chapter 4) and induced crossed modules (Chapter 5). The latter concept illustrates well the way in which low dimensional identifications in a space can influence higher dimensional homotopical information. Induced crossed modules also include free crossed modules, which are important in applications to defining and determining identities among relations for presentations of groups. This has a relation to the cohomology theory of groups.

Both of these chapters illustrate how some nonabelian calculations in homotopy theory may be carried out using crossed modules. They also show the advantages of having an invariant stronger than just an abelian group of even a module over a group. The latter are pale shadows of the structure of a crossed module.

Finally in this Part, Chapter 6 gives the proof of the HHvKT for the functor Π_2. A major interest here is that this proof requires another structure, namely that of *double groupoid with connection,* which we abbreviate to *double groupoid*. We therefore construct a functor

$$\boldsymbol{\rho}_2 : (\text{triples of spaces}) \to (\text{double groupoids}),$$

and show that this is equivalent in a clear sense to a small generalisation of our earlier Π_2 functor, to

$$\Pi_2 : (\text{triples of spaces}) \to (\text{crossed modules of groupoids}).$$

Here a triple of spaces is of the form (X, X_1, X_0), where $X_0 \subseteq X_1 \subseteq X$, and the pointed case is when X_0 is a singleton.

This substantial chapter develops the 2-dimensional groupoid theory which is then used in the proof of the HHvKT, which gives precise situations where $\boldsymbol{\rho}_2$, and hence also Π_2, preserves colimits. The surprising fact is that in this book we are able to obtain many new nonabelian calculations in homotopy theory without any of the standard machinery of algebraic topology, such as simplicial complexes, simplicial approximation, chain complexes, or homology theory.

All this theory generalises to higher dimensions, as we show in Parts II and III, but the new ideas and basic intuitions are more easily explained in dimension 2.

Chapter 1

History

In this chapter we give some of the context and historical background to the main work of this book, in order to show the tradition in which this work has been done. It is hoped that this will help you to understand and evaluate the results, and to analyse the potential for further developments and applications. We are showing how the extensions from groups, to groupoids, and then to certain multiple groupoids, and to other related structures, enables new results and understanding in algebraic topology.

It is generally accepted that the notion of abstract group is a central concept of mathematics, and one which allows the successful expression of the intuitions of reversible processes. In order to obtain the higher dimensional, nonabelian, local to global results described briefly in the Introduction, the concept of group has:

A) to be 'widened' to that of groupoid, which in a sense generalises the notion of group to allow a spatial component, and

B) to be 'increased in height' to higher dimensions.

Step A) is an essential requirement for step B).

A major stimulus for this view was work of Philip Higgins in his 1963 paper [Hig64], and this book is based largely on his resulting collaboration with Brown. Higgins writes in the Preface to [Hig71] that: "The main advantage of the transition [from groups to groupoids] is that the category of groupoids provides a good model for certain aspects of homotopy theory. In it there are algebraic models such notions as path, homotopy, deformation, covering and fibration. Most of these become vacuous when restricted to groups, although they are clearly relevant to group-theoretic problems. . . . There is another side of the coin: in applications of group theory to other topics it is often the case that the natural object of study is a groupoid rather than a group, and the algebra of groupoids may provide a more concrete tool for handling concrete problems."

In fact there is a range of intuitions which abstract groups are unable to express, and for which other concepts such as groupoid, pseudogroup and inverse semigroup have turned out to be more appropriate. As Mackenzie writes in [Mac87]:

> The concept of groupoid is one of the means by which the twentieth century reclaims the original domain of applications of the group concept. The modern, rigorous concept of group is far too restrictive for the range of geometrical applications envisaged in the work of Lie. There have thus arisen the concepts of Lie pseudogroup, of differentiable and of

Lie groupoid, and of principal bundle – as well as various related infinitesimal concepts such as Lie equation, graded Lie algebra and Lie algebroid – by which mathematics seeks to acquire a precise and rigorous language in which to study the symmetry phenomena associated with geometrical transformations which are only locally defined.

A number of these concepts related to groupoids were initiated by C. Ehresmann over many years, particularly the notion of differential or Lie groupoid ([Ehr83, Bro07]).

A failure to accept a relaxation of the concept of group made it difficult to develop a higher dimensional theory modelling some key aspects of homotopy theory. To see the reasons for this we need to understand the basic intuitions which a higher dimensional theory is trying to express, and to see how these intuitions were dealt with historically. This study will confirm a view that it is reasonable to examine and develop the algebra which arises in a natural way from the geometry rather than insist that the geometry has to be expressed within the current available concepts, schemata and paradigms.

1.1 Basic intuitions

There were two simple intuitions involved. One was the notion of an

algebraic inverse to subdivision.

That is, we know how to cut things up, but do we have available an algebraic control over the way we put them together again? This is of course a general problem in mathematics, science and engineering, where we want to represent and determine the behaviour of complex objects from the way they are put together from standard pieces. This is the 'local-to-global problem'. Any algebra which gives new insights into questions of this form, and yields new computations, clearly has arguments in its favour.

We explain this a bit more in a very simple situation. We often translate geometry into algebra. For example, a figure as follows:

$$\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \xleftarrow{c} \bullet \xrightarrow{d} \bullet$$

is easily translated into

$$abc^{-1}d.$$

Again, given a diagram as follows:

$$\begin{tikzcd} \bullet \arrow[r, "a"] \arrow[d, "c"'] & \bullet \arrow[d, "b"] \\ \bullet \arrow[r, "d"'] & \bullet \end{tikzcd} \tag{1.1.1}$$

it is easy to write

$$ab = cd, \qquad \text{or} \qquad a = cdb^{-1}.$$

All this is part of the standard repertoire of mathematics. The formulae given make excellent sense as part of say the theory of groups. We also know how to calculate with such formulae.

The problem comes when we try to express similar ideas in one dimension higher. How can one write down algebraically the following picture, where each small square is supposed labeled?

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \bullet \arrow[l] \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] & \bullet \arrow[l] \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet & \bullet \arrow[l] \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[u] & \bullet \arrow[l] \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u]
\end{tikzcd} \tag{1.1.2}$$

Again, how can one write down algebraically the formulae corresponding to the above commutative square 1.1.1 but now for the cube:

$$\begin{tikzcd}
 & \bullet \arrow[rr] \arrow[dd] & & \bullet \arrow[dd] \\
\bullet \arrow[ur] \arrow[rr] \arrow[dd] & & \bullet \arrow[ur] \arrow[dd] & \\
 & \bullet \arrow[rr] & & \bullet \\
\bullet \arrow[ur] \arrow[rr] & & \bullet \arrow[ur] &
\end{tikzcd} \tag{1.1.3}$$

What does it mean for the *faces* of the cube to commute, or for the top face to be the composition, in some sense, of the other faces?

It is interesting that the step from a linear statement to a 2-dimensional statement should need a lot of apparatus; it also took a lot of experimentation to find an appropriate formulation. As we shall see later, the 2-dimensional composition (1.1.2) requires double groupoids or double categories, while the second (1.1.3) requires double groupoids with *thin structure*, or with *connections*.

Thus the step from dimension 1 to dimension 2 is the critical one, and for this reason most of Part I of this book is devoted to the 2-dimensional case. Further reasons are that the theory in dimension 2 is more straightforward than it becomes in higher dimensions; illustrative pictures are easier to give; and the novel features of the 2-dimensional theory need to be well understood before passing to higher dimensions. It is also intriguing that so much can be done once one has the mathematics to express the intuitions, and that then the mathematical structures control the ways the calculations have to go. This requires an emphasis on *universal properties*, which are afterwards interpreted to give formulae.

1.2 The fundamental group and homology

The above questions on 2-dimensional compositions did not arise out of the void but from a historical context which we now explain.

The intuition for a *Nonabelian Algebraic Topology* was seen early on in algebraic topology, after the ideas of homology and of the fundamental group $\pi_1(X, x)$ of a space X at a base point x of X were developed.

The motivation for Poincaré's definition of the fundamental group in his 1895 paper [Poi96] seems to be from the notion of monodromy, that is the change in the value of a meromorphic function of many complex variables as it is analytically continued along a loop avoiding the singularities. This change in value depends only on the homotopy class of the loop, and this consideration led to the notion of the group $\pi_1(X, x)$ of homotopy classes of loops at x, where the group structure arises from composition of loops. Poincaré called this group the *fundamental group* ; this fundamental group $\pi_1(X, x)$, with its relation to covering spaces, surface theory, and the later combinatorial group theory, came to play an increasing rôle in the geometry, complex analysis and algebra of the next hundred years.

It also seems possible that an additional motivation arose from dynamics, in the classification of orbits in a phase space.

The utility of the group concept in homotopy theory is increased by the relations between the fundamental group considered as a functor from based topological spaces to groups

$$\pi_1 : \mathsf{Top}_* \to \mathsf{Groups}$$

and another functor called the *classifying space*

$$B : \mathsf{Groups} \to \mathsf{Top}_*,$$

which is the composite of the *geometric realisation* and the *nerve functor* N from groups to simplicial sets.

We shall review the properties of B in Section 2.4. Now let us note that B and π_1 are inverses in some sense. To be more precise, BG is a based space that has all homotopy groups trivial except the fundamental group, which itself is isomorphic to G. Moreover, if X is a connected based CW-complex and G is a group, then there is a natural bijection

$$[X, BG]_* \cong \mathrm{Hom}(\pi_1 X, G),$$

where the square brackets denote pointed homotopy classes of maps.

It follows that there is a map

$$X \to B\pi_1 X$$

inducing an isomorphism of fundamental groups. It is in this sense that *groups are said to model homotopy 1-types*, and a computation of a group G is also regarded as a computation of the 1-type of the classifying space BG.

The fundamental group of a space may be calculated in many cases using the Seifert-van Kampen Theorem (see Section 1.5), and in other cases using fibrations of spaces. The main result on the latter, for those familiar with fibrations, is that if $1 \to K \to E \to G \to 1$ is a short exact sequence of groups, then the induced sequence $BK \to BE \to BG$ is a fibration sequence of spaces. Conversely, if $F \xrightarrow{i} X \xrightarrow{p} Y$ is a fibration sequence of spaces, and $x \in F$ then there is an induced exact sequence of groups and based sets

$$\cdots \longrightarrow \pi_1(F, x) \xrightarrow{i_*} \pi_1(X, x) \xrightarrow{p_*} \pi_1(Y, px) \to \pi_0(F) \to \pi_0(X) \to \pi_0(Y).$$

This result gives some information on $\pi_1(X, x)$ if the other groups are known and even more if the various spaces are connected. We shall return to this sequence in Section 2.6, and it will be used in other contexts, with more information on exactness at the last few terms, in Section 12.4.

Higher dimensional topological information in terms of Betti numbers and torsion coefficients had been obtained much earlier than the definition of the fundamental group. These numbers were combined into the powerful idea of the abelian homology groups $H_n(X)$ of a space X defined for all $n \geqslant 0$, and which gave very useful topological information on the space. They measured the presence of 'holes' in X of various dimensions and of various types. The origins of homology theory lie in integration, the theorems of Green and Stokes, and complex variable theory.

The notion of 'boundary' and of a 'cycle' as having zero boundary is crucial in the methods and results of this theory, but was always difficult to express precisely until Poincaré brought in simplicial decompositions, and the notion of a 'chain' as a formal sum of oriented simplices. It seems that the earlier writers thought of a cycle as in some sense a 'composition' of the pieces of which it was made, but this 'composition' was, and still is, difficult to express precisely. Dieudonné in [Die89] suggests that the key intuitions can be expressed in terms of cobordism. In any case, the notion of 'formal sum' fitted well with integration, where it was required to integrate over a formal sum of domains of integration, with the correct orientation for these:

$$\int_C f\,dz + \int_{C'} f\,dz = \int_{C+C'} f\,dz$$

It was also found that if X is connected then the group $H_1(X)$ is the fundamental group $\pi_1(X,x)$ made abelian:

$$H_1(X) \cong \pi_1(X,x)^{\mathrm{ab}}.$$

So the nonabelian fundamental group gave much more information than the first homology group. However, the homology groups were defined in all dimensions. So there was pressure to find a generalisation to all dimensions of the fundamental group.

1.3 The search for higher dimensional versions of the fundamental group

According to [Die89], Dehn had some ideas on this search in the 1920's, as would not be surprising. The first published attack on this question was the work of Čech, using the idea of classes of maps of spheres instead of maps of circles. He submitted his paper on higher homotopy groups $\pi_n(X,x)$ to the International Congress of Mathematicians at Zurich in 1932. The story is that Alexandrov and Hopf quickly proved that these groups were abelian for $n \geqslant 2$, and so on these grounds persuaded Čech to withdraw his paper. All that appeared in the Proceedings of the Congress was a brief paragraph, [Čec32].

The main algebraic reason for this abelian nature was the following result, in which the two compositions $\circ_1, \circ_2$ are thought of as compositions of 2-spheres in two directions.

Theorem 1.3.1 *Let S be a set with two monoid structures $\circ_1,\ \circ_2$ each of which is a morphism for the other. Then the two monoid structures coincide and are Abelian.*

Proof The condition that the structure $\circ_1$ is a morphism for $\circ_2$ is that the function

$$\circ_1 : (S,\circ_2) \times (S,\circ_2) \to (S,\circ_2)$$

is a morphism of monoids, where $(S,\circ_2)$ denotes S with the monoid structure $\circ_2$. This condition is equivalent to the statement that for all $x,y,z,w \in S$

$$(x \circ_2 y) \circ_1 (z \circ_2 w) = (x \circ_1 z) \circ_2 (y \circ_1 w).$$

This can be interpreted as saying that the diagram

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \qquad \begin{array}{l} \ \ulcorner\!\!\to 2 \\ \downarrow \\ 1 \end{array}$$

has only one composition. Here the arrows indicate that we are using matrix conventions in which the first coordinate gives the rows, and the second coordinate gives the columns. This law is commonly called the *interchange law* .

We now use some special cases of the interchange law. Let $e_1,\ e_2$ denote the identities for the structures $\circ_1,\ \circ_2$. Consider the matrix

$$\begin{bmatrix} e_1 & e_2 \\ e_2 & e_1 \end{bmatrix}$$

This yields easily that $e_1 = e_2$. We write then e for e_1.

Now we consider the matrix composition

$$\begin{bmatrix} x & e \\ e & w \end{bmatrix}$$

Interpreting this in two ways yields

$$x \circ_1 w = x \circ_2 w.$$

So we write $\circ$ for $\circ_1$.

Finally we consider the matrix composition

$$\begin{bmatrix} e & y \\ z & e \end{bmatrix}$$

and find easily that $y \circ z = z \circ y$. This completes the proof.

Incidentally, it will also be found that associativity comes for free. We leave this to the reader. □

This result seemed to kill any possibility of "nonabelian algebraic topology", or of any generalisations to higher dimensions of the fundamental group. In 1935, Hurewicz, without referring to Čech, published the first of his celebrated notes on higher homotopy groups, [Hur35], and the latter groups are often referred to as the 'Hurewicz homotopy groups'. As the abelian higher homotopy groups came to be accepted, a considerable amount of work in homotopy theory moved as far as possible from group theory and the nonabelian fundamental group, and the original concern about the abelian nature of the higher homotopy groups came to be seen as a quirk of history, an unwillingness to accept a basic fact of life. Indeed, Alexandrov and Finikof in their Obituary Notice for Čech, [AF61], referred to the unfortunate lack of appreciation of Čech's work on higher homotopy groups, resulting from too much attention to the disadvantage of their abelian nature.

However important nonabelian work in dimension 2 was published by J.H.C. Whitehead in 1941, 1946 and 1949, with the second paper introducing the term crossed module – these crossed modules are a central theme of this book. Brown remembers Henry Whitehead remarking in 1957 that early workers in homotopy theory were fascinated by the action of the fundamental group on higher homotopy groups. Many also were dissatisfied with the fact that the composition in higher homotopy groups was independent of the direction. Deeper reasons for this independence are contained in the theory of iterated loop spaces (see the book by Adams, [Ada78], or the books and survey articles by May [May72, May77a, May77b, May82].

A new possibility eventually arose in 1967 through the notion of *groupoid*, which we discuss in the next section.

1.4 The origin of the concept of abstract groupoid

A groupoid is defined formally as a small category in which every arrow is invertible. For more details see the surveys [Bro84, Wei01], and the books [Bro06, Hig71].

There are two important, related and relevant differences between groupoids and groups. One is that groupoids have a partial multiplication, and the other is that the condition for two elements to be composable is a geometric one (namely the end point of one is the starting point of the other). This partial multiplication allows for groupoids to be thought of as "groups with many identities". The other is that the geometry underlying groupoids is that of directed graphs, whereas the geometry underlying groups is that of based sets, i.e. sets with a chosen base point. It is clear that graphs are more interesting than sets, and can reflect more geometry. Hence people find in practice that groupoids can reflect more geometry than can groups alone. It seems that the objects of a groupoid allow the addition of a spatial component to group theory.

An argument usually made for groups is that they give the mathematics of reversible processes, and hence have a strong connection with symmetry. This argument applies even more strongly for groupoids. For groups, the processes all start and return to the same position. This is like considering only journeys which start at and end at the same place. However to *analyse* a reversible process, such as a journey, we must describe the intermediate steps, the stopping places. This description requires groupoids, since in this setting the processes described are allowed to start at one point and finish at another. Groupoids clearly allow a more flexible and powerful analysis, and this confirms a basic intuition that, in dimension 1, groupoids are more convenient than groups for writing down an 'algebraic inverse to subdivision'.

The definition of groupoid arose from Brandt's attempts to extend to quaternary forms Gauss' work on a composition law of binary quadratic forms, which has a strong place in *Disquitiones Arithmeticae*. It is of interest here that Bourbaki [Bou70], p.153, cites this composition law as an influential early example of a composition law which arose not from numbers, even taken in a broad sense, but from distant analogues[1]. Brandt found that each quaternary quadratic form had a left unit and a right unit, and that two forms were composable if and only if the left unit of one was the right unit of the other. This led to his 1926 paper on groupoids [Bra26]. (A modern account of this work on composition of forms is given by Kneser *et al.* [KOK+86].)

Groupoids were then used in the theory of orders of algebras. Curiously, groupoids did not form an example in Eilenberg and Mac Lane's basic 1945 paper on category theory [EM45]. Groupoids appear in Reidemeister's 1932 book on topology [Rei49], as the edge path groupoid, and for handling isomorphisms of a family of structures. The fundamental groupoid of a space was well known by the 1950's, and Crowell and Fox write in [CF77]:

> A few [definitions], like that of a group or of a topological space, have a fundamental importance to the whole of mathematics that can hardly be exaggerated. Others are more in the nature of convenient, and often highly specialised, labels which serve principally to pigeonhole ideas. As far as this book is concerned, the notions of category and groupoid belong to the latter class. It is an interesting curiosity that they provide a convenient systematisation of the ideas involved in developing the fundamental group.

It may well be that a concept providing a 'convenient systematisation' is generally an indication of underlying power of that concept. We referred earlier to the extensive work of C. Ehresmann on groupoids in differential topology. One motivation for this work was his strong interest in local-to-global situations. Problems of this kind are often central in mathematics and in science.

The fundamental groupoid $\pi_1(X, A)$ on a set A of base points is introduced and used in [Bro06]. Its successes suggest the value of an aesthetic approach to mathematics, namely that the concept

[1] C'est vers cette même époque que, pour le premier fois en Algèbre, la notion de loi de composition s'étend, dans deux directions différents, à des élements qui ne présentent plus avec les ⟨⟨nombres⟩⟩ (au sens le plus large donné jusque-là à ce mot) que des analogies lointaines. La première de ces extensions est due à C.F.Gauss, à l'occasion de ses recherches arithmétiques sur les formes quadratiques . . .

which feels right and gives 'a convenient systematisation' is likely to be the most powerful one. In this viewpoint, much good mathematics enables difficult things to become easy, and an important part of the development of good mathematics is finding: (i) the appropriate underlying structures, (ii) the appropriate language to describe these structures, and (iii) means of calculating with these structures. Without the appropriate structures to guide us, we may take many wrong turnings.

There is no benefit today in arithmetic in Roman numerals. There is also no benefit today in insisting that the group concept is more fundamental than that of groupoid; one uses each at the appropriate place. It is as well to distinguish the sociology of the use of a mathematical concept from the scientific consideration of its relevance to the progress of mathematics.

It should also be said that the development of new concepts and language is a different activity from the successful employment of a range of known techniques to solve already formulated problems.

The notion that groupoids give a more flexible tool than groups in some situations is only beginning to be widely appreciated. One of the most significant of the books which use the notion seriously is Connes book "Noncommutative geometry", published in 1994, [Con94]. He states that Heisenberg discovered quantum mechanics by considering the *groupoid of transitions* for the hydrogen spectrum, rather than the usually considered group of symmetry of an individual state. This fits with the previously expounded philosophy. The main examples of groupoids in his book are equivalence relations and holonomy groupoids of foliations.

On the other hand, in books on category theory the role of groupoids is often fundamental (see for example Mac Lane and Moerdijk [MLM96]). In foliation theory, which is a part of differential topology and geometry, the notion of *holonomy groupoid* is widely used. For surveys of the use of groupoids, see [Bro87, Hig71, Wei01, Mac05, Bro07]. Groupoids have been used extensively by Ehresmann, [Ehr80].

1.5 The van Kampen Theorem

We believe a change of prospects for homotopy theory came about in a roundabout way, in the mid 1960s. R. Brown was writing the first edition of the book [Bro06] and became dissatisfied with the standard treatments of the van Kampen Theorem. This basic tool computes the fundamental group of a space X given as the union of two connected open subsets U_1, U_2 with connected intersection U_{12}. For those familiar with the concepts, the result is that the natural morphism

$$\pi_1(U_1, x) *_{\pi_1(U_{12},x)} \pi_1(U_2, x) \to \pi_1(X, x) \tag{1.5.1}$$

induced by inclusions is an isomorphism. The group on the left hand side of the above arrow is the free product with amalgamation; it is the construction for groups corresponding to $U_1 \cup U_2$ for spaces, as we shall see later in discussing pushouts. This version of the theorem was given by Crowell [Cro71], based on lectures by R.H. Fox. One important consequence is that the fundamental group shared the same possibilities and the same difficulties of computation as general abstract groups.

The problem was with the connectivity assumption on U_{12}, since this prevented the use of the theorem for deducing the result that the fundamental group of the circle S^1 is isomorphic to the group $\mathbb{Z}$ of integers. (See Section 1.7 where $\pi_1(S^1)$ is calculated.) If S^1 is the union of two connected open sets, then their intersection cannot be connected. So the fundamental group of the circle is usually determined by the method of covering spaces. Of course this method is basic stuff anyway, and needs to be explained, but having to make this detour, however attractive, is unaesthetic.

It was found that a uniform method could be given using nonabelian cohomology, [Bro65], but

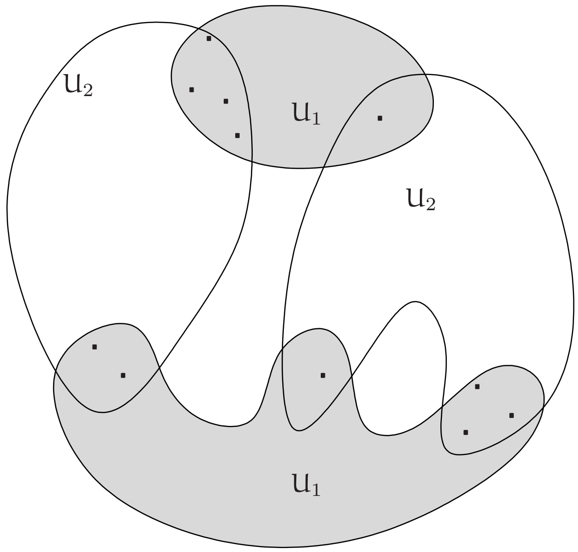


Figure 1.1: Example of spaces in a van Kampen type situation

a full exposition of this became turgid. Then Brown came across the paper by Philip Higgins entitled 'Presentations of groupoids with applications to groups' [Hig64], which among other things defined free products with amalgamation of groupoids. We will explain something about groupoids a bit later. It seemed reasonable to insert an exercise in the book on an analogous result to 1.5.1 for the fundamental groupoid $\pi_1(X)$, namely that the natural morphism of groupoids

$$\pi_1(U_1) *_{\pi_1(U_{12})} \pi_1(U_2) \to \pi_1(X) \tag{1.5.2}$$

is an isomorphism. It then seemed desirable to write out a solution to the exercise, and lo and behold! the solution was much clearer *and more powerful* than all the turgid stuff on nonabelian cohomology. Further work yielded the idea that it was sensible to generalise from the fundamental group $\pi_1(X, x)$ on one base point x to the fundamental groupoid $\pi_1(X, A)$ on a set A chosen freely according to a given geometric situation. In particular if U_{12} is not connected it is not clear from which component of U_{12} a base point should be chosen. So one hedges one's bets, and chooses a *set of base points*, one in each component of U_{12}. One finds that the natural morphism

$$\pi_1(U_1, A) *_{\pi_1(U_{12}, A)} \pi_1(U_2, A) \to \pi_1(X, A) \tag{1.5.3}$$

is also an isomorphism and that the proof of this result using groupoids is simpler than the original proof of 1.5.1 for groups. One also obtains a new range of calculations. For example, U_1, U_2, U_{12} may have respectively 27, 63, and 283 components, and yet X could be connected - a description of the fundamental group of this situation in terms of groups alone is not so easy.

In view of these results the writing of the first edition of the book [Bro06] was redirected to give a full account of groupoids and the van Kampen Theorem. A conversation with G.W.Mackey in 1967 informed Brown of Mackey's work on ergodic groupoids (see the references in [Bro87]). It seemed that if the idea of groupoid arose in two separate fields, then there was more in this than met the eye. Mackey's use of the relation between group actions and groupoids suggested the importance of strengthening the book with an account of covering spaces in terms of groupoids, following the initial lead of Higgins in [Hig64] and of Gabriel and Zisman in [GZ67].

Later Grothendieck was to write (1985):

"The idea of making systematic use of groupoids (notably fundamental groupoids of spaces, based on a given set of base points), however evident as it may look today, is to be seen as a significant conceptual advance, which has spread into the most manifold areas of mathematics. ... In my own work in algebraic geometry, I have made extensive use of groupoids - the first one being the theory of the passage to quotient by a "pre-equivalence relation" (which may be viewed as being no more, no less than a groupoid in the category one is working in, the category of schemes say), which at once led me to the notion (nowadays quite popular) of the nerve of a category. The last time has been in my work on the Teichmüller tower, where working with a "Teichmüller groupoid" (rather than a "Teichmüller group") is a must, and part of the very crux of the matter ..."

1.6 Proof of the van Kampen Theorem (groupoid case)

In this section we sketch a proof that the morphism induced by inclusions

$$\eta : \pi_1(U_1, A) *_{\pi_1(U_{12}, A)} \pi_1(U_2, A) \to \pi_1(X, A) \tag{1.6.1}$$

is an isomorphism when U_1, U_2 are open subsets of $X = U_1 \cup U_2$ and A meets each path component of U_1, U_2 and $U_{12} = U_1 \cap U_2$.

What one would expect is that the proof would construct directly an inverse to η. Alternatively, the proof would verify in turn that η is surjective and injective.

The proof we give might at first seem roundabout, but in fact it follows the important procedure of *verifying a universal property*. One advantage of this procedure is that we do not need to show that the free product with amalgamation of groupoids exists in general, nor do we need to give a construction of it at this stage. Instead we define the free product with amalgamation by its universal property, which enables us to go directly to an efficient proof of the van Kampen Theorem. It also turns out that the universal property guides many explicit calculations.

We use the notion of pushout in the category of groupoids. It is a special case of the pushout in categories that we study in the Appendix. Let us recall the definition in this case. We say that the groupoid G and the two morphisms of groupoids $G_1 \xrightarrow{j_1} G$ and $G_2 \xrightarrow{j_2} G$ are the *pushout* of the two morphisms of groupoids $G \xrightarrow{i_1} G_1$ and $G \xrightarrow{i_2} G_2$ if they satisfy

PO1) the diagram

$$\begin{tikzcd} G_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] & G_1 \arrow[d, "j_1"] \\ G_2 \arrow[r, "j_2"'] & G \end{tikzcd} \tag{1.6.2}$$

is a commutative square, i.e. $j_1 i_1 = j_2 i_2$,

PO2) the last diagram is universal with respect to this type of diagram, i.e. for any groupoid K and morphisms of groupoids $G_1 \xrightarrow{k_1} K$ and $G_2 \xrightarrow{k_2} K$ such that the following diagram is commutative

$$\begin{tikzcd} G_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] & G_1 \arrow[d, "k_1"] \\ G_2 \arrow[r, "k_2"'] & K \end{tikzcd} \tag{1.6.3}$$

there is a unique morphism of groupoids $k : G \to K$ such that $kj_1 = k_1, kj_2 = k_2$. The two diagrams are often combined into one as follows:

$$
\begin{tikzcd}
G_0 \arrow[r, "i_1"] \arrow[d, "i_2"'] & G_1 \arrow[d, "j_1"] \arrow[ddr, bend left, "k_1"] & \\
G_2 \arrow[r, "j_2"] \arrow[drr, bend right, "k_2"'] & G \arrow[dr, dashed, "k"] & \\
 & & K
\end{tikzcd}
\tag{1.6.4}
$$

We think of a pushout square as given by a standard input, the pair (i_1, i_2), and a standard output, the pair (j_1, j_2). The properties of this standard output are defined by reference to *all other* commutative squares with the same (i_1, i_2). At first sight this might seem strange, and logically invalid. However a pushout square is somewhat like a computer program: given the data of another commutative square of the right type, then the output will be a morphism (k in the above diagram) with certain defined properties.

It is a basic feature of universal properties that the standard output, in this case the pair (j_1, j_2) making the diagram commute, is determined up to isomorphism by the standard input (i_1, i_2). Further details will be given in the Appendix. See also [Bro06, Appendix A4].

Thus in our case, we have:

Theorem 1.6.1 *If* U_1, U_2 *are open subsets of* X, $X = U_1 \cup U_2$, *and* A *is a subset of* $U_{12} = U_1 \cap U_2$ *meeting each path component of* U_1, U_2, U_{12} *(and therefore of* X*), the following diagram of morphisms induced by inclusion*

$$
\begin{tikzcd}
\pi_1(U_{12}, A) \arrow[r, "i_1"] \arrow[d, "i_2"'] & \pi_1(U_1, A) \arrow[d, "j_i"] \\
\pi_1(U_2, A) \arrow[r, "j_2"'] & \pi_1(X, A)
\end{tikzcd}
$$

is a pushout of groupoids.

Proof So we suppose given a commutative diagram of morphisms of groupoids

$$
\begin{tikzcd}
\pi_1(U_{12}, A) \arrow[r, "i_1"] \arrow[d, "i_2"'] & \pi_1(U_1, A) \arrow[d, "k_1"] \\
\pi_1(U_2, A) \arrow[r, "k_2"'] & K
\end{tikzcd}
\tag{1.6.5}
$$

We have to prove that there is a unique morphism $k : \pi_1(X, A) \to K$ such that $kj_1 = k_1, kj_2 = k_2$.

Let us take an element $[\alpha] \in \pi_1(X, A)$ with $\alpha : (I, \partial I) \to (X, A)$. By the Lebesgue covering lemma ([Bro06, 3.6.4]) there is a subdivision

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

of I into intervals by equidistant points such that α maps each $[t_i, t_{i+1}]$ into U_1 or U_2 (possibly in both). Choose one of these for each i and call it U^i and α_i the restriction of α. This subdivision determines a decomposition

$$\alpha = \alpha_0 \alpha_1 \ldots \alpha_{n-1}.$$

Of course the point $\alpha(t_i)$ need not lie in A, but it lies in $U^i \cap U^{i-1}$ and this intersection can only be U_1, U_2 or U_{12}. By the connectivity conditions, for each $i = 0, 1, \cdots, n-1$, we may choose a path c_i

in $U^i \cap U^{i-1}$ joining $\alpha(t_i)$ to A. Moreover, if $\alpha(t_i)$ already lies in A (which is the case when $i = 0$ and when $i = n$), we choose c_i to be the constant path at $\alpha(t_i)$.

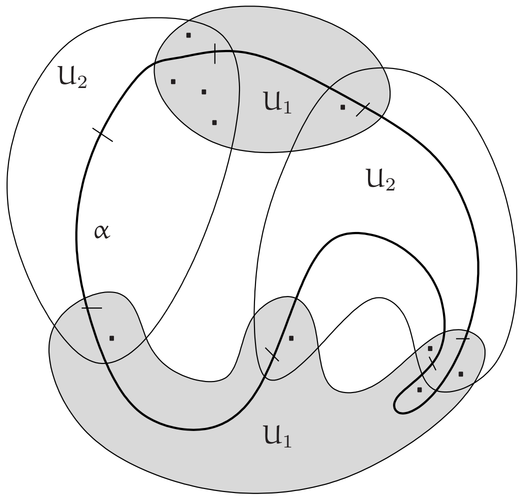


Figure 1.2: A path α in a van Kampen type situation

For each $0 \leqslant i < n$ we have the path $\beta_i = c_i^{-1}\alpha_i c_{i+1}$ in U^i joining points of A. It is clear that

$$[\alpha] = [\beta_0][\beta_1]\cdots[\beta_{n-1}] \in \pi_1(X, A).$$

Notice that β_i also represents a class in $\pi_1(U^i, A)$. Let us call $\psi_i = k_i([\beta_i])$. If the homomorphism of groups that commutes the external square k exists, the value of $k([\alpha])$ is determined, because

$$k([\alpha]) = k([\beta_0][\beta_1]\cdots[\beta_{n-1}]) = k([\beta_0])k([\beta_1])\cdots k([\beta_{n-1}]) = \psi_0\psi_1\cdots\psi_{n-1}.$$

This proves uniqueness of k. We have also proved that $\pi_1(X, A)$ is generated as a groupoid by the images of $\pi_1(U_1, A), \pi_1(U_2, A)$.

We have yet to prove that the element $k([\alpha])$ is independent of all the choices made. Before going into that, notice that the construction we have just made can be interpreted diagrammatically as follows. The starting situation looks like the bottom side of the diagram

$$\begin{tikzcd}
\bullet \arrow[r, dashed, "\beta_0"] & \bullet \arrow[r, dashed, "\beta_1"] & \bullet \arrow[r, dotted, no head] & \bullet \arrow[r, dashed, "\beta_{n-2}"] & \bullet \arrow[r, dashed, "\beta_{n-1}"] & \bullet \\
\bullet \arrow[u, "c_0"] \arrow[r, "\alpha_0"'] & \circ \arrow[u, "c_1"] \arrow[r, "\alpha_1"'] & \circ \arrow[u, "c_2"] \arrow[r, dotted, no head] & \circ \arrow[u, "c_{n-2}"] \arrow[r, "\alpha_{n-2}"'] & \circ \arrow[u, "c_{n-1}"] \arrow[r, "\alpha_{n-1}"'] & \bullet \arrow[u, "c_n"]
\end{tikzcd} \qquad (1.6.6)$$

where the solid circles denote points which definitely lie in A.

The way of getting β_i may be seen as composing with a retraction from above like the one in the Fig 1.3.

If necessary, this retraction also provides a homotopy $\alpha \simeq \beta_0\beta_1\cdots\beta_{n-1}$ rel end points. This is the first of lots of filling arguments where we have defined a map in a subset of the boundary of a cube and fill the whole cube by appropriate retractions. This is studied in all generality in Chapter 10, using 'expansions' and 'collapses'.

We shall use another filling argument in I^3 to prove independence of choices. Suppose that we have a homotopy rel end points $h : \alpha \simeq \alpha'$ of two maps $(I, \partial I) \to (X, A)$. We can perform the construction in (1.6.6) for each of α, α', and then glue the three homotopies together.

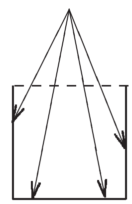
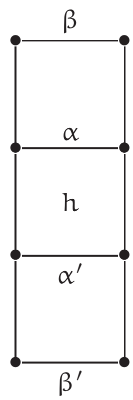


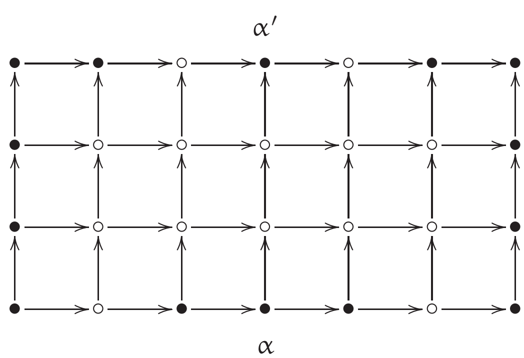
Figure 1.3: Retraction from above-centre



(1.6.7)

So, replacing βs by αs, we can assume the maps α, α' have subdivisions $\alpha = [\alpha_i]$, $\alpha' = [\alpha'_j]$ such that each α_i, α'_j has end points in A and lies in one of U_1, U_2. Since h is a map $I^2 \to X$, we may again by the Lebesgue covering lemma make a subdivision $h = [h_{lm}]$ such that each h_{lm} lies in one of U_1, U_2. Also by further subdivision as necessary, we may assume this subdivision of h refines on $I \times \partial I$ the given subdivisions of α, α'.

The problem is that none of the vertices of this subdivision are necessarily mapped into A, except those on $\partial I \times I$ (since the homotopy is rel vertices and α, α' both map ∂I to A) and those on $I \times \partial I$ determined by the subdivisions of α, α'. So the situation looks like the following:



(1.6.8)

We want to deform h to $h' : \alpha \simeq \alpha'$, a new homotopy rel end points between the same maps, having the same subdivision as does h, and such that any subsquare mapped by h into U_i, $i = 1, 2$ remains so in h', and any vertex already in A is not moved. This is done inductively by filling arguments in the cube I^3.

Let us imagine the 3-dimensional cube I^3 as $I^2 \times I$ where I^2 has the subdivision we are working with in h. Define the bottom map to be h. We have to fill I^3 so that in the top face we get a similar diagram but with all the vertices solid, i.e. in A, and each subsquare in the top face lies in the same U_i as the corresponding in the bottom one.

We start by defining the map on all 'vertical' edges, i.e. on $\{v\} \times I$ for all vertices in the partition of I^2. If the image of a vertex lies in U_{12} but not in A, then we choose a path in U_{12} joining it to a point of A. We work similarly for the case of vertices with images in $U_1 \setminus U_{12}$, $U_2 \setminus U_{12}$. Let us call e_{lm} the path we have chosen between the vertex $h(s_l, t_m)$ and A. (These e_{lm} are constant if $h(s_l, t_m)$ lies already in A.) This gives us the map on the vertical edges of I^3.

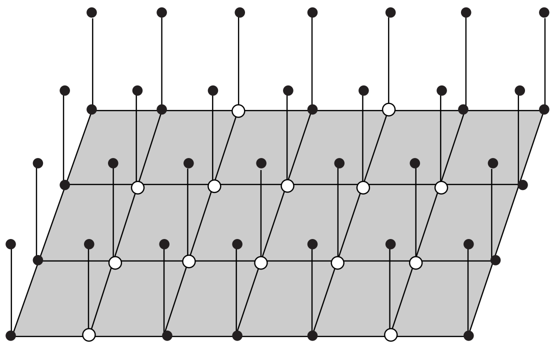


Figure 1.4: Extending to the edges

From now on, we restrict our construction to the part of I^3 over the square $S_{lm} = [s_l, s_{l+1}] \times [t_m, t_{m+1}]$. Let us call $\sigma_{lm} = h|_{[s_l,s_{l+1}]\times\{t_m\}}$ and $\tau_{lm} = h|_{\{s_l\}\times[t_m,t_{m+1}]}$. Then, using the retraction of Figure 1.3 on each lateral face, we can fill all the faces of a 3-cube except the top one. Now, using the retraction from a point on a line perpendicular to the centre of the top face, as in the following Figure 1.5

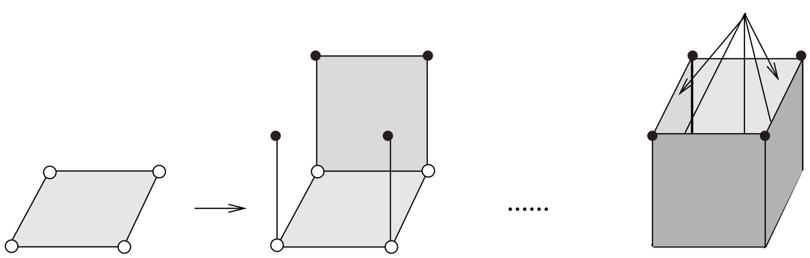
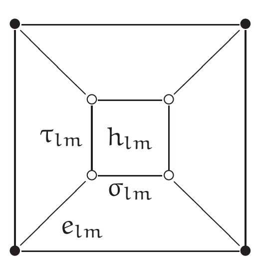


Figure 1.5: Extending to the lateral faces

we get at the top face a map that looks like



(1.6.9)

Thus, in particular, it is a map into U_i sending all vertices in A.

If we do the above construction in each square of the subdivision, we get a top face of the cube that is an homotopy rel end points between two paths in the same classes as α and α', and subdivided in such a way that each subsquare goes into some U_i sending all vertices into A. Each of these squares produces a commutative square of paths in one of $\pi_1(U_i, A)$, $i = 1, 2$. Thus the diagram can be pictured as

$$
\begin{tikzcd}
& & & \alpha' & & & \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] \\
& & & \alpha & & &
\end{tikzcd}
\tag{1.6.10}
$$

Applying k_i to each subsquare we get a commutative square l_i in K. Since $k_1 i_1 = k_2 i_2$, we get that the l_i compose in K to give a square l in K.

Now comes the vital point. Since **the composite of commutative squares in a groupoid produces a commutative square**, the external square l is commutative.

But because of the way we constructed it, two sides of this composite commutative square l in K are identities. Therefore the opposite sides of l are equal. This shows that our element $k([\alpha])$ is independent of the choices made, and so proves that k is well defined as a function on arrows of the fundamental groupoid $\pi_1(X, A)$.

The proof that k is a morphism is now quite simple, while uniqueness has already been shown. So we have shown that the diagram in the statement of the theorem is a pushout of groupoids.

This completes the sketch proof. □

There is another way of expressing the above argument on the composition of commutative squares being a commutative square, namely by working on formulae for each individual square as in the expression $a = cdb^{-1}$ for 1.1.1. Putting together two such squares as in

$$
\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, "c"'] & \bullet \arrow[r, "e"] \arrow[d, "b"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "d"'] & \bullet \arrow[r, "g"'] & \bullet
\end{tikzcd}
\tag{1.6.11}
$$

allows cancelation of the middle term

$$ae = (cdb^{-1})(bgf^{-1}) = cdgf^{-1}$$

which if $c = 1, f = 1$ reduces to $ae = dg$. This argument extends to longer gluings of commutative squares, and hence extends, by induction, and in the other direction, to a subdivision of a square.

We would like to extend the above argument to the faces of a cube, and then to an n-dimensional cube.

For a cube, the expression of one of the faces in terms of the others can be done (see the Homotopy Commutativity Lemma 6.7.6) and then can be used to prove a 2-dimensional van Kampen Theorem. That is done in Section 6.8.

It is much more difficult to follow this route in the general case and a more roundabout method is developed in Chapter 14. The algebra to carry out this argument in dimension n is given in Chapter 13. It is interesting that such a complicated and subtle algebra seems to be needed to make it all work. We emphasise the the purely algebraic work of Chapter 13 is essential for the applications in the following two chapters, and so for the whole of Part II.

Remark 1.6.2 One of the nice things about proving the theorem by verifying the universal property is that the proof uses some calculations in a general groupoid K, and groupoids have, in some sense, the minimal set of properties needed for the result. This avoids a calculation in $\pi_1(X, A)$, and somehow makes the calculations as easy as possible. The same characteristics hold in some other verifications of universal properties, for example in the computation of the fundamental groupoid of an orbit space in [Bro06, Chapter 11]. We will see a similar situation later for double groupoids.

1.7 The fundamental group of the circle

In order to interpret the last theorem, one has to set up the basic algebra of computational groupoid theory. In particular, one needs to be able to deal with presentations of groupoids. This is done to a good extent in [Hig71, Bro06]. Here we can give only the indications of the theory.

The theory of groupoids may be thought of as an algebraic analogue of the theory of groups, but based on directed graphs rather than on sets. [1]

Let us explain some basic definitions in groupoid theory. A groupoid G is called *connected* if $G(a, b)$ is non empty for all $a, b \in \mathrm{Ob}(G)$. The maximal connected subgroupoids of G are called the *(connected) components* of G .

If a is an object of the groupoid G, then the set $G(a, a)$ inherits a group structure from the composition on G, and this is called the *object group* of G at a and is written also $G(a)$. The groupoid G is called *simply connected* if all its object groups are trivial . If it is connected and simply connected, it is called 1-*connected*, or an *indiscrete groupoid* .

A standard example of an indiscrete groupoid is the groupoid $I(S)$ on a set S . This has object set S and arrow set $S \times S$, with $s, t : S \times S \to S$ being the first and second projections. The composition on $I(S)$ is given by

$$(a, b)(b, c) = (a, c), \quad \text{for all } a, b, c \in S.$$

A special case is the groupoid we will write $\mathcal{I} = I(\{0, 1\})$. This has two nonidentity elements which we write $\iota : 0 \to 1$ and $\iota^{-1} : 1 \to 0$. This apparently 'trivial' groupoid will play a key role in the theory, since it determines homotopies. It is also called the 'unit interval groupoid'.

A directed graph X is called *connected* if the free groupoid $F(X)$ on X is connected, and is called a *forest* if every object group $F(X)(a)$ of $F(X)$, $a \in \mathrm{Ob}(X)$, is trivial. A connected forest is called a *tree* . If X is a tree, then the groupoid $F(X)$ is indiscrete; an indiscrete groupoid is also called a *tree groupoid*.

Let G be a connected groupoid and let a_0 be an object of G. For each $a \in \mathrm{Ob}(G)$ choose an arrow $\tau a : a_0 \to a$, with $\tau a_0 = 1_{a_0}$. Then an isomorphism

$$\phi : G \to G(a_0) \times I(\mathrm{Ob}(G)) \tag{1.7.1}$$

is given by $g \mapsto ((\tau a)g(\tau b)^{-1}, (a, b))$ when $g \in G(a, b)$ and $a, b \in \mathrm{Ob}(G)$. The composition of ϕ with the projection yields a morphism $\rho : G \to G(a_0)$ which we call a *deformation retraction* since it is the identity on $G(a_0)$ and is in fact homotopic to the identity morphism of G, though we do not elaborate on this fact here.

It is also standard [Bro06, 8.1.5] that a connected groupoid G is isomorphic to the free product groupoid $G(a_0) * T$ where $a_0 \in \mathrm{Ob}(G)$ and T is any wide, tree subgroupoid of G. The importance of this is as follows.

Suppose that X is a graph which generates the connected groupoid G. Then X is connected. Choose a maximal tree T in X. Then T determines for each a_0 in $\mathrm{Ob}(G)$ a retraction $\rho_T : G \to G(a_0)$ and the isomorphisms

$$G \cong G(a_0) * I(\mathrm{Ob}(G)) \cong G(a_0) * F(T)$$

show that a morphism $G \to K$ from G to a groupoid K is completely determined by a morphism of groupoids $G(a_0) \to K$ and a graph morphism $T \to K$ which agree on the object a_0.

We shall use later the following proposition, which is a special case of [Bro06, 6.7.3]:

Proposition 1.7.1 *Let G, H be groupoids with the same set of objects, and let $\phi : G \to H$ be a morphism of groupoids which is the identity on objects. Suppose that G is connected and $a_0 \in \mathrm{Ob}(G)$. Choose a retraction $\rho : G \to G(a_0)$. Then there is a retraction $\sigma : H \to H(a_0)$ such that the following diagram, where ϕ' is the restriction of ϕ:*

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "\phi"'] & G(a_0) \arrow[d, "\phi'"] \\ H \arrow[r, "\sigma"'] & H(a_0) \end{tikzcd} \tag{1.7.2}$$

is commutative and is a pushout of groupoids.

This result can be combined with Theorem 1.6.1 to determine the fundamental group of the circle S^1.

Corollary 1.7.2 *The fundamental group of the circle S^1 is a free group on one generator.*

Proof We represent S^1 as the union of two semicircles E^1_+, E^1_- with intersection $\{-1, 1\}$. Then both fundamental groupoids $\pi_1(E^1_+, \{-1, 1\})$ and $\pi_1(E^1_-, \{-1, 1\})$ are easily seen to be isomorphic to the connected groupoid $\mathcal{I}$ with object set $\{-1, 1\}$ and trivial object groups. In fact this groupoid is the free groupoid on one generator $\iota : -1 \to 1$.

Also, $\pi_1(\{-1, 1\}, \{-1, 1\})$ is the discrete groupoid on these objects $\{-1, 1\}$. By an application of Theorem 1.6.1 we get a pushout of groupoids

$$\begin{tikzcd} \{-1, 1\} \arrow[r] \arrow[d] & \mathcal{I} \arrow[d] \\ \mathcal{I} \arrow[r] & \pi_1(S^1, \{-1, 1\}) \end{tikzcd}$$

From the previous Proposition, we have a pushout of groupoids

$$\begin{tikzcd} \mathcal{I} \arrow[r] \arrow[d] & \{1\} \arrow[d] \\ \pi_1(S^1, \{-1, 1\}) \arrow[r] & \pi_1(S^1, 1) \end{tikzcd}$$

Gluing them, we get a pushout of groupoids

$$\begin{tikzcd} \{-1,1\} \arrow[r] \arrow[d] & \{1\} \arrow[d] \\ \mathcal{I} \arrow[r] & \pi_1(S^1,1) \end{tikzcd}$$

and the result follows by an easy universal argument. □

Note that S^1 may be regarded as a pushout in the category of topological spaces

$$\begin{tikzcd} \{-1,1\} \arrow[r] \arrow[d] & \{1\} \arrow[d] \\ [-1,1] \arrow[r] & S^1 \end{tikzcd} \tag{1.7.3}$$

The correspondence between these last two diagrams was for R.Brown a major incentive to exploring the use of groupoids. Here we have a successful algebraic model of a space, but of a different type from that previously considered. An aspect of its success is that groupoids have structure in two dimensions, namely 0 and 1, and this is useful for modeling the way spaces are built up using identifications in dimensions 0 and 1.

Another interesting aspect is that the groupoid $\mathcal{I}$ is finite, and it is easy to explore all its properties. By contrast, the integers form an infinite set, and discussion of its properties usually requires induction.

The problem was to find analogous methods in higher dimensions.

1.8 Higher order groupoids

The successes of the use of groupoids in 1-dimensional homotopy theory and the successes in group theory as exposed in the books [Bro06, Hig71] suggested the potential interest in the use of groupoids in higher dimensional homotopy theory. In particular, it seemed possible that a Higher Homotopy van Kampen Theorem (HHvKT) could be proved if the 'right' higher homotopy groupoids could be constructed, with properties analogous to those which enabled the proof of this theorem in dimension 1.

Experiments by Brown to obtain such a construction in the years 1965-74 proved abortive. However in 1971 Chris Spencer came to Bangor as a Science Research Council Research Assistant, and in this and a subsequent period considerable progress was made on the discovering the algebra of double groupoids. It was in this time that the relation with crossed modules was made, so linking the notion of double groupoids with more classical ideas.

Crossed modules had been defined by J.H.C. Whitehead in 1946 [Whi46] in order to express the properties of the properties of the boundary map

$$\partial : \pi_2(X, X_1, x) \to \pi_1(X_1, x)$$

of the second relative homotopy group, a group which is in general nonabelian. He gave the first nontrivial determination of this group in showing that when X is formed from X_1 by attaching 2-cells, then $\pi_2(X, X_1, x)$ is isomorphic to the free crossed $\pi_1(X_1, x)$-module on the characteristic maps of the 2-cells.

This result was a crucial clue to Brown and Higgins in 1974. On the one hand it showed that a universal property, namely freeness, did exist in 2-dimensional homotopy theory. Also, if our proposed theory was to be any good, it should have this theorem as a corollary. However, Whitehead's theorem was about *relative homotopy groups*, which suggested that we should look at a relative theory, i.e. a space X with a subspace X_1. With the experience obtained by then, we quickly found a satisfactory, even simple, construction of a relative homotopy double groupoid $\rho_2(X, X_1, x)$ and a proof of a 2-dimensional van Kampen Theorem, as envisaged.

The equivalence between these sorts of double groupoids and crossed modules proved earlier by Brown and Spencer, then gave the required van Kampen type theorem for the second homotopy crossed module, and so new calculations of second relative homotopy groups.

So we have a pattern of proof:

A) construct a homotopically defined multiple groupoid;

B) prove it is equivalent to a more familiar homotopical construction;

C) prove a van Kampen Theorem in the multiple groupoid context; and

D) interpret this theorem in the more familiar context.

These combined give new nonabelian, higher dimensional, local-to-global results. This pattern has been followed in the corresponding result for crossed complexes, which is dealt with in our Part II, and results for the cat^n-groups of Loday [Lod82]. However, we do not discuss the latter in this book.

Crossed modules had occurred earlier in other places. In the mid 1960s the great school of Grothendieck in Paris had considered sets with two structures, that of group and of groupoid, and had proved these were equivalent to crossed modules. However this result was not published, and so was known only to a restricted group of people.

It is now clear that once one moves to higher version of groupoids, the presence of crossed modules is inevitable, and is an important part of the theory and applications. This is why Part I is devoted entirely to the area of crossed modules and double groupoids.

Notes

[1]p. 20 For some discussion of the philosophy of moving from sets to directed graphs, see [Bro94]. We refer to [Bro06, Hig71] for the construction of a free groupoid over a directed graph.

Chapter 2

Homotopy theory and crossed modules

In this chapter we explain how crossed modules over groups arose in topology in the first half of the last century, and give some of the later developments.

The topologist J.H.C. Whitehead (19041960) was steeped in the combinatorial group theory of the 1930's, and much of his work can be seen as trying to extend the methods of group theory to higher dimensions, keeping the interplay with geometry and topology. These attempts led to greatly significant work, such as the theory of simple homotopy types [Whi50b], the algebraic background for which started the subject of algebraic K-theory. His ideas on crossed modules have taken longer to come into wide use, but they can be regarded as equally significant.

One of his starting points was the van Kampen Theorem for the fundamental group. This tells us in particular how the fundamental group is affected by the attaching of a 2-cell, or of a family of 2-cells, to a space. Namely, if $X = A \cup \{e_i^2\}_{i \in I}$, where the 2-cell e_i^2 is attached by a map which for convenience we suppose is $f_i : (S^1, 1) \to (A, x)$, then each f_i determines an element ϕ_i in $\pi_1(A, x)$, and a consequence of the van Kampen Theorem for the fundamental group is that the group $\pi_1(X, x)$ is obtained from the group $\pi_1(A, x)$ by adding the relations ϕ_i, $i \in I$.

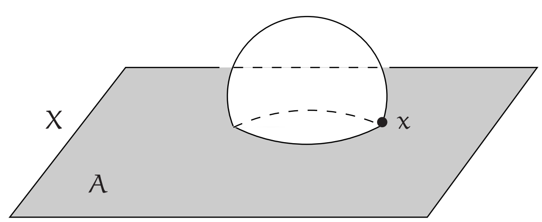


Figure 2.1: Picture of an attached 2-cell

The next problem was clearly to determine the effect on the higher homotopy groups of adding cells to a space. So his 1941 paper [Whi41] was entitled 'On adding relations to homotopy groups'. If we could solve this problem in general then we would in particular be able to calculate all homotopy groups of spheres. Work over the last 70 years has shown the enormous difficulty of this task.

In this paper he gave important results in higher dimensions, but he was also able to obtain information on the second homotopy groups of $X = A \cup \{e_i^2\}_{i \in I}$. His results were clarified by him in two subsequent papers using the notion of *crossed module* [Whi46], and then *free crossed* *module*, [Whi49b]. This formulation became the key for Brown and Higgins to Higher Homotopy van Kampen Theorems, as we shall see later. His basic method of proof (an exposition is given in [Bro80]) uses what is now called transversality, and has become the foundation of a technique called 'pictures', [HAM93]. His algebraic methods have also been exploited rather differently and in a more algorithmic way in [BRS99] to compute second homotopy modules, see Subsection 11.2.4.

We begin this chapter by giving a definition of the fundamental crossed module

$$\Pi_2(X, A, x) = (\partial : \pi_2(X, A, x) \to \pi_1(A, x))$$

of a pointed pair of spaces and explaining some of Whitehead's work. Then we state two central results:

- the 2-dimensional van Kampen theorem, in Section 2.3;
- the notion of classifying space of a crossed module, in Section 2.4.

It is these two combined which give many of the important homotopical applications of crossed modules (including Whitehead's results). However the construction of the classifying space, and the proof of its properties, needs the methods of crossed complexes of Part II, and is given in Chapter 10. We give applications of the 2-dimensional van Kampen theorem in Chapters 4 and 5 and prove it in Chapter 6. This sets the scene for the corresponding higher dimensional results of Part II, and the substantial proofs of Part III.

Section 2.5 shows that crossed modules are equivalent to another algebraic structure, that of cat^1-groups. This is used in Section 2.6 to obtain the cat^1-group of a fibration, which yields an alternative way of obtaining the fundamental crossed module.

Section 2.7 shows that crossed modules are also equivalent to 'categories internal to groups', or, equivalently, to groupoids internal to groups. This is important philosophically, because groupoids are a generalisation of equivalence relations, and equivalence relations give an expression of the idea of quotienting, a fundamental process in mathematics and science, because it is concerned with classification. We can think of groupoids as giving ways of saying not only that two objects *are equivalent*, but also *how they are equivalent*: the arrows between two objects give different 'equivalences' between them., which can sometimes be regarded as 'proofs' that the objects are equivalent.

Moving now to the case of groups, to obtain a quotient of a group P we need not just an equivalence relation, but this equivalence relation needs to be a *congruence*, i.e. not just a subset but also a subgroup of $P \times P$. An elementary result in group theory is that a congruence on a group P is determined completely by a normal subgroup of P. The corresponding result for groupoids is that a groupoid with a group structure is equivalent to a crossed module $M \to P$ where P is the group of objects of the groupoid.

This family of equivalent structures – crossed modules, cat^1-groups, group objects in groupoids – gives added power to each of these structures. In fact in Chapter 6 we will use crucially another related structure, that of *double groupoids with connection*. This is equivalent to an important generalisation of a crossed module, that of *crossed module of groupoids*, which copes with the varied base points of second relative homotopy groups.

2.1 Homotopy groups and relative homotopy groups

Recall that two maps $f, g : X \to Y$ between two topological spaces are said to be homotopic if f can be continuously deformed to g. Formally, they are *homotopic*, and this is denoted by $f \simeq g$, if there is a map

$$F : X \times I \to Y$$

such that $F_0(x) = F(x, 0) = f(x)$ and $F_1(x) = F(x, 1) = g(x)$. The map F is called a *homotopy* from f to g.

This definition gives an equivalence relation among the set of maps from X to Y. The quotient set is denoted $[X, Y]$ and the equivalence class of a map f is denoted by $[f]$.

Sometimes we are interested in considering only deformations that keep some subset fixed. If $A \subseteq X$, we say that two maps as above are *homotopic relative to* A, and denote this by $f \simeq g \operatorname{rel} A$, if there is a homotopy F from f to g satisfying $F(a, t) = f(a)$ for all $a \in A$, $t \in I$. This definition gives another equivalence relation among the set of maps from X to Y. The quotient set is denoted $[X, Y]_A$ and the equivalence class of a map f is again denoted by $[f]$.

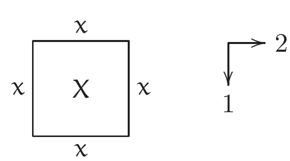
Since all maps homotopic relative to, or rel to, A must agree with a map $u : A \to Y$, this set for a fixed u is written $[X, Y; u]$. Thus $[X, Y]_A$ is the union of the disjoint sets $[X, Y; u]$ for all $u : A \to X$.

A particular case of this definition is when we study maps sending a fixed subset A of X to a given point $y \in Y$. Then the quotient set corresponding to maps from X to Y sending all A to y with respect to homotopy rel A, is written as $[(X, A), (Y, y)]$ or, when $A = \{x\}$, as $[X, Y]_*$.

To define the *homotopy groups of a space*, we consider homotopy classes of maps from particular spaces. Namely if $x \in X$, the n-th homotopy group of X based at x is defined as

$$\pi_n(X, x) = [(I^n, \partial I^n), (X, x)]$$

where ∂I^n is the boundary of I^n. The elements of $\pi_n(X, x)$ are classes of maps that can be pictured for $n = 2$ as in the following diagram:



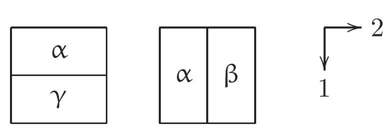
(2.1.1)

where we use throughout all the book a matrix like convention for directions. One of the reasons for this will become clear in Chapter 6.

In the case $n = 1$ we obtain the fundamental group $\pi_1(X, x)$. For all $n \geqslant 1$ there initially seem to be n group structures on this set induced by the composition of representatives given for $1 \leqslant i \leqslant n$ by

$$(f +_i g)(t_1, t_2, \ldots, t_n) = \begin{cases} f(t_1, t_2, \ldots, 2t_i, \ldots, t_n) & \text{if } 0 \leqslant t_i \leqslant 1/2, \\ g(t_1, t_2, \ldots, 2t_i - 1, \ldots, t_n) & \text{if } 1/2 \leqslant t_i \leqslant 1. \end{cases}$$

Remark 2.1.1 For the case $n = 2$ the following diagrams picture the two compositions:



$\alpha +_1 \gamma$ $\qquad$ $\alpha +_2 \beta$

□

Theorem 2.1.2 *If $n \geqslant 2$, then any of the multiplications $+_i$, $i = 1, \cdots, n$ on $\pi_n(X, x)$ induce the same group structure, and all these group structures are abelian.*

Proof By Theorem 1.3.1, we need only to verify the interchange law for the compositions $+_i, +_j, 1 \leqslant i < j \leqslant n$. It is easily seen that if $f, g, h, k : (I^n, \partial I^n) \to (X, x)$ are representatives of elements of $\pi_n(X, x)$, then the two compositions obtained by evaluating the following matrix in two ways

$$\begin{array}{|c|c|}\hline f & g \\ \hline h & k \\ \hline\end{array} \qquad \begin{array}{l} \rightarrow j \\ \downarrow \\ i \end{array}$$

in fact coincide. The verification consists in checking the formula for such a multiple composition. □

Of course this argument is the same as in Theorem 1.3.1.

We shall need later that π_n is functorial in the sense that to any map $\phi : X \to Y$ there is associated a homomorphism of groups

$$\phi_* : \pi_n(X, x) \to \pi_n(Y, \phi(x))$$

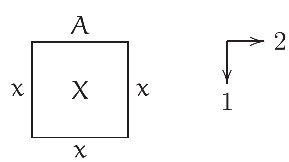
defined by $\phi_*[f] = [\phi f]$, and which satisfies the usual functorial properties $(\phi\psi)_* = \phi_*\psi_*, 1_* = 1$.

Now we may repeat everything for maps of triples and homotopies among them. By a *based pair of spaces* (X, A, x) is meant a space X, a subspace A of X and a base point $x \in A$. The n^{th} *relative homotopy group* $\pi_n(X, A, x)$ of the based pair (X, A, x) is defined as the homotopy classes of maps of triples

$$\pi_n(X, A, x) = [(I^n, \partial I^n, J^{n-1}), (X, A, x)]$$

where $J^{n-1} = \{1\} \times I^{n-1} \cup I \times \partial I^{n-1}$. That is we consider maps $\alpha : I^n \to X$ such that $\alpha(\partial I^n) \subseteq A$ and $\alpha(J^{n-1}) = \{x\}$ and homotopies through maps of this kind.

The picture we shall have in mind as representing elements of $\pi_n(X, A, x)$ is



(2.1.2)

As before, a multiplication on $\pi_n(X, A, x)$ is defined by the compositions $+_i$ in any of the last $(n-1)$ directions. It is not difficult to check that any of these multiplications gives a group structure and analogously to Theorem 2.1.2 these all agree and are abelian if $n \geqslant 3$. Also, for any maps of based pairs $\phi : (X, A, x) \to (Y, B, y)$, there is a homomorphism of groups

$$\phi_* : \pi_n(X, A, x) \to \pi_n(Y, B, y)$$

as before .

The homotopy groups defined above fit nicely in an exact sequence called the *homotopy exact sequence of the pair* as follows:

$$\begin{aligned} \cdots \to \pi_n(X, x) &\xrightarrow{j_*} \pi_n(X, A, x) \xrightarrow{\partial_n} \pi_{n-1}(A, x) \xrightarrow{i_*} \pi_{n-1}(X, x) \to \cdots \\ &\xrightarrow{j_*} \pi_2(X, A, x) \xrightarrow{\partial_2} \pi_1(A, x) \xrightarrow{i_*} \pi_1(X, x) \xrightarrow{j_*} \\ &\xrightarrow{j_*} \pi_1(X, A, x) \xrightarrow{\partial_1} \pi_0(A, x) \xrightarrow{i_*} \pi_0(X, x) \end{aligned} \tag{2.1.3}$$

where i_* and j_* are the homomorphisms induced by the inclusions, and the *boundary map of a pair* ∂ is given by restriction, i.e. for any $[\alpha] \in \pi_n(X, A, x)$ represented by a map $\alpha : (I^n, \partial I^n, J^{n-1}) \to (X, A, x)$, we define $\partial[\alpha] = [\alpha']$ where α' is the restriction of α to the face $\{0\} \times I^{n-1}$, which we identify with I^{n-1}.

This exact sequence is of abelian groups and homomorphisms until $\pi_2(X, x)$, of groups and homomorphisms until $\pi_1(X, x)$, and of based sets for the last three terms. The amount of exactness for the last terms is the same as for the exact sequence of a fibration of groupoids, see [Bro06, 7.2.9], and which we use again in Section 12.4 (Theorem 12.4.1.) .

The final interesting piece of structure is the existence of a $\pi_1(A, x)$-*action* on all the terms of the above exact sequence which are groups. Let us define this action. For any $[\alpha] \in \pi_n(X, A, x)$ and any $[\omega] \in \pi_1(A, x)$, we define the map

$$F = F(\alpha, \omega) : I^n \times \{0\} \cup J^{n-1} \times I \to X$$

given by α on $I^n \times \{0\}$ and by ω on $\{t\} \times I$, for any $t \in J^{n-1}$. Then we have defined F on the subset of I^{n+1} indicated in Figure 2.2

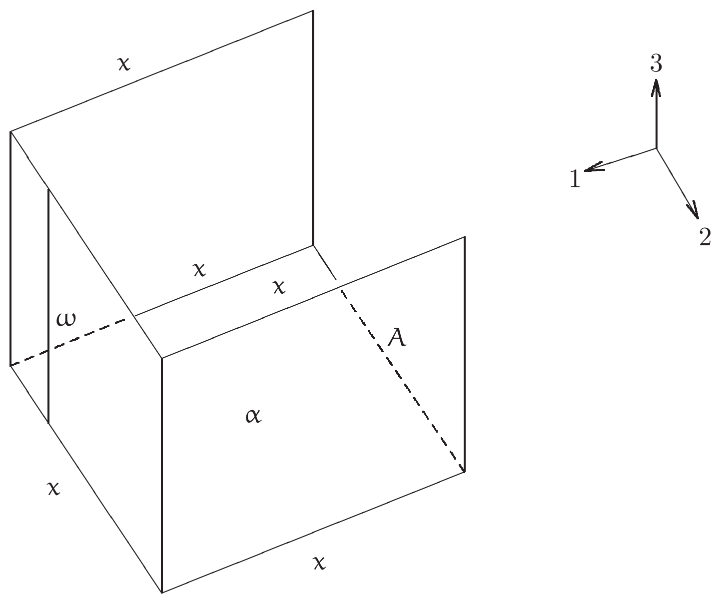


Figure 2.2: Action of $\pi_1(A, x)$

Now, we compose with the retraction

$$r : I^{n+1} \to I^n \times \{0\} \cup J^{n-1} \times I$$

given by projecting from a point $P = (0, \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, 2)$ and indicated in Figure 2.3, getting a map $Fr : I^{n+1} \to X$ extending F.

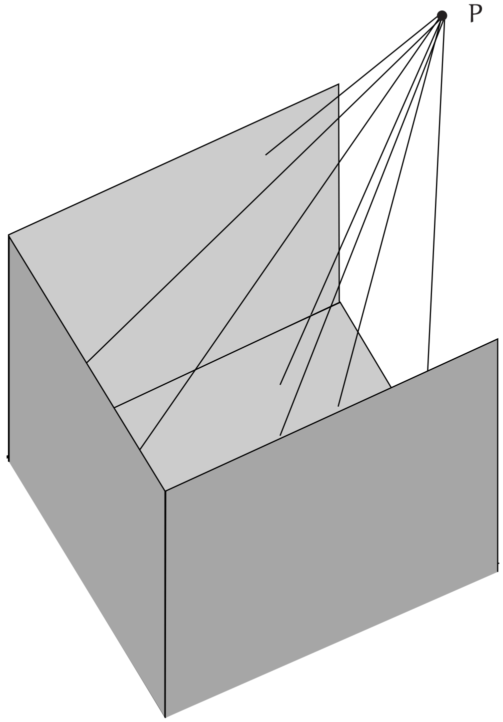


Figure 2.3: Retraction from above-lateral

The “restriction” map

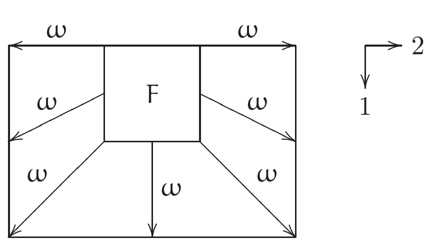
$$I^n \cong I^n \times \{1\} \hookrightarrow I^{n+1} \xrightarrow{Fr} X$$

represents an element $[\alpha]^{[\omega]} \in \pi_n(X, A, x)$.

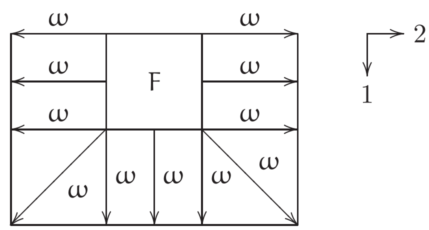
We leave the reader to develop proofs that the action is an action of a group on a group, that is that various axioms are satisfied. However all this will follow in a more algebraic fashion using the theory given in chapter 14.

Notice that in this definition we use another of the filling arguments that we have started using in the proof of Theorem 1.6.1 in Section 1.6. Arguments of the same kind prove that the assignment just defined is independent of the several choices involved (α, ω and the extension of F), and that it defines an action.

Remark 2.1.3 Notice that when $n = 2$ the map representing $[\alpha]^{[\omega]}$ could be drawn



or, equivalently we could have chosen the one described as follows



□

In a similar way, we may define an action of $\pi_1(X, x)$ on $\pi_n(X, x)$. In our case, this gives an action of $\pi_1(A, x)$ on both $\pi_n(A, x)$ and $\pi_n(X, x)$. Moreover, all maps in the homotopy exact sequence are maps of $\pi_1(A, x)$-groups.

All the above constructions can be repeated for *based* r*-ads* $X_* = (X; X_1, X_2, \ldots, X_r, x)$, where all X_i are subspaces of X. Homotopy groups $\pi_n X_*$ are defined for $n \geqslant r+1$ and are abelian for $n \geqslant r+2$. There are various long exact sequences relating the homotopy groups of $(r+1)$-ads and r-ads. An account of these is in [Hu59]. The homotopy groups of an $(r+1)$-ad are also a special case of the homotopy groups of an r-cube of spaces [Lod82, BL87b, Gil87] . All these groups are important for discussing the failure of excision for relative homotopy groups, to which we have referred earlier, and whose analysis in some cases using nonabelian methods will be an important feature of this book.

2.2 Whitehead's work on crossed modules

We start with the basic definition of crossed module. (From Chapter 6 onwards we will need crossed modules over a groupoid, but until then we stick to the group case.)

Definition 2.2.1 A *crossed module (over a group)* $\mathcal{M} = (\mu : M \to P)$ is a morphism of groups $\mu : M \to P$ called the boundary of $\mathcal{M}$ together with an action $(m, p) \mapsto m^p$ of P on M satisfying the two axioms

CM1) $\mu(m^p) = p^{-1}\mu(m)p$

CM2) $n^{-1}mn = m^{\mu n}$

for all $m, n \in M$, $p \in P$. □

When we wish to emphasise the codomain P, we call $\mathcal{M}$ a *crossed* P*-module*.

Basic algebraic examples of crossed modules are:

- A *conjugation crossed module* is an inclusion of a normal subgroup $N \triangleleft G$, with action given by conjugation. In particular, for any group P the identity map $\mathrm{Id}_P : P \to P$ is a crossed module with the action of P on itself by conjugation. T. Porter has remarked that the concept of crossed module can be seen as an 'externalisation' of the concept of normal subgroup. That is, an inclusion is replaced by a homomorphism with special properties. This process occurs in other algebraic situations.
- If M is a group, its *automorphism crossed module* has the form $(\chi : M \to \mathrm{Aut}(M))$ where χm is the inner automorphism mapping n to $m^{-1}nm$. If A satisfies $\mathrm{Inn}(M) \leqslant A \leqslant \mathrm{Aut}(M)$ and $\chi(M) \subseteq A$, we also call the automorphism crossed module to $(\chi : M \to A)$.

- A P-*module crossed module* has zero boundary and M is a P-module.
- A *central extension crossed module* $(\mu : M \to P)$ has surjective boundary with kernel contained in the centre of M and $p \in P$ acts on $m \in M$ by conjugation with any element of $\mu^{-1}p$.
- Any homomorphism $(\mu : M \to P)$, with M abelian and $\operatorname{Im}\mu$ in the centre of P, provides a crossed module with P acting trivially on M.

The category $\mathsf{XMod}/\mathsf{Groups}$ of crossed modules has as objects all crossed modules over groups. Morphisms in $\mathsf{XMod}/\mathsf{Groups}$ from $\mathcal{M}$ to $\mathcal{N}$ are pair of group homomorphisms (g, f) forming commutative diagrams with the two boundaries,

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\mu"'] & N \arrow[d, "\nu"] \\ P \arrow[r, "f"'] & Q \end{tikzcd}$$

and preserving the action in the sense that for all $m \in M, p \in P$ we have $g(m^p) = (gm)^{fp}$. If P is a group, then the category XMod/P of *crossed* P-*modules* is the subcategory of $\mathsf{XMod}/\mathsf{Groups}$ whose objects are the crossed P-modules and whose morphisms are the group homomorphisms $g : M \to N$ such that g preserves the action (i.e. $g(m^p) = (gm)^p$, for all $m \in M,\ p \in P$), and $\nu g = \mu$.

Here are some elementary general properties of crossed modules which we will often use.

Proposition 2.2.2 *For any crossed module* $\mu : M \to P$*,* μM *is a normal subgroup of* P*, i.e.* $\mu M \lhd P$.

Proof This is immediate from CM1). □

The *centraliser* $C(S)$ of a subset S of a group M is the set of elements of M which commute with all elements of S. In particular, $C(M)$ is written ZM and called the *centre* of M and is abelian. Any subset of ZM is called *central* in M .

The *commutator* of elements m, n of a group M is the element $[m, n] = m^{-1}n^{-1}mn$. The *commutator subgroup* $[M, M]$ of M, is the normal subgroup of M generated by all commutators. We write M^{ab} for the abelian group $M/[M, M]$, the *abelianisation* of M.

Proposition 2.2.3 *Let* $\mu : M \to P$ *be a crossed module, and let* $C = \operatorname{Cok}\mu$*. Then*

(i) $\operatorname{Ker}\mu$ *is central in* M.

(ii) $\mu(M)$ *acts trivially on* ZM.

(iii) ZM *and* $\operatorname{Ker}\mu$ *inherit an action of* C *to become* C*-modules.*

(iv) P *acts on* M^{ab} *and* $\mu(M)$ *acts trivially on* M^{ab} *which inherits an action of* C *to become a* C*-module.*

Proof Axiom CM2) shows that if $m, n \in M$ and $\mu n = 1$ then $mn = nm$. This proves (i). On the other hand, and by CM2) and CM1), $mn = nm$ implies $m^{\mu n} = m$, and this proves (ii). Then (iii) follows using these and Proposition 2.2.2, which implies $C = P/\mu(M)$.

Since $[m, n]^p = [m^p, n^p]$ for $m, n \in M,\ p \in P$, we have $[M, M]$ is P-invariant, so that P acts on M^{ab}. However in this action $\mu(M)$ acts trivially since if $m, n \in M$ then

$$m^{\mu n} = n^{-1}mn = m \bmod [M, M].$$

□

Thus for any crossed module $(\mu : M \to P)$ with $C = \operatorname{Cok}\mu,\ \pi = \operatorname{Ker}\mu$ we have an exact sequence of C-modules

$$\pi \longrightarrow M^{\mathrm{ab}} \longrightarrow (\mu M)^{\mathrm{ab}} \longrightarrow 1.$$

The first map is not injective in general. To see this, consider the crossed module $\chi : M \to \mathsf{Aut}(M)$ associated to a group M. Then $\pi = \operatorname{Ker}\chi = ZM$, the centre of M. There are groups M for which

$$1 \neq ZM \subseteq [M, M],$$

for example the quaternion group, the dihedral groups and many others. For all these the composite map $\pi \to ZM \to M^{\mathrm{ab}}$ is trivial and so not injective. These examples give point to the following useful result.

Proposition 2.2.4 *If there is a section $s : \mu M \to M$ of μ which is a group homomorphism (but not necessarily a P-map) then M is isomorphic as a group to $\pi \times \mu M$. Further $[M, M] \cap \pi = 1$, and the map $\pi \to M^{\mathrm{ab}}$ is injective.*

Proof Because s is a section (i.e. μs is the identity on μM) we have that $M = (\pi)(\operatorname{Im} s)$ and $\pi \cap (\operatorname{Im} s) = \{1\}$. Because the action of $\operatorname{Im} s$ on π is trivial, we have an internal product decomposition $M = (\pi) \times (\operatorname{Im} s)$. Furthermore, by Proposition 2.2.3 we know that π is abelian so $[M, M] = [\operatorname{Im} s, \operatorname{Im} s]$.

So, $[M, M] \cap \pi = \{1\}$ and $\pi \to M^{\mathrm{ab}}$ is injective. □

An important example where the section s exists is when $\mu(M)$ is a free group. The well known Schreier Subgroup Theorem of combinatorial group theory, that a subgroup of a free group is itself free (see books on combinatorial group theory, for example [LS01, Joh97] and also [Hig71], or [Bro06, 10.8.2], for a groupoid proof) assures us that this is the case when M itself is free.

The results of the following exercise will be used in Chapter 11, Example 11.2.20.

Exercise 2.2.5 Let $\mu : M \to P$ be a crossed module.
(i) If M is abelian, then μM acts trivially on M so that M can be seen as a $\operatorname{Cok}\mu$ module; and
(ii) if M has a single generator as P-group, and P is abelian, then M is abelian. □

Example 2.2.6 Crossed modules $\mu : M \to P$ in which both M and P are abelian form an interesting subcategory of that of crossed modules. As an example, let $\mu : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_4$ be the morphism which maps each $\mathbb{Z}_2$ summand injectively, and where $\mathbb{Z}_4$ operates by switching the two summands. This gives a crossed module which as we shall see in Chapter 12 Theorem 12.2.8 is in a key sense non trivial.

Other algebraic examples of crossed modules arise from two key constructions: *coproducts* of crossed P-modules in Chapter 4 and *induced* crossed modules in Chapter 5.

The major geometric example of a crossed module is the following, where the basic definitions were given in the last Section. Let (X, A, x) be a based pair of spaces, that is X is a topological space and $x \in A \subseteq X$. Whitehead showed that the boundary map

$$\partial : \pi_2(X, A, x) \to \pi_1(A, x), \tag{2.2.1}$$

from the second relative homotopy group of (X, A, x) to the fundamental group $\pi_1(A, x)$, together with the standard action of $\pi_1(A, x)$ on $\pi_2(X, A, x)$, has the structure of crossed module. This result and its proof will be seen in various lights in this book. Because of this example it is convenient and sensible to regard crossed modules $\mu : M \to P$ as 2-dimensional versions of groups, with P, M being respectively the 1- and 2-dimensional parts. This analogy also will be pursued in more detail later. At this stage we only note that the full description of the 2-dimensional part requires specification of its 1-dimensional foundation and of the way the two parts fit together: that is, we need the whole structure of crossed module.

Now we see that we have a functor from based pairs of topological spaces to crossed modules

$$\Pi_2 : \mathsf{Top}^2_* \to \mathsf{XMod/Groups} \tag{2.2.2}$$

which sends the based pair (X, A, x) to the crossed module given in 2.2.1 above. (Later we shall formulate a groupoid version of this functor, allowing the base point to vary, but it is best to get familiar with this special case at first.)

The work of Whitehead on crossed modules over the years 1941-1949 contained in [Whi41, Whi46, Whi49b] and mentioned in the Introduction to this Chapter can be summarised as follows.

He started trying to obtain information on how the higher homotopy groups of a space are affected by adding cells. For the fundamental group, the answer is a direct consequence of the van Kampen Theorem:

> adding a 2-cell corresponds to adding a relation to the fundamental group, adding an n-cell for $n \geqslant 3$ does not change the fundamental group.

So the next question is:

> how is the second homotopy group affected by adding 2-cells?, i.e. if $X = A \cup \{e^2_i\}$, what is the relation between $\pi_2(A)$ and $\pi_2(X)$?

In the first paper ([Whi41]), he formulated a geometric proof of a theorem in this direction. In the second paper ([Whi46]) he gave the definition of crossed module and showed that the second relative homotopy group $\pi_2(X, A, x)$ of a pair of spaces could be regarded as a crossed module over the fundamental group $\pi_1(A, x)$. In the third paper ([Whi49b]) he introduced the notion of free crossed module and showed that his previous work could be reformulated as showing that the second relative homotopy group $\pi_2(X, A, x)$ was isomorphic to the *free crossed module* on a set of generators corresponding to the 2-cells. This concept of free crossed module will be studied in detail in Section 3.4.

He was not in fact able to obtain any detailed computations of second homotopy groups from this result, but it was fundamental to his work on the classification of homotopy 2-types, and, together with the concept of chain complex with operators that we shall develop in the second part, on a range of realisation problems [Whi41, Whi46].

The proof he gave was difficult to read, since it was spread over three papers, with some notation changes, and that is why a repackaged version of the proof by Brown was accepted for publication [Bro80]. The main ideas of the proof included knot theory, and also transversality, techniques of which became fashionable only in the 1960s (see also [HAM93]). A number of other proofs have been given, including one we give in this book (see Corollary 5.4.8) in which the result is seen as a special case of a 2-dimensional van Kampen type theorem.

The way this work was developed by Whitehead seems a very good example of what Grothendieck has called ‘struggling to bring new concepts out of the dark’ through the search for the underlying structural features of a geometric situation.

Whitehead’s work on free crossed modules parallelled independent work by Reidemeister and his student Renee Peiffer at about the same time on the closely related notion of identities among relations [Rei49, Pei49], which we deal with in Section 3.1. Whitehead also acknowledged in [Whi41] that some of his results on second homotopy groups were also obtainable from work of Reidemeister on chain complexes with operators, now recognised as given by the complex of cellular chains of the universal cover of the space, and which has been extensively used for example in simple homotopy theory [Coh73].

2.3 The 2-dimensional van Kampen Theorem

Whitehead's theorem on free crossed modules referred to in the last section demonstrated that a particular universal property was available for homotopy theory in dimension 2. This suggested that there was scope for some broader kind of universal property at this level.

It also gave a clue to a reasonable approach. Such a universal property, in order to be broader, would clearly have to include Whitehead's theorem. Now this theorem is about the fundamental crossed module of a particular pair of spaces. So the broader principle should be about the fundamental crossed modules of *pairs* of spaces. The simplest property would seem to be, in analogy to the van Kampen Theorem, that the functor

$$\Pi_2 : \mathsf{Top}^2_* \to \mathsf{XMod}/\mathsf{Groups}$$

described in (2.2.2) preserves certain pushouts. This led to the formulation of the next theorem. Also there had been a long period of experimentation by Brown and Spencer on the relations between crossed modules and double groupoids [BS76b, BS76a], and by Higgins on calculation with crossed modules, so that the proof of the theorem, and the deduction of interesting calculations, came fairly quickly in 1974.

The next two theorems correspond to Theorem C of this Brown and Higgins paper ([BH78]). We separate the statement into two theorems for an easier understanding. The first one is about coverings by two (open) subspaces, the second one about adjunction spaces.

First, we say the based pair (X, A) is *connected* if A and X are path connected and for $x \in A$ the induced map of fundamental groups $\pi_1(A, x) \to \pi_1(X, x)$ is surjective, or, equivalently, using the homotopy exact sequence, when $\pi_1(X, A, x) = 0$.

Having in mind that all pairs are based but not including the base point in the statement, we have:

Theorem 2.3.1 *Let A, U_1, and U_2 be subspaces of X such that the total space X is covered by the interiors of U_1 and U_2. We define $U_{12} = U_1 \cap U_2$, and $A_\nu = A \cap U_\nu$ for $\nu = 1, 2, 12$. If the pairs (U_ν, A_ν) are connected for $\nu = 1, 2, 12$, then:*

(Con) *The pair (X, A) is connected.*

(Iso) *The following diagram induced by inclusions*

$$\begin{tikzcd} \Pi_2(U_{12}, A_{12}) \arrow[r] \arrow[d] & \Pi_2(U_2, A_2) \arrow[d] \\ \Pi_2(U_1, A_1) \arrow[r] & \Pi_2(X, A) \end{tikzcd} \tag{2.3.1}$$

is a pushout of crossed modules.

Remark 2.3.2 Recall that this statement means that the above mentioned diagram is commutative and has the following universal property: For any crossed module $\mathcal{M}$ and morphisms of crossed modules $\phi_\nu : \Pi_2(U_\nu, A_\nu) \to \mathcal{M}$ for $\nu = 1, 2$ making the external square commutative, there is a unique morphism of crossed modules $\phi : \Pi_2(X, A) \to \mathcal{M}$ such that the diagram

$$\begin{tikzcd}
\Pi_2(U_{12}, A_{12}) \arrow[r] \arrow[d] & \Pi_2(U_1, A_1) \arrow[d] \arrow[ddr, bend left, "\phi_1"] & \\
\Pi_2(U_2, A_2) \arrow[r] \arrow[drr, bend right, "\phi_2"'] & \Pi_2(U, A) \arrow[dr, dotted, "\phi"] & \\
& & \mathcal{M}
\end{tikzcd}$$

commutes. □

There is a slightly more general version of the theorem for adjunction spaces that can be deduced from the preceding theorem by using general mapping cylinder arguments.

Theorem 2.3.3 *Let X and Y be spaces, A a subset of X and $f : A \to Y$ a map. We consider subspaces $X_1 \subseteq X$ and $Y_1 \subseteq Y$ and define $A_1 = X_1 \cup A$ and $f_1 = f| : A_1 \to Y_1$. If the inclusions $A \subseteq X$ and $A_1 \subseteq X_1$ are closed cofibrations and the pairs $(Y, Y_1), (X, X_1), (A, A_1)$ are connected, then:*

(Con) *The pair $(X \cup_f Y, X_1 \cup_{f_1} Y_1)$ is connected.*

(Iso) *The following diagram induced by inclusions*

$$\begin{tikzcd}
\Pi_2(A, A_1) \arrow[r] \arrow[d] & \Pi_2(Y, Y_1) \arrow[d] \\
\Pi_2(X, X_1) \arrow[r] & \Pi_2(X \cup_f Y, X_1 \cup_{f_1} Y_1)
\end{tikzcd} \tag{2.3.2}$$

is a pushout of crossed modules.

Remark 2.3.4 The term closed cofibration included in the hypothesis of the theorem is satisfied in a great number of useful cases. It can be intuitively interpreted as saying that the placing of A in X and of A_1 in X_1 are 'locally not wild'. □

The interest in these theorems is at least seven fold:

- The theorem does have Whitehead's Theorem as a consequence (see Corollary 5.4.8).
- The theorem is a very useful computational tool and gives information unobtainable so far by other sources.
- The theorem is an example of a local-to-global theorem. Such theorems play an important rôle in mathematics and its applications.
- The theorem deals with nonabelian objects, and so cannot be proved by traditional means of algebraic topology.
- The two available proofs use groupoid notions in an essential way.
- The existence of the theorem confirms the value of the crossed module concept, and of the methods used in its proof. We should be interested in algebraic structures for which this kind of result is true.

- It shows the difficulty of homotopy theory since one has, it appears, to go through all this just to determine, as we explain in Section 5.8, the second homotopy groups of certain mapping cones.

A further point is that the proof we shall give later does not assume the general existence of pushouts of crossed modules. What it does is directly verify the required universal property in this case.

We conclude this section by stating an analogue of Theorem 2.3.1, but for general covers of a space X; this also will be deduced from Theorem 6.8.2.

Let Λ be an indexing set and suppose we are given a family $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ of subsets of X such that the interiors of the sets of $\mathcal{U}$ cover X. For each $\nu = \{\nu_1, \cdots, \nu_n\} \in \Lambda^n$, we write

$$U_\nu = U_{\nu_1} \cap \cdots \cap U_{\nu_n}.$$

Let A be a subspace of X, and define $A_\nu = U_\nu \cap A$, for any $\nu \in \Lambda^n$. Suppose also given a base point $x \in A$ which is contained in every X_λ.

Theorem 2.3.5 *Assume that for every $\nu \in \Lambda^n, n \geqslant 1$, the pair (U_ν, A_ν) is connected. Then*

(Con) *the pair (X, A) is connected, and*

(Iso) *the crossed module $\Pi_2(X, A)$ satisfies the following universal property: For any crossed module $\mathcal{M}$ and any family of morphisms of crossed modules $\{\phi_\lambda : \Pi_2(U_\lambda, A_\lambda) \to \mathcal{M} \mid \lambda \in \Lambda\}$ such that for any $\lambda, \mu \in \Lambda$ the diagram*

$$\begin{tikzcd}
\Pi_2(U_{\lambda\mu}, A_{\lambda\mu}) \arrow[r] \arrow[d] & \Pi_2(U_\lambda, A_\lambda) \arrow[d, "\phi_\lambda"] \\
\Pi_2(U_\mu, A_\mu) \arrow[r, "\phi_\mu"] & \mathcal{M}
\end{tikzcd}$$

commutes, there is a unique morphism of crossed modules $\phi : \Pi_2(X, A) \to \mathcal{M}$ such that all triangles of the form

$$\begin{tikzcd}
\Pi_2(U_\lambda, A_\lambda) \arrow[r] \arrow[dr] & \Pi_2(X, A) \arrow[d, "\phi"] \\
 & \mathcal{M}
\end{tikzcd}$$

commute.

The universal property of the theorem can be expressed as what is called a 'co-equaliser condition' (see Appendix).

Remark 2.3.6 It can be easily seen from the proof that the conditions on n-fold intersections for all $n \geqslant 1$ can be relaxed to path connectivity of all 4-fold intersections, and 1-connectivity of all pairs given by 8-fold intersections. More refinements of the arguments, using Lebesgue covering dimension, reduce these numbers to 3 and 4 respectively. These improvements were originally shown by Razak Salleh in his thesis [RS76]. □

The proof of Theorem 2.3.5 will be given later via another intermediate algebraic structure, that of double groupoids, since these have properties which are more appropriate than are those of crossed modules for expressing the geometry of the proof, which is analogous to that of the 1-dimensional theorem.

2.4 The classifying spaces of a group and of a crossed module

We are going to give in the second part of this book the construction of the classifying space of a crossed complex; this includes as particular cases the classifying spaces of a group and of a crossed module.

Nevertheless, this is a good point to state some of the properties of these special cases. In particular we want to stress that these classifying spaces classify the 1-type and the 2-type of a space.

The *classifying space of a group* P is a functorial construction

$$B : \mathsf{Groups} \to \mathsf{Top}_*$$

assigning a reduced CW-complex BP to each group P so that

Proposition 2.4.1 *The homotopy groups of the classifying space of the group* P *are given by*

$$\pi_i(BP) \cong \begin{cases} P & \textit{if } i = 1, \\ 0 & \textit{if } i \geqslant 2. \end{cases}$$

This gives a natural equivalence from $\pi_1 B$ to the identity. There is also some relation between $B\pi_1$ and the identity. It is given by

Proposition 2.4.2 *Let* X *be a reduced* CW*-complex and let* $\phi : \pi_1(X) \to P$ *be a homomorphism of groups. Then there is a map*

$$X \to BP$$

inducing the homomorphism ϕ *on fundamental groups.*

As a consequence we get that $B\pi_1$ captures all information on fundamental groups.

Theorem 2.4.3 *Let* X *be a reduced* CW*-complex and let* $P = \pi_1(X)$*. Then there is a map*

$$X \to BP$$

inducing an isomorphism of fundamental groups.

It is because of these results that groups are said to model pointed, connected homotopy 1-types.

Next, we state some properties of the *classifying space of a crossed module*. It is a functor

$$B : \mathsf{XMod} \to \mathsf{Top}_*$$

assigning to a crossed module $\mathcal{M} = (\mu : M \to P)$ a pointed CW-space $B\mathcal{M}$ with the following properties:

Proposition 2.4.4 *The homotopy groups of the classifying space of the crossed module* $\mathcal{M}$ *are given by*

$$\pi_i(B\mathcal{M}) \cong \begin{cases} \textit{Coker } \mu & \textit{for } i = 1 \\ \textit{Ker } \mu & \textit{for } i = 2 \\ 0 & \textit{for } i > 2. \end{cases}$$

There is a twofold relation with the classifying space of a group defined before. On the one hand, it is a generalisation, i.e.

Proposition 2.4.5 *If* P *is a group then the classifying space* $B(1 \to P)$ *is exactly the classifying space* BP *discussed before.*

On the other hand

Proposition 2.4.6 *Let* $M \triangleleft P$ *be a normal subgroup of the group* P*. Then the morphism of crossed modules* $(M \to P) \to (1 \to P/M)$ *induces a homotopy equivalence of classifying spaces* $B(M \to P) \to B(P/M)$.

This follows from Whitehead's theorem, that a map of CW-spaces inducing an isomorphism of all homotopy groups is a homotopy equivalence.

Proposition 2.4.7 *The classifying space* BP *is a subcomplex of* $B\mathcal{M}$*, and there is a natural isomorphism of crossed modules*

$$\Pi_2(B\mathcal{M}, BP) \cong \mathcal{M}. \tag{2.4.1}$$

Theorem 2.4.8 *Let* X *be a reduced* CW*-complex, and let* $\Pi_2(X, X^1)$ *be the crossed module* $\pi_2(X, X^1) \to \pi_1(X^1)$*, where* X^1 *is the 1-skeleton of* X*. Then there is a map*

$$X \to B(\Pi_2(X, X^1)) \tag{2.4.2}$$

inducing an isomorphism of π_1 *and* π_2.

It is because of these results that it is reasonable to say that crossed modules model all pointed connected homotopy 2-types. This result is originally due to Mac Lane and Whitehead [MLW50] (they use the term 3-type for what later came to be called 2-type), and with a different proof.

Later we shall give by means of crossed complexes an elegant description of the cells of the classifying space $B(M \to P)$. The existence and properties of the classifying space show that calculations of pushouts of crossed modules, such as those required by the 2-dimensional van Kampen Theorem, may also be regarded as calculations of homotopy 2-types. This is evidence that the fundamental crossed module of a pair is an appropriate candidate for a 2-dimensional version of the fundamental group, as sought by an earlier generation of topologists.

The situation we have for crossed modules and pairs of spaces comes under a format very similar to the main diagram (MD) in the Preface:

$$\begin{tikzcd}
\text{(topological data)} \arrow[rr, shift left, "\Pi"] \arrow[dr, "U"'] & & \text{(algebraic data)} \arrow[ll, shift left, "\mathcal{B}"] \arrow[dl, "B"] \\
& \mathsf{Top} &
\end{tikzcd} \tag{2.4.3}$$

We suppose the following properties:

(i) The functor Π preserves certain colimits.

(ii) There is a natural equivalence $\Pi\mathcal{B} \simeq 1$.

(iii) $B = U\mathcal{B}$.

(iv) There is a convenient natural transformation $1 \simeq \mathcal{B}\Pi$ preserving some homotopy properties.

Property (i) is a form of the van Kampen Theorem. This enables some computations to get started.
Property (ii) shows that the algebraic data forms a reasonable mirror of the topological data.
Property (iii) allows the classifying space to be defined: U is some kind of forgetful functor.
Property (iv) is difficult to state precisely in general terms. The intention is to show that the structure $\mathcal{B}\Pi$ captures some slice of the homotopy properties of the original topological data.

We shall not use any general format of or deduction from these properties, but it should be realised that the material we give on groups and on crossed modules forms part of a much more general pattern.

Let us finish this section by giving also some indications of how to go up one dimension further. First we give a theorem about the behaviour of that classifying space of crossed modules functor when applied to a short exact sequence. This theorem will be deduced from a more general theorem on the classifying space of crossed complexes, where more machinery is available for the proof.

Theorem 2.4.9 *Suppose the commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "i"] \arrow[d, "\lambda"] & M \arrow[r, "p"] \arrow[d, "\mu"] & N \arrow[r] \arrow[d, "\nu"] & 1 \\ 1 \arrow[r] & K \arrow[r, "j"'] & P \arrow[r, "f"'] & Q \arrow[r] & 1 \end{tikzcd} \tag{2.4.4}$$

is such that the vertical arrows are crossed modules, the squares are morphisms of crossed modules, and the rows are exact sequences of groups. Then the diagram of induced maps of classifying spaces

$$B(L \to K) \to B(M \to P) \to B(N \to Q)$$

is a fibration sequence.

In the above situation we say that the crossed module $L \to K$ is a kernel of the morphism (p, f) of crossed modules. Note that the groups L, K are essentially normal subgroups of M, P respectively. There is an additional property, that if $k \in K, m \in M$, then $p(m^{-1}m^{j(k)}) = 1$, so that $m^{-1}m^{j(k)} \in \operatorname{Im} i$. This gives rise to a function $h : K \times M \to L$. The properties are summarised by saying that the first square of diagram 2.4.4 is a *crossed square* [Lod82]. This structure gives the next stage after crossed modules for modeling homotopy types, that is they model homotopy 3-types. There seem to be good reasons why the analysis of kernels should give rise to a higher order structure modeling a further level of homotopy types. These ideas are quite subtle and require notions of 'crossed squares' which cannot be pursued this book (see [BL87b] and [Por93]).

2.5 Cat^1-groups.

There are several algebraic and combinatorial categories that are equivalent to the category of crossed modules. Some of these equivalences were already known to Verdier in the late 60's, but the first published account seems to have been by Brown and Spencer in 1976 [BS76b]. Later, these equivalences have been generalised by Porter, [Por87], to a more categorical setting, and by Ellis and Steiner, [ES87], to an n-cube setting.

Of the categories equivalent to XMod/Groups, perhaps the most used is the category Cat^1-Groups of cat^1-groups. One of its advantages is the naturality of the generalisation to higher dimensions and in this way was used for Loday in [Lod82]. It is also useful in some cases when describing the colimits used in the 2-dimensional van Kampen Theorem.

In this section, we explain this equivalence and some of the applications. Let us begin by expressing the basic properties of a crossed module $\mathcal{M} = (\mu : M \to P)$ in an alternative way.

The action of P on M can be encoded using the semidirect product $P \ltimes M$, and the projection $s : P \ltimes M \to P, (p, m) \mapsto p$. The map μ gives a homomorphism $t : P \ltimes M \to P \ltimes M$ by the rule $(p, m) \mapsto (p\mu(m), 1)$; that t is a homomorphism of groups follows from CM1).

It is a bit more difficult to find the way CM2) can be translated, but after playing for a while it can be seen that it gives that the elements of $\mathrm{Ker}\, s$ and those of $\mathrm{Ker}\, t$ commute in the semidirect product. This is the kind of algebraic object we need.

A *cat^1-group* is a triple $\mathcal{G} = (G, s, t)$ such that G is a group and $s, t : G \to G$ are group homomorphisms satisfying

CG1) $st = t$ and $ts = s$

CG2) $[\mathrm{Ker}\, s, \mathrm{Ker}\, t] = 1$.

A *homomorphism of cat^1-groups* between (G, s, t) and (G', s', t') is a homomorphism of groups $f : G \to G'$ preserving the structure, i.e. such that $s'f = fs$ and $t'f = ft$. These objects and morphisms define the category Cat^1-Groups of cat^1-groups.

Example 2.5.1 The category of groups, Groups, can be considered a full subcategory of Cat^1-Groups using the inclusion functor

$$I : \mathsf{Groups} \to \mathsf{Cat}^1\text{-}\mathsf{Groups}$$

given by $I(G) = (G, Id, Id)$. □

Having in mind the discussion at the beginning of this section, we define a functor

$$\lambda : \mathsf{XMod/Groups} \to \mathsf{Cat}^1\text{-}\mathsf{Groups}$$

given by $\lambda(\mu : M \to P) = (P \ltimes M, s, t)$, where $s(g, m) = (g, 1)$ and $t(g, m) = (g(\mu m), 1)$.

Proposition 2.5.2 *If $\mu : M \to P$ is a crossed module, then $\lambda(\mu : M \to P)$ is a cat^1-group.*

Proof It is clear that s is a homomorphism. To check that t is also a homomorphism , let us consider elements $(g, m), (g', m') \in P \ltimes M$. Then, we have

$$\begin{aligned} t((g,m)(g',m')) &= t(gg', m^{g'}m') \\ &= (gg'\mu(m^{g'})\mu m', 1) = (gg'g'^{-1}\mu m g'\mu m', 1) \quad \text{by CM1)} \\ &= (g\mu m g'\mu m', 1) = t(g,m)t(g',m'). \end{aligned}$$

It is also easy to prove that s, t satisfy CG1).

To check CG2), let us consider generic elements $(1, m) \in \mathrm{Ker}\, s$ and $(\mu m', m'^{-1}) \in \mathrm{Ker}\, t$. Then, we have

$$\begin{aligned} (1,m)(\mu m', m'^{-1}) &= (\mu m', m^{\mu m'} m'^{-1}) = (\mu m', m'^{-1} m m' m'^{-1}) \quad \text{by CM2)} \\ &= (\mu m', m'^{-1} m) = (\mu m', m'^{-1})(1, m). \end{aligned}$$

□

Example 2.5.3 Thus, associated to any normal subgroup M of G, we have a cat[1]-group $M \ltimes G$, where G acts on M by conjugation. □

To define the functor back, let us check that all cat[1]-groups have a semidirect product decomposition.

Proposition 2.5.4 *For any cat[1]-group* (G, s, t)*:*

i) The maps s, t *have the same range, i.e.* $s(G) = t(G) = N$*, and are the identity on* N.

ii) The morphisms s *and* t *are "projections", i.e.* $t^2 = t$ *and* $s^2 = s$.

Proof i) As $st = t$, we have $\operatorname{Im} t \subseteq \operatorname{Im} s$ and as $ts = s$, we have $\operatorname{Im} s \subseteq \operatorname{Im} t$.

ii) We have $ss = sts = ts = s$. Similarly, $tt = t$. □

As an easy consequence, we have:

Corollary 2.5.5 *There are two split short exact sequences*

$$1 \to \operatorname{Ker} s \hookrightarrow G \xrightarrow{s} N \to 1 \qquad \text{and} \qquad 1 \to \operatorname{Ker} t \hookrightarrow G \xrightarrow{t} N \to 1.$$

Remark 2.5.6 Thus G is isomorphic to both semidirect products $N \ltimes \operatorname{Ker} s$ and $N \ltimes \operatorname{Ker} t$, where N acts on both kernel by conjugation. The map $N \ltimes \operatorname{Ker} s \to G$ is just the product and the inverse isomorphism $G \to N \ltimes \operatorname{Ker} s$ is given by $g \mapsto (s(g), s(g^{-1})g)$. □

We can also define an inverse functor

$$\gamma : \mathsf{Cat^1\text{-}Groups} \to \mathsf{XMod/Groups}$$

given by $\gamma(G, s, t) = (t| : \operatorname{Ker} s \to \operatorname{Im} s)$ where $\operatorname{Im} s$ acts on $\operatorname{Ker} s$ by conjugation.

Proposition 2.5.7 *If* (G, s, t) *is a cat[1]-group, then* $\gamma(G, s, t)$ *is a crossed module.*

Proof With respect to CM1), for all $g \in \operatorname{Im} s$ and $m \in \operatorname{Ker} s$, we have

$$t(m^g) = t(g^{-1}mg) = (tg)^{-1}(tm)(tg).$$

Now, since $g \in \operatorname{Im} s = \operatorname{Im} t$, by Proposition 2.5.4 we have $tg = g$. Thus, $t(m^g) = g^{-1}(tm)g$.

On the other hand, with respect to CM2) for all $m, m' \in \operatorname{Ker} s$, we have

$$m'^{(tm)} = (tm^{-1})m'(tm) = (tm^{-1})m'(tm)m^{-1}m.$$

Now, since $(tm)m^{-1} \in \operatorname{Ker} s$ and $m' \in \operatorname{Ker} s$, they commute, giving

$$m'^{(tm)} = (tm^{-1})(tm)m^{-1}m'm = m^{-1}m'm.$$

□

Proposition 2.5.8 *The functors* λ *and* γ *give an equivalence of categories.*

Proof On one hand we have $\lambda\gamma(G, s, t) = (\operatorname{Im} t \ltimes \operatorname{Ker} s, s', t')$ where $s'(g, m) = (g, 1)$ and $t'(g, m) = (gt(m), 1)$. Clearly there is a natural isomorphism of groups $\phi : G \to \operatorname{Im} t \ltimes \operatorname{Ker} s$ given by $\phi(g) = (s(g), s(g)^{-1}g)$ that is an isomorphism of cat[1]-groups.

On the other hand, $\gamma\lambda(\mu : M \to P) = (\operatorname{Ker} \xrightarrow{t} \operatorname{Im} s)$ where $s : P \ltimes M \to P \ltimes M$ is given by $s(g, m) = (g, 1)$. There are obvious natural isomorphisms $\operatorname{Ker} s \cong M$ and $\operatorname{Im} s \cong P$ giving a natural isomorphism of crossed modules. □

2.6 The fundamental crossed module of a fibration

In this section the proofs will be omitted or be sketchy, since background in fibrations of spaces is needed. Throughout we assume that 'space' means 'pointed space'.

In this section we are going to give a proof that for any fibration $\mathcal{F} = (F \xrightarrow{i} E \xrightarrow{p} X)$ the induced map

$$i_* : \pi_1(F) \to \pi_1(E)$$

is a crossed module $\Pi_2(\mathcal{F})$ which we call the *fundamental crossed module* of the fibration $\mathcal{F}$. This is an observation first made by Quillen and from it can be deduced the fundamental crossed module of a pair of spaces.

Perhaps it is better first to recall in some detail the action of $\pi_1(E)$ on $\pi_1(F)$ for any fibration $\mathcal{F}$.

Let us consider $[\mu] \in \pi_1(F)$ and $[\alpha] \in \pi_1(E)$. The projection to X of the loop $\alpha^{-1}\mu\alpha$ is homotopic to the constant through a homotopy of loops $H : I \times I \to X$. Since p is a fibration, using the homotopy lifting property, we get a homotopy of loops $\overline{H} : I \times I \to E$ from $\alpha^{-1}\mu\alpha$ to a loop projecting to the constant, i.e. $\operatorname{Im} \overline{H}_1 \subseteq F$. We define

$$[\mu]^{[\alpha]} = [\overline{H}_1] \in \pi_1(E).$$

We omit the proof that this action is well defined. This is a good exercise on fibration theory.

To prove that i_* is a crossed module, we proceed in a roundabout way. Clearly, it is equivalent to prove that the semidirect product $\pi_1(E) \ltimes \pi_1(F)$ given by the action just defined is a cat^1-group. Again, this is not done directly, but instead we prove that there is a natural isomorphism of groups

$$\pi_1(E \times_X E) \cong \pi_1(E) \ltimes \pi_1(F)$$

and that the former is a cat^1-group, where $E \times_X E$ is the pullback of p along p, i.e.

$$E \times_X E = \{(e, e') \in E \times E : p(e) = p(e')\}.$$

First, let us prove that $\pi_1(E \times_X E)$ decomposes in the expected semidirect product.

Proposition 2.6.1 *For any fibration $\mathcal{F} = (F \xrightarrow{i} E \xrightarrow{p} X)$, there are two splitting short exact sequences*

$$1 \to \pi_1(F) \xrightarrow{i_{1*}} \pi_1(E \times_X E) \xrightarrow{p_{1*}} \pi_1(E) \to 1 \quad \textit{and} \quad 1 \to \pi_1(F) \xrightarrow{i_{2*}} \pi_1(E \times_X E) \xrightarrow{p_{2*}} \pi_1(E) \to 1$$

where i_l is the inclusion of F in the l^{th} factor. Moreover both are natural with respect to maps of fibrations.

Proof Recall that the projection in the first factor $E \times_X E \to E$ is a fibration with fibre F since it is the pullback of p along itself. Also, the diagonal map gives a section of this fibration. Thus, its homotopy exact sequence decomposes into a sequence of splitting short exact sequences. In particular,

$$1 \to \pi_1(F) \xrightarrow{i_{1*}} \pi_1(E \times_X E) \xrightarrow{p_{1*}} \pi_1(E) \to 1$$

is a splitting short exact sequence. The same is true in the second case. □

Now, we are able to prove that $(\pi_1(E \times_X E), s, t)$ where s (resp. t) is the homomorphism induced on the fundamental groups by the composition of the projection in the first (resp. second) factor and the diagonal is a cat^1-group for any fibration $\mathcal{F}$. We shall call it the *fundamental cat^1-group of the fibration $\mathcal{F}$*.

Proposition 2.6.2 *Let* $\mathcal{F} = (F \xrightarrow{i} E \xrightarrow{p} X)$ *be a fibration. Then* $(\pi_1(E \times_X E), s, t)$ *is a cat*^{1}*-group.*

Proof It clearly satisfies CG1) since the maps s, t are essentially projections.

To prove CG2), using the exact sequence of Proposition 2.6.1, we have $\operatorname{Ker} s = \operatorname{Im} i_{1*}$ and $\operatorname{Ker} t = \operatorname{Im} i_{2*}$

Also by Proposition 2.6.1 the elements of $\operatorname{Ker} s$ are of the form $[(ct, \mu)]$ where μ is a loop in the fibre and the elements of $\operatorname{Im} s$ are of the form $[(\alpha, \alpha)]$ where α is a loop in E.

We choose elements $[(ct, \mu)] \in \operatorname{Ker} s$ and $[(\nu, ct)] \in \operatorname{Ker} t$ where μ and ν are loops in the fibre. The commutativity of these elements is now clear, since

$$[(\nu, ct)][(ct, \mu)] = [(ct, \mu)][(\nu, ct)] = [(\nu, \mu)].$$

□

Now, we proceed to identify the crossed module associated with $(\pi_1(E \times_X E), s, t)$.

Proposition 2.6.3 *The crossed module* $(t| : \operatorname{Ker} s \to \operatorname{Im} s)$ *associated to the cat*^{1}*-group* $\pi_1(E \times_X E)$ *is naturally isomorphic to* $\Pi_2\mathcal{F} = (\pi_1(F) \to \pi_1(E))$.

Proof There are natural isomorphisms $\pi_1(F) \cong \operatorname{Ker} s$ and $\pi_1(E) \cong \operatorname{Im} s$, given by $[\mu] \mapsto [(ct, \mu)]$ and $[\alpha] \mapsto [(\alpha, \alpha)]$ respectively. It remains only to check that these isomorphisms preserve actions.

The action of $\operatorname{Ker} s$ on $\operatorname{Im} s$ is given by conjugation in $\pi_1(E \times_X E)$. Under these isomorphisms the result of the action of $[\alpha] \in \pi_1(E)$ on $[\mu] \in \pi_1(F)$, is the homotopy class of any loop ν in F satisfying

$$[(ct, \nu)] = [(\alpha^{-1}\alpha, \alpha^{-1}\mu\alpha)].$$

Recalling the definition of the $\pi_1(E)$ action on $\pi_1(F)$ at the beginning of the section, we see that $[\mu]^{[\alpha]}$ is represented by just this same element. □

To define the fundamental cat^1-group functor on maps of general topological spaces we need some more homotopy theory. There is no space to develop this here in full, and so we just sketch the ideas, which are well covered in books on abstract homotopy theory, for example [KP97].

A standard procedure in homotopy theory is to factor any map $f : Y \to X$ through a homotopy equivalence i and a fibration $\overline{f} : \overline{Y} \to X$ where $\overline{Y} = \{(\lambda, y) \in X^I \times Y : \lambda(1) = f(y)\}$ and $\overline{f}(\lambda, y) = \lambda(0)$.

This gives a functor $\mathtt{Fib} : f \mapsto \overline{f}$ from maps to fibrations. We define the cat^1-group functor on maps of general topological spaces by composition with the cat^1-group of fibrations functor.

Let us sketch a direct description of the composite functor

$$\mathsf{Maps} \to \mathsf{Cat}^1\text{-}\mathsf{Groups}$$

following ideas of Gilbert in [Gil87].

The functor is defined by

$$(f : Y \to X) \mapsto (\pi_1(\overline{Y} \times_X \overline{Y}), p_{1*}, p_{2*}).$$

Using the homeomorphism

$$\overline{Y} \times_X \overline{Y} \equiv \{(y_1, \lambda, y_2) \in Y \times X^I \times Y : \lambda(0) = f(y_1) \text{ and } \lambda(1) = f(y_2)\}$$

the projections in the factors correspond to the maps

$$\begin{array}{rclcrcl} p_1(y_1, \lambda, y_2) & = & (y_1, \lambda_1), & \text{where} & \lambda_1(t) & = & \lambda(t/2) \\ p_2(y_1, \lambda, y_2) & = & (y_2, \lambda_2), & \text{where} & \lambda_2(t) & = & \lambda(1 - (t/2)). \end{array}$$

Via the same homeomorphism, the elements of $\pi_1(\overline{Y} \times_X \overline{Y})$ correspond to homotopy classes of triples, $[(\alpha, \mu, \beta)]$, where $\mu : I \times I \to X$ maps $I \times \{0, 1\}$ to the base point and $\alpha, \beta : I \to Y$ are loops on Y lifting $\mu(0, -)$ and $\mu(1, -)$ respectively. The homotopies correspond to triples, (F, H, G), the map $H : I \times I \times I \to X$ sending $I \times \{0, 1\} \times I$ to the base point, and $F, G : I \times I \to Y$ being homotopies of loops, relative to the end points, lifting $H(0, -, -)$ and $H(1, -, -)$, respectively.

The description of p_{1*} and p_{2*} follows easily.

For the sake of coherence let us point out that if f is already a fibration, both definitions of the fundamental cat^1-group produce the same group up to isomorphism.

If f is a fibration, f and $\overline{f}$ are fibre homotopy equivalent. It can be checked directly that $Y \times_X Y$ and $\overline{Y} \times_X \overline{Y}$ are also homotopy equivalent, but it is also a consequence of the following cogluing theorem which is a special case of the results of [BH70]. The dual of this result, namely a 'gluing theorem', is proved in [Bro06] and in an abstract setting in [KP97].

Theorem 2.6.4 *Suppose given maps over* X

$$\begin{tikzcd} Y \arrow[rr, "i"] \arrow[dr, "f"'] & & \overline{Y} \arrow[dl, "\overline{f}"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[rr, "j"] \arrow[dr, "g"'] & & \overline{Z} \arrow[dl, "\overline{g}"] \\ & X & \end{tikzcd}$$

such that $f, \overline{f}, g, \overline{g}$ *are fibrations, and* i, j *are homotopy equivalences. Then the induced map on pullbacks*

$$i \times_X j : Y \times_X Z \to \overline{Y} \times_X \overline{Z}$$

is also a homotopy equivalence, and in fact a fibre homotopy equivalence.

In the particular case in which we are mostly interested, we consider a pair of topological spaces (X, A). Associated to the inclusion $i : A \to X$ there is the fibration $\overline{A} \to X$ where $\bar{A}$ is the space of paths in X starting at some point of A and the map sends each path to its end point. The fibre of this fibration is the space

$$F_i = \{\lambda \in X^I : \lambda(0) \in A \quad \text{and} \quad \lambda(1) = *\}$$

whose homotopy groups are, by definition, those of the pair (X, A), i.e.

$$\pi_n(F_i) = \pi_{n+1}(X, A).$$

In particular, the *fundamental crossed module of a pair functor*

$$\Pi_2 : \mathsf{Top}^2_* \longrightarrow \mathsf{Fib} \longrightarrow \mathsf{XMod/Groups}$$

is given by

$$\Pi_2(X, A) = (\partial : \pi_2(X, A) \to \pi_1(A))$$

with the usual action already known and used by Whitehead.

Finally in this section, we mention some relations of crossed modules with algebraic K-theory, for those familiar with that area.

Let R be a ring. A basic structure for algebraic K-theory is the homotopy fibration

$$F(R) \to BGL(R) \to BGL(R)^+.$$

This yields the crossed module

$$\pi_1(F(R)) \to \pi_1(BGL(R))$$

which is equivalent to

$$\mathrm{St}(R) \to \mathrm{GL}(R)$$

which has cokernel $K_1(R)$ and kernel $K_2(R)$.

Now let I be an ideal of R, and let $\mathrm{GL}(R, I)$, the *congruence subgroup*, be the kernel of $\mathrm{GL}(R) \to \mathrm{GL}(R/I)$. By the same trick, we get a crossed module

$$\mathrm{St}(R, I) \to \mathrm{GL}(R, I)$$

which has cokernel $K_1(R, I)$ and kernel $K_2(R, I)$. This is Loday's definition of the relative K_2 [Lod78]. It differs from that of Milnor [Mil71] by relations corresponding to those of the second rule CM2) for a crossed module.

One advantage of this procedure is the generalisation to multirelative groups $K_2(R; I_1, \dots, I_n)$ [GWL81, Ell88]. The relevant algebra is that of crossed n-cubes of groups. All this was one of the motivations for the van Kampen Theorem for n-cubes of spaces [BL87b].

2.7 The category of categories internal to groups

In this section, we study another category equivalent to XMod/Groups, namely the category of categories internal to groups, written Cat[Groups]. This category has easy generalisations both to higher dimensions and to other algebraic settings.

This category has two features that make it very interesting. On the one hand it can be used as an intermediate step to get a simplicial equivalent of crossed modules which can be generalised to crossed n-cubes. (This has been done by T. Porter in [Por93]).

On the other hand, we shall see that the category Cat[Groups] is formed by groupoids, being also the category of group-groupoids. This will be generalised in Chapter 6 to an equivalence from the category XMod of crossed modules over groupoids to a category of double groupoids.

First, let us recall from the Appendix that the definition of a category $\mathcal{C}$ is given by two sets, the object set, $\mathrm{Ob}\ \mathcal{C}$, and the morphism set, $\mathrm{Mor}\,\mathcal{C}$, and four maps, the identity i, the source and target s, t, and the composition of morphisms $\circ$, satisfying several axioms. Note that $\circ$ is considered as a function on its domain.

We say that $\mathcal{C}$ is a *category internal to* Groups, if both $\mathrm{Ob}\ \mathcal{C}$ and $\mathrm{Mor}\,\mathcal{C}$ have a group structure and the maps s, t, i and $\circ$ are homomorphisms of groups. Thus, a category internal to Groups is also a *group in the category of all small categories*. This principle for algebraic structure that 'an $\mathcal{A}$ in a $\mathcal{B}$ is also a $\mathcal{B}$ in an $\mathcal{A}$' is of wide applicability.

Similarly, a functor $f : \mathcal{C} \to \mathcal{C}'$ between two categories, is a pair of maps $\mathrm{Ob}\ f$ and f commuting with the structure maps (source, target, identity and composition).

A functor between categories internal to Groups is a *functor internal to* Groups if both maps are homomorphisms of groups.

Then, Cat[Groups] is the category whose objects and morphisms are categories and functors internal to Groups.

For any object $\mathcal{C}$ in Cat[Groups], we will write the product in $\mathrm{Mor}\,\mathcal{C}$ additively and the product in $\mathrm{Ob}\ \mathcal{C}$ multiplicatively. Then, if 1 and 0 are the identities in $\mathrm{Ob}\ \mathcal{C}$ and $\mathrm{Mor}\,\mathcal{C}$, we have $i(1) = 0$, $s(0) = 1$ and $t(0) = 1$. So, the elements of $\mathrm{Ker}\, s$ (resp. $\mathrm{Ker}\, t$) are the morphisms with source 1 (target 1).

The next property shows that, for any category internal to Groups, we can define the composition of morphisms in terms of the other structure maps.

Proposition 2.7.1 *For any two composable morphisms, u and v, we have*

(i) $$v \circ u = v - itu + u = v - isv + u,$$

(ii) $$v \circ u = u - itu + v = u - isv + v.$$

Proof (i) We have

$$v \circ u = (v + 0) \circ (itu + (-itu + u)) = (v \circ itu) + (0 \circ (-itu + u)) = v - itu + u.$$

The second equality is immediate because the morphisms are composable, so $itu = isv$.

(ii) is proved in a similar way. □

Remark 2.7.2 Thus, to prove that a category where the objects and morphisms are groups, and the source target and identity are homomorphisms, is internal to groups, all we have to check is that the composition defined using Proposition 2.7.1 is a homomorphism. □

There is also an expression for the inverse of a morphism, proving that all categories internal to groups are groupoids.

Proposition 2.7.3 *For any morphism in a category internal to groups we have*

$$u^{-1} = isu - u + itu.$$

Proof Let us define u^{-1} by this formula. We can easily check that it has the appropriate source and target and that both compositions are the identity. □

Remark 2.7.4 As a consequence of this property, a category internal to groups is a groupoid internal to groups, or, equivalently, a group in the category of groupoids.

Considering that a group is just a groupoid with only one object, we could try to study the category of "groupoids of groupoids", or "double groupoids". We shall do this in Chapter 6. □

To end this section, we state the relation of Cat[Groups] to the previous categories. The equivalence with Cat^1- Groups is easily defined.

In one direction, we assign to the cat^1-group (G, s, t) the category having $\mathrm{Im}\, s = \mathrm{Im}\, t$ as set of objects, G as set of morphisms, s and t as source and target, identity the inclusion $\mathrm{Im}\, s \subseteq G$ and composition defined by $g' \circ g = g' - itg + g$, for any $g, g' \in G$ with $tg = sg'$. It can be easily checked that this gives a category internal to Groups.

In the other direction, to any category $\mathcal{C}$ internal to Groups we assign the cat^1-group $(\mathrm{Mor}\ \mathcal{C}, i \circ s, i \circ t)$.

Thus, the categories XMod/Groups and Cat[Groups] are equivalent, since both are equivalent to Cat^1- Groups. However, it is convenient to record for further use the functors giving this equivalence.

The functor one way is defined as $\mathcal{C} \mapsto (s| : \mathrm{Ker}\, t \to \mathrm{Ob}\ \mathcal{C})$, where $\mathcal{C}$ is a cat^1-group. The reverse functor assigns to any crossed module $\mathcal{M} = (\mu : M \to P)$ the category having P as set of objects,

$P \ltimes M$ as set of morphisms; identity map given by the inclusion; source and target maps given by $s(g, m) = g$ and $t(g, m) = g(\mu m)$ and composition given by any of the formulae in Proposition 2.7.1.

Nevertheless, there is a simpler expression for the composition in this case. Notice first that with the definition of source and target, two morphisms $(g', m'), (g, m) \in P \ltimes M$ are composable when $g\mu m = g'$.

Proposition 2.7.5 *The composition of morphisms in* $P \ltimes M$*,*

$$\circ : P \ltimes M\,{}_s\!\times_t P \ltimes M \to P \ltimes M$$

is given by $(g(\mu m), m') \circ (g, m) = (g, mm')$.

Proof This is not difficult to prove using the definition of composition given in Proposition 2.7.1 i). □

With this property we can get another model of the category internal to Groups associated to a crossed module.

Proposition 2.7.6 *The map* $A : \mathrm{Mor}\, \mathcal{C}\,{}_s\!\times_t \mathrm{Mor}\, \mathcal{C} \to P \ltimes M \ltimes M$ *defined by* $A((g', m'), (g, m)) = (g, m, m')$*, is an isomorphism carrying the composition to the map*

$$\circ' : P \ltimes M \ltimes M \to P \ltimes M$$

sending (g, m, m') *to* (g, mm').

Proof Clearly A is bijective and transforms the composition to the afore mentioned map. It remains to check that A is a homomorphism and that is left as an exercise. □

Let us consider now the composite functor

$$\mathsf{Fib} \to \mathsf{Cat}^1\text{-}\mathsf{Groups} \to \mathsf{Cat}[\mathsf{Groups}]$$

i.e., mapping $\mathcal{F}$ to the category internal to Groups associated to the cat^1-group $\pi_1(E \times_X E)$.

Using the isomorphism $\mathrm{Im}\, p_{i*} \cong \pi_1(E)$, this category is isomorphic to the category that has $\pi_1(E)$ as objects, $\pi_1(E \times_X E)$ as morphisms, source and target given by projections, identity given by the diagonal and composition the only one possible to make this a category internal to groups.

As seen before, this category is also isomorphic to the one associated to $\pi_1(E) \ltimes \pi_1(F)$, that has $\pi_1(E)$ as objects, $\pi_1(E) \ltimes \pi_1(F)$ as morphisms, $([\alpha], [\mu]) \mapsto [\alpha]$ and $([\alpha], [\mu]) \mapsto [\alpha] * i_*([\mu])$ as source and target maps and composition given by

$$([\alpha] * i_*[\mu], [\mu']) \circ ([\alpha], [\mu]) = ([\alpha][\mu * \mu']).$$

We finish by stating a description of the composition in $\pi_1(E \times_X E)$.

Proposition 2.7.7 *Let* $[(\alpha, \beta)], [(\beta', \gamma')] \in \pi_1(E \times_X E)$ *be such that* $[\beta] = [\beta']$*,i.e. there is a homotopy* $G : \beta' \cong \beta$*. Since* p *is a fibration there is a homotopy* H *lifting* pG *and starting with* γ'*. Then*

$$[(\beta', \gamma')] \circ [(\alpha, \beta)] = [(\alpha, H_1)].$$

Proof It is clear that $[(\beta', \gamma')]$ and $[(\beta, H_1)]$ are homotopic using the homotopy (G, H). Then, $[(\beta', \gamma')] \circ [(\alpha, \beta)] = [(\beta, H_1)] \circ [(\alpha, \beta)]$. So, we only have to consider the composition in the case $[(\beta, \gamma)] \circ [(\alpha, \beta)]$. Using that $\mathcal{F}$ is a fibration there are unique $[\mu], [\mu'] \in \pi_1(F)$ with

$$[(\alpha, \beta)] = A([\alpha], [\mu]) = [(\alpha * ct, \alpha * \mu)]$$

and

$$[(\beta,\gamma)] = A([\beta],[\mu']) = [(\beta * ct, \beta * \mu')].$$

Clearly, $[\beta] = [\alpha] * i_*([\mu])$, and

$$\begin{aligned}
[(\beta,\gamma)] \circ [(\alpha,\beta)] &= A([\beta],[\mu']) \circ A([\alpha],[\mu]) \\
&= [(\beta * ct, \beta * \mu')] \circ [(\alpha * ct, \alpha * \mu)] \\
&= [(\alpha * ct, \alpha * \mu' * \mu)] \\
&= [((\alpha * ct) * ct, \alpha * \mu' * \mu)] \\
&= [(\alpha * ct, \beta * \mu')] \\
&= [(\alpha,\gamma)].
\end{aligned}$$

$\square$

This proof is related to a proof in [BJ04] which shows that in the construction of a double homotopy groupoid of a map of spaces, a composition defined geometrically agrees with that derived from Generalised Galois Theory.

We can also describe easily the functor

$$\mathsf{Maps} \to \mathsf{Cat}[\mathsf{Groups}].$$

Notice that $\pi_1(\overline{Y})$ is isomorphic to $\pi_1(Y)$ under the projection. So the associated category internal to groups is equivalent to the one having $\pi_1(Y)$ as objects, $\pi_1(\overline{Y} \times_X \overline{Y})$ as morphisms, source and target given by $[(\alpha,\mu,\beta)] \to [\alpha]$ and $[(\alpha,\mu,\beta)] \to [\beta]$, and composition given by $[(\beta,\mu',\gamma)] \circ [(\alpha,\mu,\beta)] = [(\alpha,\mu' * \mu,\gamma)]$.

Note that if ν is an homotopy from β to β', the composition of $[(\alpha,\mu,\beta)]$ with $[(\beta',\mu',\gamma)]$ is given by $[(\alpha,\mu' * \nu * \mu,\gamma)]$ since $[(\beta',\mu',\gamma)] = [(\beta,\mu' * \nu,\gamma)]$.

Chapter 3

Basic algebra of crossed modules

In this chapter we analyse what historically was the second source of crossed modules over groups: identities among relations in presentations of groups. This also leads to the area of *crossed resolutions of groups, and groupoids* with which we deal in Chapter 11.

A central problem in mathematics is the representation of infinite objects in manipulable, and preferably finite, terms. One method of doing this is by what is called a *resolution*. There is not a formal definition of this, but we can see several examples.

This notion first arose in the 19th century study of invariants. *Invariant theory* deals with subalgebras of polynomial algebras $\Lambda = k[x_1, \ldots, x_n]$, where k is a ring. Consider for example, the subalgebra A of $\mathbb{Z}[a, b, c, d]$ generated by

$$a^2 + b^2,\ c^2 + d^2,\ ac + bd,\ ad - bc.$$

It is called an *invariant subalgebra* since it is invariant under the action of $\mathbb{Z}_2$ which switches the variables a, b and c, d. As pointed out in [Gar80, p.247], "these generators satisfy the relation

$$(ac + bd)^2 + (ad - bc)^2 = (a^2 + b^2)(c^2 + d^2)$$

which is classically called a *syzygy*, and the algebra A of invariant polynomials turns out to be the homomorphic image of the polynomial algebra in four variables given by the quotient algebra

$$\mathbb{Z}[x, y, z, w]/(z^2 + w^2 - xy).$$

In particular, the algebra is finitely generated by four explicit polynomials, and the ideal of relations is finitely generated by a single explicit relation."

Hilbert solved also the so-called second main problem of invariant theory, in showing that the ideal of relations among the invariants was also finitely generated. [2] On [Gar80, p.253-4] we have: "Since the second main problem had succumbed so easily, it was natural to turn to chains of syzygies, studying relations among the generating set of relations and so on. More precisely, this work involved the sequence of finitely generated $k[x_1, \ldots, x_n]$-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_1 & \longrightarrow & F_1 & \longrightarrow & B_1 & \longrightarrow & 0 \\
0 & \longrightarrow & J_2 & \longrightarrow & F_2 & \longrightarrow & J_1 & \longrightarrow & 0 \\
 & & \cdots & & \cdots & & \cdots & & \\
0 & \longrightarrow & J_q & \longrightarrow & F_q & \longrightarrow & J_{q-1} & \longrightarrow & 0,
\end{array}$$

where the F_i are free with rank equal to the minimal number of generators of the i'th syzygy J_i. Hilbert's main theorem on the chains of syzygies says that if k is a field then $J_q = 0$ if $q > n$. In effect, this launched the theory of homological dimension of rings."

It was also natural to splice the morphisms $F_q \to J_{q-1} \to F_{q-1}$ together to get a sequence

$$\cdots \xrightarrow{\partial_{q+1}} F_q \xrightarrow{\partial_q} F_{q-1} \xrightarrow{\partial_{q-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} B_1$$

which was *exact* in the sense that

$$\operatorname{Ker} \partial_q = \operatorname{Im} \partial_{q+1}$$

for all q. This sequence was called a *free resolution* of the module B_1.

A basic question was the dependence of this sequence on the choices made. It was found that given any two such free resolutions $F_* \to B_1,\ F'_* \to B_1$, then there was a morphism $F_* \to F'_*$ and any two such morphisms were *homotopic*. It was also later found that the condition 'free' could conveniently be replaced by the condition *projective*.

Another source for homological algebra was the homology and cohomology theory of groups. As pointed out in [ML78], the starting point for this was the 1942 paper of Hopf [Hop42]. Let X be an aspherical space (i.e. connected and with $\pi_i X = 0$ for $i > 1$), and let

$$1 \to R \to F \to \pi_1 X \to 1$$

be an exact sequence of groups with F free. Hopf proved the formula

$$H_2 X \cong (R \cap [F,F])/[F,R].$$

We shall see in Section 5.5 that this formula follows from our 2-dimensional van Kampen Theorem for crossed modules. Thus we see the advantage for homotopy theory of having a 2-dimensional algebraic model of homotopy types.

Later work of Eilenberg-Mac Lane [EML47] found an algebraic formula for $H_n X,\ n \geqslant 2$ as follows. Produce sequences of $\mathbb{Z}G$-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_1 & \longrightarrow & F_1 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
0 & \longrightarrow & J_2 & \longrightarrow & F_2 & \longrightarrow & J_1 & \longrightarrow & 0 \\
 & & \cdots & & \cdots & & \cdots & & \\
0 & \longrightarrow & J_q & \longrightarrow & F_q & \longrightarrow & J_{q-1} & \longrightarrow & 0,
\end{array}$$

in which $\mathbb{Z}$ is the trivial $\mathbb{Z}G$-module, and each F_n is a free $\mathbb{Z}G$-module. Splice these together to give a *free resolution of* $\mathbb{Z}$:

$$F_* : \ldots \to F_n \to F_{n-1} \to \ldots \to F_2 \to F_1 \to \mathbb{Z}.$$

Form the chain complex $C = F \otimes_{\mathbb{Z}G} \mathbb{Z}$. Then $H_n C \cong H_n X$. Using particular choices of the F_n, the Hopf formula may be deduced [Bro94, p.46].

Thus we see an input from the homotopy and homology theory of spaces into the development of homological algebra. The use of homological methods across vast areas of mathematics is a feature of 20th century mathematics. It seems the solution of Fermat's last theorem depended on it, but it has also been applied in differential equations, coding theory and theoretical physics.

In its 20th century form, homological algebra is primarily an abelian theory. There is considerable work on nonabelian homological algebra, but this is only beginning to link with work in homotopical algebra, differential topology, and related areas. This book has an aim of showing one kind of start to a more systematic background to such an area.

Now the elementary, computational and example-oriented approach to groups considers presentations $\langle X; R\rangle$ of a group Q: that is X is a subset of Q and there is an exact sequence

$$1 \to N \to FX \xrightarrow{p} Q \to 1 \qquad (*)$$

where FX is the free group on generators $[x],\ x \in X;\ p$ is defined by $p[x] = x,\ x \in X$; and R is a set of generators of N as normal subgroup of FX. Thus, each element of N is a *consequence*

$$c = (r_1^{\epsilon_1})^{u_1} \dots (r_n^{\epsilon_n})^{u_n},$$

$r_i \in R, \epsilon_i = \pm 1, u_i \in FX$ and $a^b = b^{-1}ab$. However, this representation of elements of N, and the persistent use of N and FX as non-abelian groups (rather than of modules derived from them) plays a small role in the homological algebra of groups. One would expect, *a priori*, that the sequence (*) would be the beginning of a "nonabelian resolution" of the group Q. We will show that this is so in a later chapter.

Another curiosity is that there are a number of results in homotopy theory which are satisfactory for 1-connected spaces, but for which no formulation has been given when this assumption has been dropped, particularly when some non-abelian group has to be described. As long as interest was focussed on high-dimensional, or stable, problems, this restriction seemed not to matter. In many problems of current interest (for example low-dimensional topology, low-dimensional homology of groups, algebraic K-theory) this restriction has proved irksome, but few appropriate constructions have been generally seen to be available. This is one of the reasons for promoting the subject matter of this book.

In Section 3.1 we recall what is a presentation $\langle X \mid \omega\rangle$ of a group P, and show that the 'identities among the relations' can be seen as the elements of the kernel of a morphism $\theta : F(R \times P) \to P$ which satisfies CM1) in the definition of crossed modules.

This gives good reason to relax the concept of crossed module. In Section 3.3 we define precrossed modules in terms of axiom CM1) and also the functor that associates to every precrossed module a crossed module. This construction $(-)^{cr}$ is adjoint to the inclusion of categories $\mathsf{XMod/Groups} \hookrightarrow \mathsf{PXMod/Groups}$.

The morphism $\theta : F(R \times P) \to P$ has some extra freeness properties, making it what is called a 'free precrossed module'. These are studied in Section 3.4.

The chapter ends with the definition of a category of algebraic objects equivalent to that of precrossed modules and generalising the equivalence defined in Section 2.5.

3.1 Presentation of groups and identities among relations.

We now show how crossed modules arise in combinatorial group theory.

A group G is of course defined as a set with a multiplication satisfying certain axioms. In some cases this multiplication can be specified by a formula involving the elements: notable examples are certain matrix groups, such as the Heisenberg group H of matrices of the form

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$$

for $x, y, z \in \mathbb{Z}$. Thus the elements of H are given by triples (x, y, z) of integers with multiplication

$$(x, y, z)(u, v, w) = (x + u, y + v + xw, z + w).$$

This is known as a 'polynomial group law'. So we have a formula for the elements of the group H and for the multiplication.

The reader should not be surprised that this could raise difficulties in other cases. Part of the problem may be to give a useful formula for the elements of the group, let alone a formula for the multiplication. In mathematics as a whole, the question of 'presenting' information on a structure is often a key part of a problem.

An often useful way of representing the elements of a group is by giving generators for the group.

Example 3.1.1 Let D_4 be the dihedral group of order 8, i.e. the group of symmetries of the square. This group is generated by the elements x, y where x is rotation anticlockwise through $90°$ and y is reflection in a vertical bisector of the square. The elements of D_4 can then be written as

$$1, x, x^2, x^3, y, yx, yx^2, yx^3$$

and this is quite a convenient labeling of the elements. However if you try to work out the multiplication table in terms of this labeling you find you need more information, namely *relations* among the generators, for example

$$x^4 = 1,\ y^2 = 1,\ xyxy = 1.$$

If you are not already familiar with these, particularly the last one, then you are expected to verify them using some kind of model of a square. It turns out that every relation you might need in working out the multiplication table is a consequence only of these three. Thus we can specify the group completely also in terms of what we call a 'presentation'

$$\mathcal{P} = \langle x, y \mid x^4, y^2, xyxy\rangle.$$

If there is any need, we shall write $D_4 = \operatorname{gp}\mathcal{P}$. We need a definition of this idea of a presentation. □

The first thing to note is that the term x^4 in the presentation $\mathcal{P}$ is not an element of the group D_4, since the 4th power of the element x in D_4 is 1. Rather, as is common with the mathematical use of $=$, one side of the $=$ sign in $x^4 = 1$ is in fact an instruction: 'multiply x by itself 4 times', while the other side tells you what will be the result. A convenient language to express both an 'instruction for a procedure' and the result of the procedure is that of a morphism defined on a free group.

A *free group* $F(X)$ on a set X is intuitively a group $F(X)$ together with an inclusion mapping $i : X \to F(X)$ such that X generates the group $F(X)$ and 'there are no relations among these generators'. There are two useful ways of expressing this precisely.

One of them is to give what is called a 'universal property': this is that a morphism $g : F(X) \to G$ to a group G is entirely determined by its values on the set X. Put in another way, given any function $f : X \to G$, there is a unique morphism $g : F(X) \to G$ such that $gi = f$. This 'external' definition thus defines a free group by its relation to all other groups, and is a model for the notion of 'freeness' in other algebraic situations. A set X generating a free group plays a rôle similar to that of a basis for a vector space, and we also talk about X as a basis for the free group $F(X)$. However, unlike vector spaces, not every group is free. The simplest example is the group $\mathbb{Z}_2$ with two elements: it is not free because there is only one morphism $\mathbb{Z}_2 \to \mathbb{Z}$, the zero morphism.

The other 'internal' way of specifying a free group is to specify its elements and the multiplication, and this can be done in terms of 'reduced words': every non identity element of $F(X)$ is uniquely expressible in the form

$$x_1^{r_1} x_2^{r_2} \dots x_n^{r_n} \tag{3.1.1}$$

where $n \geqslant 1, x_i \in X, r_i \in \mathbb{Z}, r_i \neq 0$, and for no i is $x_i = x_{i+1}$, i.e. no cancelation in the expression 3.1.1 is possible. In this specification, work is needed to give the multiplication since adjoining two reduced words often yields a non reduced word, and the reduction process has to be given. [3]Accounts of this are in many books on combinatorial group theory, for example [Joh97, LS01, Coh89]. Reduced words are commonly used to store elements of a free group in computer implementations of combinatorial group theory.

We assume now that we have free groups, and this allows us to give our first definition of a presentation of a group Q.

Definition 3.1.2 A *presentation* $\mathcal{P} = \langle X \mid R \rangle$ of a group Q consists of a set X and a subset R of the free group $F(X)$ together with a surjective morphism $\phi : F(X) \to Q$ such that $\operatorname{Ker}\phi$ is the normal closure in $F(X)$ of the set R.

If there is any need, we shall write $Q = \operatorname{gp}\mathcal{P}$.

We explain in more detail the notion of normal closure, since this gives a useful model of an important general process, and we will use a more general form later for presentations of groupoids. First recall that for any normal subgroup $K \lhd P$, the group P acts on the group K by conjugation: we write k^p for $p^{-1}kp, k \in K, p \in P$. A basic aspect of group theory is that a normal subgroup is a kernel of a morphism (in this case, for example, of the quotient morphism $P \to P/K$), and that the kernel of any morphism from P to a group is normal in P.

If R is a subset of the group P then the *normal closure* $N^P(R)$ of R in P is the smallest normal subgroup of P containing R. We write conjugation of p by q as $p^q = q^{-1}pq$ for all $p, q \in P$. The elements of $N^P(R)$ are all *consequences* of R in P, namely all products

$$c = (r_1^{\varepsilon_1})^{p_1} \dots (r_m^{\varepsilon_m})^{p_m} \tag{3.1.2}$$

where $r_i \in R$, $\varepsilon_i = \pm 1$, $p_i \in P$ and $m \geqslant 1$. An important point is that if $\phi : P \to Q$ is any morphism to a group Q such that $\phi(R) = \{1\}$, then $\phi(N^P(R)) = \{1\}$, since $\operatorname{Ker}\phi$ is normal. Thus ϕ factors as $P \to P/N^P(R) \to Q$ where the first morphism is the quotient morphism.

Now we can see that there might be *identities among consequences*. Intuitively, such an identity is a 'formal' product such as 3.1.2 which is 1 when evaluated in the group P. A definition is given below. Here we consider some examples.

Example 3.1.3 For any elements r, s of R, we have the identities

$$r^{-1}s^{-1}rs^r = 1,$$
$$rs^{-1}r^{-1}s^{(r^{-1})} = 1.$$

These identities hold always, whatever R. □

Example 3.1.4 Suppose $r \in R, p \in P$ and $r = p^m, m \in \mathbb{Z}$. Then $rp = pr$, i.e. p belongs to the centraliser $C(r)$ of r in P. We have the identity

$$r^{-1}r^p = 1. \tag{3.1.3}$$

It is known that if the group P is free and $r \in R$ then there is a unique element p of P such that $r = p^m$ with $m \in \mathbb{N}$ maximal and then $C(r)$ is the infinite cyclic group generated by p. This element p is called the *root* of r and if $m > 1$ then r is called a *proper power*. □

Example 3.1.5 Suppose the commutators $[p, q] = p^{-1}q^{-1}pq, [q, r], [r, p]$ are among the elements of R. Then the well known rule

$$[p, q][p, r]^q\,[q, r][q, p]^r\,[r, p][r, q]^p = 1 \tag{3.1.4}$$

is an identity among the consequences of R, since $[q, p] = [p, q]^{-1}$. □

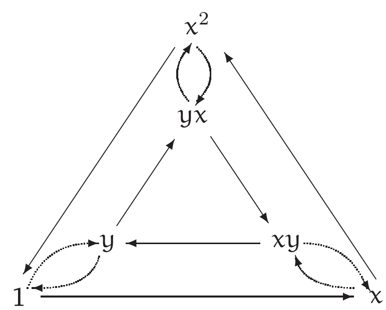
Example 3.1.6 Let S_3 be the symmetric group on three letters with presentation $\langle x, y \mid r, s, t\rangle$ where $r = x^3, s = y^2, t = xyxy$. The fact that each relation is a proper power gives rise to three identities among relations, namely

$$r^{-1}r^x, s^{-1}s^y, t^{-1}t^{xy}.$$

However there is also a fourth identity namely

$$(s^{-1})^{x^{-1}}ts^{-1}(r^{-1})^{y^{-1}}t^x(s^{-1})^x r^{-1}t^{x^{-1}}.$$

We leave it to you to verify that this is an identity among relations by writing out the formula in the free group on x, y. This identity can also be interpreted as a kind of composition of 2-cells in the following picture:



The Cayley graph of S_3

We shall discuss this a bit more in the next Section in terms of van Kampen diagrams. The general notion of 'composition of 2-cells' makes more sense with our discussion of computing identities among relations in Subsection 11.2.4 of Part II. □

Note that in all these examples conjugation is crucial. This is related to the fact that the kernel K of a morphism from a group P should be thought of not just as a subgroup K of the group P but also as a subgroup with an action of P on K. This principle, that a kernel in nonabelian situations has more structure than just that of subobject, is of general applicability. It is of direct applicability to the definition of an identity among the consequences of a subset R of the group P.

One extra formality is needed. We wish to allow for the consideration of repeated elements of R. One reason for this is that we may have some difficulty in recognising that two specified elements of P are in fact the same. In the context of presentations, we wish to allow for repeated relations. In the geometric context, we allow repeated attaching of cells by the same map (for example a constant map). Therefore we replace the subset R of P by a function $\omega : R \to P$ and define a *presentation* to be $\mathcal{P} = \langle X \mid \omega\rangle$. Nevertheless, we keep the notation $\langle X \mid R\rangle$ whenever $R \subseteq F(X)$ and ω is the inclusion.

Now in order to say that an identity among consequences is a *formal* product such as 3.1.2 which is 1 when evaluated in the group P, we need to define the free object in which such a 'formal product' should lie.

We adopt a more general notation and define the *free* P*-group on* R to be the free group on the set $R \times P$. We denote this P-group by H. The action of P on $R \times P$ is given by the product, i.e. by

$$(r, p)^q = (r, pq)$$

and this determines an action of P on the free group H. By another use of the universal property of a free group there is a morphism $\theta : H \to P$ defined on generators by

$$\theta(r, p) = p^{-1}\omega(r)p.$$

It is easy to see that the image of θ is the normal closure in P of $\omega(R)$. In symbols:

$$\theta(H) = N^P(\omega(R)).$$

It is clear also that the map θ preserves the action of P: that is, for any $h \in H, p \in P$

$$\theta(h^p) = p^{-1}(\theta h)p. \tag{3.1.5}$$

You will recognise this as the axiom CM1) for a crossed module; however H with θ does not necessarily satisfy axiom CM2).

The elements of H will be called *formal consequences* of $\omega : R \to P$ in P.

There is an alternative description of H which we give for those familiar with the group theoretic background.

Proposition 3.1.7 *The group H is isomorphic to the normal closure of R in the free product $P * F(R)$.*

Proof This is a simple consequence of the Kurosch subgroup Theorem for free products.[4] $\square$

Our first definition is that an *identity among the consequences* of $\langle X \mid \omega\rangle$ in P is an element of $E = \operatorname{Ker}\theta$. Equivalently, an identity among consequences is a formal consequence which gives 1 when evaluated as an actual consequence in P.

The idea of specifying an identity among consequences is thus very similar to that of specifying a relation as an element of the free group FX, but taking into account the action of FX. This leads to an appropriate concept of 'free'. However, we are not yet at our final position.

It is easy to see that certain identities are always present in E. We define the *basic Peiffer elements* to be the elements of E of the form

$$a^{-1}b^{-1}ab^{\theta(a)}$$

where $a, b \in R \times P$. Note that

$$(r', p')^{\theta(r,p)} = (r', p'p^{-1}(wr)p).$$

More generally, if $h, k \in H$ we will write

$$[\![h, k]\!] = h^{-1}k^{-1}hk^{\theta(h)}$$

and call such an element a *Peiffer element*. These should be thought of as 'twisted commutators'. [5]

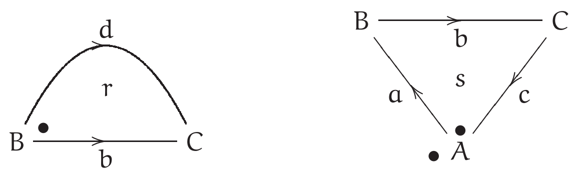
3.2 van Kampen diagrams

These diagrams give a geometric method of deducing consequences of relations, and can, as we shall see, be used to show exactly how to write a word as a consequence of the relations. We do not give a general definition or description, but illustrate it with examples. The idea has been used extensively in some sophisticated theorems in combinatorial group theory. For our purposes, the idea illustrates geometric aspects of the use of crossed modules.

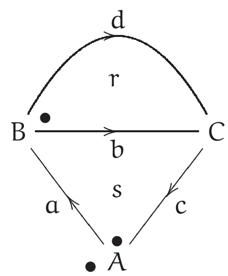
The idea of these diagrams come from the fact that a relation in a presentation can be represented by a based cell whose sides are labeled by the letters of the relation in such way that when they are read clockwise from the base point we get the relation.

Then, we can get new relations by gluing two or more of these cell along some common sides. Let us consider a simple case.

Example 3.2.1 Suppose for a given presentation we have the relations $r = bd^{-1}$ and $s = abc$. They can be represented as based cells as follows:



We write $\delta s = abc,\ \delta t = db^{-1}$. Now, we glue r and s alongside b getting



The boundary of this new cell is

$$adc = abc \cdot c^{-1}b^{-1} \cdot db^{-1} \cdot bc = (\delta s)(\delta(t^{bc})).$$

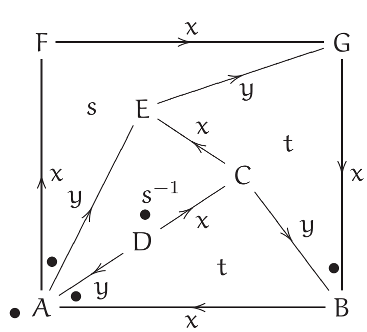
Of course t^{bc} makes sense in the context of crossed modules of groupoids, since t is based at B whereas t^{bc} is based at A. □

Here is a more complex example.

Example 3.2.2 The quaternion group of order 8 is usually presented in the form

$$Q_8 = \mathrm{gp}\,\langle x, y \mid x^4, x^2y^{-2}, y^{-1}xyx \rangle.$$

However the following diagram shows that the relation x^4 is a consequence of the other two relations. Set $r = x^4, s = x^2y^{-2}, t = y^{-1}xyx$ and consider the drawing



In this diagram, each cell has a base point, represented by a •, which is where the reading of the boundary starts in clockwise direction. This explains why we have an s and s^{-1}, since the latter is s read counterclockwise.

Now we have to show how we can deduce from this diagram the expression we want.

We take the outside loop starting from A (which has a base point for the outside 'cell') and then change it to traverse the boundary of each internal cell, obtaining the rule which you can easily verify:

$$xxy^{-1}y^{-1} \cdot yx^{-1}x^{-1}y \cdot y^{-1}xyx \cdot x^{-1} \cdot y^{-1}xyx \cdot x = x^4.$$

This can be reread as:

$$s \cdot yx^{-1}x^{-1}y \cdot t \cdot x^{-1} \cdot t \cdot x = x^4.$$

But $yx^{-1}x^{-1}y = yx^{-1}x^{-1} \cdot yyx^{-1}x^{-1} \cdot xxy^{-1} = (s^{-1})^{xxy^{-1}}$. So our final result is that

$$s \cdot (s^{-1})^{xxy^{-1}} \cdot t \cdot t^x \cdot r^{-1}$$

is an identity among relations, or, alternatively, shows in a precise way how x^4 is a consequence of the other relations. □

One context for van Kampen diagrams is clarified by the notion of *shelling* of such a diagram. This is a sequence of 2-dimensional subcomplexes $K_0, K_1, \ldots, K_n$ each of which is formed of 2-dimensional cells, with K_0 consisting of a chosen basepoint $*$, K_1 being a 2-cell s_1 with $*$ on its boundary, and such that for $i = 2, \ldots, n$, K_i is obtained from K_{i-1} by adding a 2-cell s_i such that $s_i \cap K_{i-1}$ is a non empty union of 1-cells which form a connected and 1-connected set, i.e. a path. Such a shelling will yield a formula for the boundary of K_n in terms of the boundaries of each individual cell, provided each cell is given a base point and orientation.

Here is a clear way of getting the formula (explained to us by Chris Wensley):

Choose $* = K_0$ as base point for all the K_i. The relation for K_0 is the trivial word. If B_1 is the base point for s_1 and P_1 is the anticlockwise path around s_1 from B_1 to $*$ and w_1 is the word in the generators read off along P_1, then the relation for K_1 is $\delta({s_1}^{w_1})$. For $i \geqslant 2$, let B_i be the base point for s_i, and let U_i, V_i be the first and last vertices in the intersection $s_i \cap K_{i-1}$ met when traversing the boundary of K_{i-1} in a clockwise direction (so that the intersection is a path $U_i \ldots V_i$). Then if B_i lies on $U_i \ldots V_i$ let P_i be the path $B_i \ldots U_i \ldots *$, otherwise let P_i be the path $B_i \ldots V_i \ldots U_i \ldots *$ (traversing the boundary of s_i in an anticlockwise direction and the boundary of K_{i-1} clockwise). If w_i is the word in the generators read off along P_i then

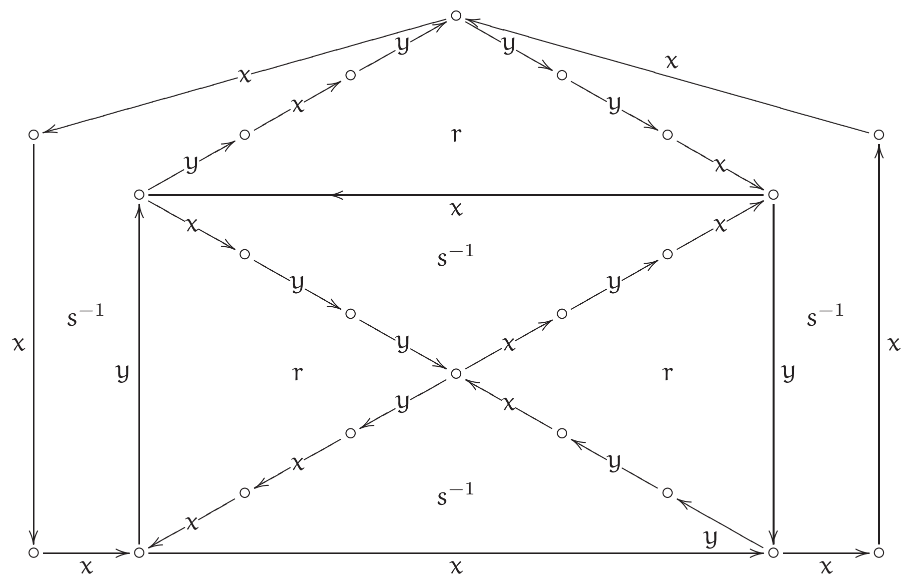
$$(\text{relation for } K_i) = (\text{relation for } K_{i-1}).\delta({s_i}^{w_i}).$$

We finish this short section by a more involved example

Example 3.2.3 Let us prove the non obvious fact that the relations

$$r = x^2yxy^3, \quad s = y^2xyx^3$$

have x^7 as a consequence using the next picture. We leave it as an exercise to check that base points and orientations can be assigned, and, harder, to give x^7 as a consequence of r, s. [6]



□

Here is a more formal definition of a van Kampen diagram.

A *complete Higher Homotopy van Kampen diagram* is a finite regular CW-structure K on a compact subset of the sphere S^2. *Regularity* here means that each attaching map $f_s : (S^1, 1) \to (K^1, K^0)$ of a 2-cell s is a homeomorphism into. By omitting one 2-cell s_∞ from K and using stereographic projection we can also regard $K \setminus s_\infty$ as a subset of the plane $\mathbb{R}^2$. The projection of $K \setminus s_\infty$ gives a planar van Kampen diagram.

Whitehead's Theorem (Corollary 5.4.8) says essentially that $\Pi(K, K^1, K^0)$ is the free crossed $\pi_1(K^1, K^0)$-module on the characteristic maps of the 2-cells of K.

3.3 Precrossed and crossed modules

Following the concepts introduced in the first section, it seems a good idea to study morphisms having the same formal properties as $\theta : H \to P$. One way of describing the distinctive feature of θ is to say that θ is a morphism of P-groups, where P acts on itself by conjugation.

So, let M and P be groups such that P acts on M on the right and let $\mu : M \to P$ be a homomorphism of groups. We say that $\mathcal{M} = (\mu : M \to P)$ is a *precrossed module* if it satisfies CM1) of section 2.2, that is:

CM1) $\mu m^p = p^{-1} \mu m p = (\mu m)^p$ for all $m \in M$ and $p \in P$,

i.e., μ is a morphism of P-groups when P acts on itself by conjugation.

A *morphism* between two precrossed modules $\mathcal{M} = (\mu : M \to P)$ and $\mathcal{N} = (\nu : N \to Q)$ is a pair (g, f) of homomorphisms of groups $g : M \to N$ and $f : P \to Q$ such that

i) the diagram

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\mu"'] & N \arrow[d, "\nu"] \\ P \arrow[r, "f"'] & Q \end{tikzcd}$$

commutes, i.e. $f\mu = \nu g$, and

ii) the actions are preserved, i.e. $g(m^p) = (gm)^{fp}$ for any $p \in P$ and $m \in M$.

The above objects and morphisms define the category PXMod/Groups of precrossed modules and morphisms.

Example 3.3.1 It is easy to see that if $\langle X \mid R\rangle$ is a presentation of a group, then using the notation of Section 3.1, $\theta : H \to F(X)$ is a precrossed module. □

Analogously to our method in this example, we can define Peiffer elements in any precrossed module. Let $\mathcal{M} = (\mu : M \to P)$ be a precrossed module and let m, m' be elements of M. Their *Peiffer commutator* is defined as

$$[\![m, m']\!] = m^{-1}m'^{-1}mm'^{\mu m}.$$

The precrossed modules in which all Peiffer commutators are trivial are precisely the crossed modules. Thus the category of crossed modules is the full subcategory of the category of precrossed modules whose objects are crossed modules.

Since the Peiffer elements are always defined in a precrossed module, it is a natural idea to factor out by the normal subgroup that they generate and consider the induced map from the quotient. Let us check that this produces a crossed module.

The *Peiffer subgroup* $[\![M, M]\!]$ of M is the subgroup of M generated by all Peiffer commutators. We now prove that this subgroup inherits the P-action and is a normal subgroup.

Theorem 3.3.2 *For any precrossed module* $\mu : M \to P$*, the Peiffer subgroup* $[\![M, M]\!]$ *of* M *is a* P*-invariant normal subgroup.*

Proof The Peiffer subgroup is P-invariant since for any $m, m' \in M$ and $p \in P$, we have

$$\begin{aligned} [\![m, m']\!]^p &= (m^{-1}m'^{-1}mm'^{\mu m})^p \\ &= (m^p)^{-1}(m'^p)^{-1}m^p m'^{(\mu m)p} \\ &= (m^p)^{-1}(m'^p)^{-1}m^p m'^{p(\mu m)^p} \\ &= (m^p)^{-1}(m'^p)^{-1}m^p m'^{p(\mu m^p)} \\ &= [\![m^p, m'^p]\!]. \end{aligned}$$

It is also normal since for any $m, m', n \in M$ we have

$$\begin{aligned} n^{-1}[\![m, m']\!]n &= n^{-1}m^{-1}m'^{-1}mm'^{\mu m}n \\ &= n^{-1}m^{-1}m'^{-1}m(nm'^{\mu mn}(m'^{-1})^{\mu mn}n^{-1})m'^{\mu m}n \\ &= ((mn)^{-1}m'^{-1}mnm'^{\mu mn})(((m'^{\mu m})^{\mu n})^{-1}n^{-1}m'^{\mu m}n) \\ &= [\![mn, m']\!][\![n, m'^{\mu m}]\!]^{-1}. \end{aligned}$$

□

Now for any precrossed module $\mu : M \to P$ we define

$$M^{cr} = M/[\![M, M]\!].$$

By the previous property, M^{cr} is a P-group. Let us see that the homomorphism μ induces a crossed module.

Proposition 3.3.3 *For any precrossed module $\mu : M \to P$, the induced map gives a crossed module*

$$\mathcal{M}^{cr} = (\mu^{cr} : M^{cr} \to P)$$

which we call the crossed module associated to μ.

Proof It is easy to see that for each $m, m' \in M$, $\mu[\![m, m']\!] = 1$, so μ induces a homomorphism of groups μ^{cr}.

Clearly μ^{cr} satisfies CM1) because it was already satisfied by μ. It also satisfies CM2) because all Peiffer commutators have been quotiented out. □

The association of the crossed module $M^{cr} \to P$ to a precrossed module $M \to P$ gives a functor

$$(-)^{cr} : \mathsf{PXMod/Groups} \to \mathsf{XMod/Groups}.$$

That is, a morphism (g, f) of precrossed modules yields a morphism (g^{cr}, f) of crossed modules, and this association satisfies the usual functorial rules.

Moreover let us prove that $(-)^{cr}$ is a left adjoint of the inclusion $\mathsf{XMod/Groups} \hookrightarrow \mathsf{PXMod/Groups}$ by checking the appropriate universal property.

Proposition 3.3.4 *Let $\mathcal{M} = (\mu : M \to P)$ be a precrossed module. For any crossed module $\mathcal{N} = (\nu : N \to Q)$ and any morphism of precrossed modules $(g, f) : \mathcal{M} \to \mathcal{N}$ there is a unique morphism of crossed modules*

$$(g^{cr}, f) : (\mu^{cr} : M^{cr} \to P) \longrightarrow (\nu' : N \to Q)$$

such that $g = g^{cr} \circ \theta$ where θ is the quotient homomorphism $\theta : M \to M^{cr}$.

Proof Obviously, g^{cr} can only be the homomorphisms induced by g on the quotient, and this is well defined since $g[\![m, m']\!] = 1$ for any elements m, m' of M. □

For future computations it is interesting to have a set of generators of the Peiffer subgroup as small as possible. The following property is useful for this.

Proposition 3.3.5 *Let $\mu : M \to P$ be a precrossed module and let V be a subset of M which generates M as a group and is also P-invariant. Then the Peiffer subgroup $[\![M, M]\!]$ of M is the normal closure in M of the set of Peiffer commutators*

$$\{[\![a, b]\!] \mid a, b \in V\}.$$

Proof Let Z be the normal closure of $W = \{[\![a, b]\!] \mid a, b \in V\}$. Since $[\![M, M]\!]$ is normal and contains W, it is clear that $Z \subseteq [\![M, M]\!] \subseteq \operatorname{Ker} \mu$. On the other hand W is P-invariant since $[\![a, b]\!]^p = [\![a^p, b^p]\!]$ as was proved in Theorem 3.3.2. So Z is also P-invariant. Thus μ induces a homomorphism of groups $\overline{\mu} : M/Z \to P$ which is P-invariant, so that we have a precrossed module. Let us check that it is also a crossed module.

Let $\overline{V}$ be the image of V in M/Z, i.e. $\overline{V}$ is the set of cosets of all elements in V. Notice that we have

$$y^{\overline{\mu} x} = x^{-1} y x \tag{$**$}$$

for any x and y lying in $\overline{V}$, which is a set of generators of M/Z. It is easy to see that for a fixed x in M/Z the set P_x of y's satisfying this equation (**) is a subgroup containing $\overline{V}$ so P_x has to be all of M/Z.

Consider now the set Q_x of x in M/Z satisfying (**) for all y in M/Z. It is closed under multiplication (since

$$y^{xx'} = (y^x)^{x'} = (x^{-1}yx)^{x'} = (x^{-1})^{x'}y^{x'}x^{x'} = x'^{-1}x^{-1}x'x'^{-1}yx'x'^{-1}xx' = x'^{-1}x^{-1}yxx')$$

and also under inversion (since if $w = y^{x^{-1}}$, we have $w^x = y$ and $w^x = x^{-1}wx$, so that $x^{-1}wx = y$ and $w = xyx^{-1}$). So $Q_x = M/Z$ and thus $\overline{\mu} : M/Z \to P$ is a crossed module. It follows that $[\![M, M]\!] \subseteq Z$. $\square$

Corollary 3.3.6 *Let* $\omega : R \to P$ *be a function to the group* P *and let* $\theta : H \to P$ *be the associated precrossed module. Then the Peiffer subgroup* $[\![H, H]\!]$ *of* H *is the normal closure in* H *of the basic Peiffer elements* $[\![a, b]\!] = a^{-1}b^{-1}ab^{\theta a}$ *where* $a, b \in R \times P$.

3.4 Free precrossed and crossed modules

Another crucial property satisfied by the precrossed module associated to a presentation of a group is that it is, in some sense, free. We need to make this property explicit.

As explained in the Appendix, a free construction in a category is usually the left adjoint of some forgetful functor. The appropriate forgetful functor in this case goes from the category of crossed modules to the category of *sets over a group* forgetting the algebra of the top group and considering only the underlying boundary map. We recall the appropriate categories.

Let P be a group. We have defined the category XMod/P of crossed P-modules in Section 2.2. In a similar way, we define the category PXMod/P by restricting to precrossed modules over P.

Let P be a set. We define Sets/P to be the category whose objects are P-sets, i.e. maps $\mathcal{S} = (\nu : S \to P)$, and whose morphisms are P-maps, Sets/P i.e. maps $\alpha : S \to S'$ making commutative the diagram

$$\begin{tikzcd} S \arrow[rr, "\alpha"] \arrow[dr, "\nu"'] & & S' \arrow[dl, "\nu'"] \\ & P & \end{tikzcd} .$$

We have a forgetful functor

$$U : \mathsf{XMod}/P \to \mathsf{Sets}/P.$$

Thus, the *free crossed module* construction is a functor

$$F : \mathsf{Sets}/P \to \mathsf{XMod}/P$$

such that for any P-set $\mathcal{S} = (\nu : S \to P)$ and for any crossed P-module $\mathcal{M} = (\mu : M \to P)$ there is a natural bijection

$$(\mathsf{Sets}/P)(\mathcal{S}, U\mathcal{M}) \cong (\mathsf{XMod}/P)((F\mathcal{S}, \mathcal{M}),$$

i.e. there is a P-inclusion $i : S \to FS$, corresponding to the morphism Id_{FS} of crossed P-modules such that for any P-map $f : S \to M$ there exist a unique extension to a morphism $f' : FS \to M$ of crossed P-modules.

In the same way we may define the *free precrossed module* using the forgetful functor between PXMod/P and Sets/P.

To determine when a crossed module is *free*, let $\mathcal{M} = (\mu : M \to P)$ be a crossed module, R a set and $\omega : R \to M$ an injective map (equivalently, let $\{m_r \mid r \in R\}$ be an indexed family of elements of M). We say that $\mathcal{M}$ is the *free crossed* P*-module* on ω (also, that ω is a *basis* of M) if the unique morphism of crossed modules $F\omega : FR \to M$ extending ω is an isomorphism, or, equivalently, if it satisfies the following universal property: for any crossed module $\mathcal{M}' = (\mu' : M' \to P)$ and map $\omega' : R \to M'$ such that $\mu\omega = \mu'\omega'$ there exists a unique morphism $h : \mathcal{M} \to \mathcal{M}'$ of P-crossed modules such that $h\omega = \omega'$.

There is a similar definition of the free precrossed module on ω. As always in universal constructions, the free crossed and precrossed P-modules on ω are unique up to isomorphism. We study now their existence.

We consider a group P and a function $\omega : R \to P$, or equivalently, an indexed family $\{p_r \mid r \in R\}$ of elements of P. First we create the *free* P*-group with basis* R. To this end, we define $E = F(R \times P)$. It is the free group on the formal elements $\{r^p \mid r \in R, p \in P\}$. We think of (r, p) as r^p. Then, any element of E can be seen as a formal product

$$(r_1^{p_1})^{\varepsilon_1} \cdots (r_n^{p_n})^{\varepsilon_n}$$

with $n \in \mathbb{N}$, $\varepsilon_i = \pm 1$, $p_i \in P$ and $r_i \in R$.

This representation makes clear the definition of the P-action on generators, since to be an action it has to satisfy $(r^p)^{p'} = r^{pp'}$. Thus, we define a P-action on E by

$$(r, p)^{p'} = (r, pp')$$

on generators and we extend it in the only possible way.

We define a map $\theta : E \to P$ by the only possible definition to make it a P-map, i.e. $\theta(r, p) = p^{-1}\omega(r)p$ on generators.

Let us check that the map θ just constructed gives the free precrossed module.

Proposition 3.4.1 $\mathcal{E} = (\theta : E \to P)$ *is the free precrossed module on* $\omega : R \to E$ *where* $\omega(r) = (r, 1)$.

Proof It is clear that P acts on E and also that θ is a homomorphism by the way they are defined.

It is easy to check that $\theta : E \to P$ is a precrossed module,

$$\theta(r, p)^{p'} = \theta(r, pp') = p'^{-1}p^{-1}\omega(r)pp' = p'^{-1}\theta(r, p)p'.$$

To prove the universal property, consider $\mathcal{M}' = (\mu' : M' \to P)$ a precrossed P-module and a map $\omega' : R \to M'$. We can define the map

$$\begin{array}{rcl} R \times P & \to & M' \\ (r, p) & \mapsto & (\omega' r)^p \end{array}$$

that extends to a unique homomorphism $h : E \to M'$ that is a morphism of precrossed modules since

i) $\mu' h(r, p) = \mu'(\omega' r)^p = p^{-1}(\mu'\omega' r)p = \theta(r, p)$ and

ii) $h(r, p)^{p'} = h(r, pp') = (\omega' r)^{pp'} = ((\omega' r)^p)^{p'} = h(r, p)^{p'}$.

Actually, this is the only possible definition of h to make it a map of P-groups. So h is unique. □

Corollary 3.4.2 *The crossed* P*-module* $\mathcal{E}^{cr} = (\theta^{cr} : E^{cr} \to P)$ *is the free crossed module on* $\omega : R \to E^{cr}$.

Proof Obviously $\mathcal{E}^{cr}$ is a crossed module. Let us check the universal property.

For any crossed P-module $\mathcal{M}' = (\mu' : M' \to P)$ and any map $\omega' : R \to M'$ there is a unique morphism of precrossed modules $\alpha : E \to M'$ satisfying $\omega'\alpha = \omega$. Thus, the induced map $\alpha^{cr} : E^{cr} \to M'$ is the only morphism of crossed modules satisfying $\omega'\alpha = \omega$. □

Exercise 3.4.3 Use the notion of normal subgroupoid and quotient groupoid in [Bro06, Section 8.3] to generalise the above work on free crossed modules to the case where P is a groupoid rather than just a group. Free crossed modules over groupoids are studied later in Section 7.2.4. □

Remark 3.4.4 For any crossed module $\mathcal{M} = (\mu : M \to P)$ such that $\mu(M)$ is a free group, there is a section $s : \mu M \to M$ which is a homomorphism of groups. Then, the Proposition 2.2.4 applies.

So if $\mu : M \to P$ is the free crossed P-module associated to a presentation $(\omega : R \to P)$ of a group G then there is a short exact sequence of G-modules

$$0 \to \pi = \operatorname{Ker}\mu \longrightarrow M^{ab} \xrightarrow{\mu^{ab}} (\mu M)^{ab} \to 0.$$

□

From the construction of the free precrossed module as a free group, it is clear that $\omega : R \to E$ is injective. It is not so clear that $\omega : R \to E^{cr}$ is also injective. This is a consequence of the following property:

Proposition 3.4.5 *Given a free crossed* P*-module* $\mathcal{M} = (\mu : M \to P)$ *on* $\omega : R \to M$*, with* G *the cokernel of* μ*, then* M^{ab} *is a free* G*-module with basis* $\omega^{ab} : R \to M^{ab}$.

Proof We know by Proposition 2.2.3 ii) that M^{ab} is a G-module. To see that M^{ab} is free we will prove that it satisfies the universal property of a free G-module.

Let M' be a G-module. The projection $P \times M' \to P$ becomes a crossed P-module when P acts on $P \times M'$ by conjugation on P and the G action on M'. For any map $\nu : R \to M'$ we define $\nu' = (\mu\omega, \nu) : R \to P \times M'$. Since $\mu : M \to P$ is a free crossed P-module we get a unique morphism of P-crossed modules $\phi : M \to P \times M'$ such that $\nu' = \phi\omega$. The composite $M \to M'$ factors through a G-morphism $\overline{\phi} : M^{ab} \to M'$ which is the only morphism of G-modules satisfying $\overline{\phi}\omega^{ab} = \nu$. □

We now give an example and proposition [7] which illustrate some of the difficulties of working with free crossed modules.

Example 3.4.6 Let $(\partial : C(R) \to F(X))$ be the free crossed module on the subset R of $F(X)$ and suppose that Y is a subset of X, and S a subset of R. Let M be the subgroup of $C(R)$ generated by $F(Y)$ operating on S, and assume that $\partial(M) \subseteq F(Y)$. Let $\mathcal{M}' = (\partial' : M \to F(Y))$ be the crossed module given by restricting ∂ to M. Then $\mathcal{M}'$ is not necessarily a free crossed module.

Let $X = Y = \{x\}, R = \{a, b\}, S = \{b\}$ be such that $\partial a = x, \partial b = 1$. Since $\partial b = 1$, we have $ab = ba$, whence $b^x b^{-1} = a^{-1}bab^{-1} = 1$. Therefore $\mathcal{M}'$ is not a free crossed module. □

Proposition 3.4.7 *Let* G, G' *be the cokernels of* ∂, ∂' *respectively, and let* $\eta : G \to G'$ *be the morphism induced by the inclusion* $i : F(Y) \to F(X)$*. If* η *is injective, then* $\mathcal{M}'$ *is the free crossed* $F(Y)$*-module on* S.

Proof Let $d : C(S) \to F(Y)$ be the free crossed $F(Y)$-module on S. It is clear that $d(C(S)) = \partial(M)$. Let $j : C(S) \to M$ be the morphism of crossed $F(Y)$-modules. Clearly j is surjective, and the result is proved when we have shown that j is injective.

Suppose that $u \in C(S)$ and $j(u) = 1$. Then $d(u) = 1$. Let $k : C(S)^{ab} \to C(R)^{ab}$ be the induced morphism of the abelianised groups. These abelianised groups are in fact free modules over G, G' respectively on the bases S, R respectively. Since η is injective, it follows that k is injective. Let $\overline{u}$ denote the class of u in $C(S)^{ab}$. Then $k\overline{u} = 0$, and hence $\overline{u} = 0$. But the morphism $C(S) \to C(S)^{ab}$ is injective on $\operatorname{Ker} d$. It follows that $u = 1$. □

3.5 Precat^1-groups and the existence of colimits

In the two previous section we have seen that when working with crossed modules it is sometimes convenient to consider the weaker structure of precrossed modules and see the category $\mathsf{XMod/Groups}$ as a full subcategory of $\mathsf{PXMod/Groups}$.

In Section 2.5 we have seen that the category $\mathsf{Cat}^1\!-\mathsf{Groups}$ of cat^1-groups is equivalent to the category $\mathsf{XMod/Groups}$. It is an easy exercise to put both together and construct a category bigger than and equivalent to $\mathsf{PXMod/Groups}$.

So, a *precat*^{1}*-group* is a triple (G, s, t) where G is a group and $s, t : G \to G$ are endomorphisms satisfying $st = t$ and $ts = s$. Thus we are omitting CG2) from the axioms of a cat^1-group, i.e. we do not impose commutativity between elements of $\operatorname{Ker} s$ and $\operatorname{Ker} t$.

As before, a *morphism* between pre-cat^1-groups is just a homomorphism of groups commuting with the *s*'s and *t*'s. These objects and morphisms define the category $\mathsf{PCat}^1\!-\mathsf{Groups}$. It contains $\mathsf{Cat}^1\!-\mathsf{Groups}$ as a full subcategory.

Proposition 3.5.1 *The categories* $\mathsf{PCat}^1\!-\mathsf{Groups}$ *and* $\mathsf{PXMod/Groups}$ *are equivalent, by an equivalence extending that between* $\mathsf{Cat}^1\!-\mathsf{Groups}$ *and* $\mathsf{XMod/Groups}$.

Proof The definitions of both functors are the same as in Section 2.5, namely

$$\lambda : \mathsf{PXMod/Groups} \to \mathsf{PCat}^1\!-\mathsf{Groups}$$

is given by $\lambda(\mu : M \to P) = (P \ltimes M, s, t)$, s and t being defined as before, and

$$\gamma : \mathsf{PCat}^1\!-\mathsf{Groups} \to \mathsf{PXMod/Groups},$$

is defined by $\gamma(G, s, t) = (t| : \operatorname{Ker} s \to \operatorname{Im} s)$.

It is easily checked that both functors are well defined and both compositions are naturally equivalent to the identity. □

As in the Section 3.3, we may define a functor associating to each pre-cat^1-group a cat^1-group

$$(-)^{cat} : \mathsf{PCat}^1\!-\mathsf{Groups} \to \mathsf{Cat}^1\!-\mathsf{Groups}$$

defined by $(G, s, t)^{cat} = (G/N, s', t')$, where $N = [\text{Ker } s, \text{Ker } t]$.

It is easy to see that the functor $(-)^{cat}$ corresponds through the equivalences of categories to

$$(-)^{cr} : \mathsf{PXMod/Groups} \to \mathsf{XMod/Groups}.$$

Then, it follows

Proposition 3.5.2 *The functor* $(-)^{cat}$ *is a left adjoint of the inclusion.*

Using this last property we can prove the existence of colimits in $\mathsf{Cat}^1\!-\mathsf{Groups}$.

Since left adjoint functors preserve colimits (see [ML71] or Appendix A), for any indexed family $\mathcal{G}_\lambda = (G_\lambda, s_\lambda, t_\lambda)$ of cat^1-groups and morphisms between them, we have

$$\mathrm{colim}_{cat}\{\mathcal{G}_\lambda\} = (\mathrm{colim}_{pre}\{\mathcal{G}_\lambda\})^{cat}.$$

So, the existence of colimits in $\mathsf{Cat}^1\!-\mathsf{Groups}$ has been reduced to the existence of colimits in $\mathsf{PCat}^1\!-\mathsf{Groups}$.

It is not difficult now to check that in $\mathsf{PCat}^1\!-\mathsf{Groups}$ the colimits are as expected, i.e. for an indexed family $\{\mathcal{G}_\lambda \mid \lambda \in \Lambda\}$ of pre-cat^1-groups $\mathcal{G}_\lambda = (G_\lambda, s_\lambda, t_\lambda)$ and morphisms between them,

$$\mathrm{colim}_{pre}\{\mathcal{G}_\lambda\} = (\mathrm{colim}_{gr}\{G_\lambda\}, \mathrm{colim}_{gr}\{s_\lambda\}, \mathrm{colim}_{gr}\{t_\lambda\}).$$

From the existence of colimits in $\mathsf{Cat}^1 - \mathsf{Groups}$ follows the existence of colimits in $\mathsf{XMod}/\mathsf{Groups}$ using the equivalence between both categories.

Remark 3.5.3 We have just added another way of computing colimits of crossed modules. So, if we have an indexed family of crossed modules $\{\mu_\lambda : M_\lambda \to P_\lambda\}$, we construct the associated family of cat^1-groups $\{(M_\lambda \ltimes P_\lambda, s_\lambda, t_\lambda)\}$ getting their colimit (G, s, t) and the colimit crossed module is $t| : \mathrm{Ker}\, s \to \mathrm{Im}\, t$.

Even if it seems a long way around, it is worthwhile because for example $M_\lambda \ltimes P_\lambda$ may be finitely generated, even if M_λ and P_λ are not. Also, there are some efficient computer-assisted ways of getting colimits, kernels and images of finitely generated groups and homomorphisms. $\square$

3.6 Implementation of crossed modules in GAP

Nowadays is almost impossible to make any serious computational work in group theory without use of a computational group theory package. Some of these packages have evolved to accommodate more structures becoming veritable computational discrete algebra packages. The one we have been using along the book is GAP (see [Gro02] for more information). The package GAP has been developed primarily for combinatorial group theory, and has the significant advantage of free availability of the library code, thus enabling the user to modify a function so as to return additional information.

Work at Bangor (in particular by M.Alp and C.D. Wensley) has produced the GAP module XMOD which includes a number of constructions on crossed modules, cat^1-groups and their morphisms. In particular: derivations, kernels and images; the Whitehead group; cat^1-groups and their relation with crossed modules; induced crossed modules.

This package has already been in use for some time, and has been incorporated into GAP4. [8]

In Section 5.9 we will show how XMOD has been used to determine explicitly some induced crossed modules whose computation do not follow from general theorems and seem too hard to compute by hand.

Notes

[2]p. 51 This finiteness no longer holds for groups. A group may be finitely generated, but not finitely presented; finitely presented but not finitely identified; and so on. For more information see Wikipedia on finitely presented groups.

[3]p. 55

[4]p. 57 This theorem may be found in [Bro06, Hig71] and books on combinatorial group theory.

[5]p. 57In this spirit, there is a 'Peiffer commutator calculus' whose study has been advanced considerably by Baues and Conduché [BC90].

[6]p. 60 These examples are from David Johnson's book [Joh97]. Other examples on van Kampen diagrams are in that book, and may also be found by a web search. The geometric and metric analysis of van Kampen diagrams has proved important in aspects of combinatorial group theory.

[7]p. 65 These are due to Whitehead in [Whi50b].

[8]p. 67 We note that Alp and Wensley have in [AW00] used this programme to list many finite cat^1-groups.

General notes

The exposition of identities among relations follows to some extent that in [BH82].

Proposition 3.3.5 is taken from Brown-Huebschmann [BH82].

Chapter 4

Coproducts of crossed P-modules

In this chapter we start to show how the 2-dimensional van Kampen theorem and the algebra of crossed modules allows specific nonabelian calculations in homotopy theory in dimension 2. To this end, we study the coproduct of crossed modules (mainly of two crossed modules) over the same group P. We construct the coproduct of crossed P-modules, check some properties and, using the 2-dimensional van Kampen Theorem, we apply these general results to some topological cases.

In the first section (4.1) we construct the coproduct of crossed P-modules. First, we see what the definition of coproduct in a general category means in this case, and then we prove its existence by a two step procedure. As a first step, we prove that the free product of groups gives the coproduct in the category of precrossed P-modules. Then, using the fact that the functor $(-)^{cr}$ preserves coproducts, we see that its associated crossed P-module is the coproduct in the category of P-modules.

This procedure is a bit complicated to implement because the free product is always a very big group (it is normally infinite even if all groups are finite). So in Section 4.2 we give an alternative description of the coproduct of two crossed P-modules. This is obtained by dividing the construction of the associated crossed module in this case into two steps, of which the first gives a semidirect product. Thus the coproduct of two crossed P-modules is a quotient of a semidirect product. Hence we can get presentations of the coproduct using the known presentations of the semidirect product.

This has some topological bearings as explained in Section 4.3. First, we know that the coproduct of two crossed P-modules is just the pushout of these two crossed modules with respect to the trivial crossed module $1 \to P$. Thus in the case that we have a topological space X that is the union of two open subsets U_1, U_2 such that both (U_i, U_{12}) are 1-connected, the fundamental crossed module $\Pi_2(X, U_{12})$ is the coproduct $\Pi_2(U_1, U_{12}) \circ \Pi_2(U_2, U_{12})$ (Theorem 4.3.2) and we can use the previous results to get information on the second homotopy group of some spaces. We end this section by studying some consequences in this case.

In the last section (4.4) we study the coproduct in a particular case that we shall use later. We begin with two crossed P-modules $\mathcal{M} = (\mu : M \to P)$ and $\mathcal{N} = (\nu : N \to P)$ satisfying the condition

$$(*) \ : \ \nu(N) \subseteq \mu(M) \text{ and there is an equivariant section of } \mu.$$

In this case, we get a description of their coproduct using the displacement subgroup N_M (Theorem 4.4.8). This case is not uncommon and we get some topological applications when the space X is got from Y by attaching a cone CA, that is, X is a mapping cone. We finish this last section with a description of the coproduct for an arbitrary set of indexes satisfying the above condition (*). This result will be used at the end of the next Chapter (see Section 5.8).

4.1 The coproduct of crossed P-modules

We give a construction of coproducts in the category XMod/P of crossed modules over the group P. We do this for a general family of indices since this causes no more difficulty than the case of two crossed modules.

From the general definition of the coproduct in a category given in the Appendix, we see that the coproduct of a family $\{\mathcal{M}_t \mid t \in T\}$ of crossed modules over P is given by a crossed module $\mathcal{M}$ and a family of morphisms of crossed P-modules $\{i_t : \mathcal{M}_t \to \mathcal{M} \mid t \in T\}$ satisfying the following universal property: for any family $\{u_t : \mathcal{M}_t \to \mathcal{M}' \mid t \in T\}$ of morphisms of crossed modules over P, there is a unique morphism $u : \mathcal{M} \to \mathcal{M}'$ of crossed modules over P such that $u_t = ui_t$ for each $t \in T$. Diagrammatically, there exists a unique dashed arrow such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{M}_t \arrow[r, "i_t"] \arrow[dr, "u_t"'] & \mathcal{M} \arrow[d, dashed, "u"] \\ & \mathcal{M}'. \end{tikzcd}$$

As with any universal construction, the coproduct is unique up to isomorphism.

As we have seen in Section 3.3, the functor $(-)^{\mathrm{cr}}$ from precrossed modules to crossed modules, obtained by factoring out the Peiffer subgroup, is left adjoint to the inclusion of crossed modules into precrossed modules, and so takes coproducts into coproducts. Thus to construct the coproduct of crossed P-modules we construct the coproduct in PXMod/P, the category of precrossed modules over the group P and apply the functor crs to it. The coproduct in PXMod/P is simply obtained using the coproduct in the category Groups of groups, and this is the well known free product $*_t G_t$ of a family $\{G_t\}$ of groups [LS01].

Proposition 4.1.1 *Let T be an indexing set and, for each $t \in T$ let $\mathcal{M}_t = (\mu_t : M_t \to P)$ be a precrossed P-module. We define $*_t M_t$ to be the free product of the groups M_t, $t \in T$. There is an action of P on $*_t M_t$ defined by the action of P on each M_t. Consider the morphism*

$$*_t\mathcal{M}_t = (\partial' : *_t M_t \to P),$$

together with the homomorphisms $i_t : M_t \to *_t M_t$ given by the inclusion in the free product, and where $\partial' = *_t\mu_t$ is the homomorphism of groups induced from the homomorphisms μ_t using the universal property of the coproduct of groups. Then the above defined $*_t\mathcal{M}_t$ is a precrossed P-module and the homomorphisms i_t are morphisms of precrossed modules over P giving the coproduct in the category PXMod/P.

Proof Let $\mathcal{M} = *_t\mathcal{M}_t$. If we represent by $p_\#$ the action by $p \in P$, then the action $p_\# : \mathcal{M} \to \mathcal{M}$ of p is defined by the composite morphisms $\mathcal{M}_t \xrightarrow{p\#} \mathcal{M}_t \xrightarrow{i_t} \mathcal{M}$.

In terms of the normal form of an element of the free product, this means that the action is given by the formula

$$(m_{t_1} \dots m_{t_n})^p = (m_{t_1})^p \dots (m_{t_n})^p, \ m_{t_i} \in M_{t_i}.$$

As already pointed out, the homomorphisms μ_t extend uniquely to a homomorphism $*_t\mu_t$. So

$$\begin{aligned}(*_t\mu_t)((m_{t_1} \dots m_{t_n})^p) &= (*_t\mu_t)(m^p_{t_1} \dots m^p_{t_n}) \\ &= (\mu_{t_1}(m^p_{t_1})) \dots (\mu_{t_n}(m^p_{t_n})) \\ &= p^{-1}(\mu_{t_1} m_{t_1})p \dots p^{-1}(\mu_{t_n}(m_{t_n}))p \\ &= p^{-1}((\mu_{t_1} m_{t_1}) \dots (\mu_{t_n}(m_{t_n}))p\end{aligned}$$

and $*_t\mu_t$ is a precrossed module.

The verification of the universal property is easy. □

We now easily obtain:

Corollary 4.1.2 *If* $\mathcal{M}_t = (\mu_t : M_t \to P), t \in T$ *is a family of crossed* P*-modules, then applying the functor* $(-)^{cr}$ *to* $*_t\mathcal{M}_t$ *to give*

$$\partial'^{cr} : (*_t\mathcal{M}_t)^{cr} \to P$$

with the morphisms $j_t : M_t \xrightarrow{i_t} *_t\mathcal{M}_t \to (*_t\mathcal{M}_t)^{cr}$*, where the second morphism is the quotient homomorphism, gives the coproduct of crossed* P*-modules.*

We denote this coproduct by

$$\bigcirc_t\mathcal{M}_t = (\partial : \bigcirc_t M_t \to P)$$

where the morphisms $j_t : \mathcal{M}_t \to \bigcirc_t\mathcal{M}_t$ are understood to be part of the structure. These morphisms need not be injective. In the case $T = \{1, 2, \dots, n\}$, this coproduct will be written $M_1 \circ \cdots \circ M_n \to P$. As is standard for coproducts in any category, the coproduct in XMod/P is associative and commutative up to natural isomorphisms.

4.2 The coproduct of two crossed P-modules

Throughout this section we suppose given two crossed P-modules $\mathcal{M} = (\mu : M \to P)$ and $\mathcal{N} = (\nu : N \to P)$, and we develop at some length the study of their coproduct in XMod/P

$$\mathcal{M} \circ \mathcal{N} = (\mu \circ \nu : M \circ N \to P)$$

and the canonical morphisms from M, N into $M \circ N$. This is the case that has been analysed more deeply in the literature. Most of the results of this section were in print for the first time in a paper by Brown ([Bro84]). Further results were obtained in [GH89], and some more applications and results are also given in [HAM93]. However this construction as a quotient of the free product really goes back to Whitehead [Whi49b].

The basic observation in [Bro84] is that $M \circ N$ may be obtained as a quotient of the semidirect product group $M \ltimes N$ where M operates on N *via* P. This result makes the coproduct of two crossed modules computable and from this we get some topological computations.

For convenience, we assume M, N are disjoint. To study $M \circ N = (M * N)^{cr}$ in some detail we should have a closer look at $[\![M * N, M * N]\!]$, the Peiffer subgroup of $M * N$. As seen in Section 3.3, $[\![M * N, M * N]\!]$ is the subgroup of $M * N$ generated by all Peiffer commutators

$$[\![k, k']\!] = k^{-1}k'^{-1}kk'^{(\mu*\nu)k}$$

for all $k, k' \in M * N$.

Notice that by Proposition 3.3.5, $[\![M * N, M * N]\!]$ is also the normal subgroup generated by the Peiffer commutators of any given P-invariant set of generators. Now $M \cup N$ generates $M * N$ and is P-invariant. Since $\mathcal{M}$ and $\mathcal{N}$ are crossed modules, redwe have $[\![m, m']\!] = 1$ and $[\![n, n']\!] = 1$, for all $m, m' \in M$ and $n, n' \in N$. Thus $[\![M * N, M * N]\!]$ is the normal subgroup of $M * N$ generated by the elements

$$r(m, n) = n^{-1}m^{-1}nm^n, \text{ and } s(m, n) = m^{-1}n^{-1}mn^m$$

for all $m \in M$, $n \in N$.

It is useful to divide the process of quotienting out by the Peiffer subgroup into two steps. First, we consider the quotient of $M * N$ by the group U generated by $\{s(m, n) \mid m \in M, n \in N\}$ all Peiffer commutators of the second kind. A useful observation already developed in [Bro84] is that this quotient is the well known semidirect product.

Proposition 4.2.1 *The precrossed* P*-module*

$$\frac{\mathcal{M} * \mathcal{N}}{U} = (\mu * \nu : (M * N)/U \to P)$$

where U *is the normal* P*-invariant subgroup generated by the set* $\{m^{-1}n^{-1}mn^m \mid m \in M, n \in N\}$ *is isomorphic to*

$$\mathcal{M} \ltimes \mathcal{N} = (\mu \ltimes \nu : M \ltimes N \to P)$$

where the semidirect product is associated to the action of M *on* N *via* μ *and the* P*-action.*

Proof The inclusions $M \to M \ltimes N$ and $N \to M \ltimes N$ extend to a homomorphism of groups

$$\varphi : M * N \to M \ltimes N.$$

Let us check that $\varphi(U) = 1$ by computing φ on all generators,

$$\begin{aligned}
\varphi(m^{-1}n^{-1}mn^m) &= (m^{-1}, 1)(1, n^{-1})(m, 1)(1, n^m) \\
&= (m^{-1}, n^{-1})(m, n^m) \\
&= (m^{-1}m, (n^{-1})^m n^m) \\
&= (1, 1).
\end{aligned}$$

So we have an induced homomorphism of P-groups

$$\overline{\varphi} : (M * N)/U \to M \ltimes N.$$

We define a homomorphism in the other direction

$$\psi : M \ltimes N \to (M * N)/U$$

by $\psi(m, n) = [mn]$ the equivalence class of the element $mn \in M * N$. To check the homomorphism property, we compute

$$\begin{aligned}
\psi(m', n')^{-1}\psi(m, n)^{-1}\psi((m, n)(m', n')) &= [n'^{-1}m'^{-1}][n^{-1}m^{-1}]\psi(mm', n^{m'}n') \\
&= [n'^{-1}m'^{-1}n^{-1}m^{-1}mm'n^{m'}n'] \\
&= [n'^{-1}(m'^{-1}n^{-1}m'n^{m'})n'] \\
&= [1]
\end{aligned}$$

since $m'^{-1}n^{-1}m'n^{m'} \in U$.

Clearly $\overline{\varphi}\psi = 1$. Since $\psi\overline{\varphi}$ is a homomorphism, to prove that it is 1 it is enough to check this on the generators $\psi\overline{\varphi}[mn]$, $m \in M$, $n \in N$, and this is clear.

It now follows, as may be proved directly, that $\mu \ltimes \nu : M \ltimes N \to P$, $(m, n) \mapsto (\mu m)(\nu n)$ is a homomorphism which with the action of P given by $(m, n)^p = (m^p, n^p)$ is a precrossed P-module. □

So $\mathcal{M} \ltimes \mathcal{N}$ is a precrossed module containing $\mathcal{M}$ and $\mathcal{N}$ as submodules. Let us see that it satisfies a universal property with respect to maps of the crossed modules $\mathcal{M}$ and $\mathcal{N}$ to any given crossed module $\mathcal{M}'$.

Proposition 4.2.2 *Let* $\mathcal{M}' = (\mu' : M' \to P)$ *be a crossed* P*-module and let* $f : M \to M'$ *and* $g : N \to M'$ *be morphisms of crossed* P*-modules. Then there is a unique map of precrossed* P*-modules extending* f *and* g*, namely* $f \ltimes g : M \ltimes N \to M',\ (m, n) \mapsto (fm)(gn)$.

Proof Uniqueness is obvious.

To prove existence we have to check that the morphism of precrossed P-modules

$$f * g : M * N \to M'$$

sends all elements of U to 1, where U is the subgroup specified in Proposition 4.2.1. On generators of U we have

$$(f * g)(m^{-1}n^{-1}mn^m) = f(m^{-1})g(n^{-1})f(m)g(n^m) = g(n^{-1})^{\mu' fm}g(n)^{\mu m} = 1$$

since $\mu' : M' \to P$ is a crossed module and $\mu' f = \mu$. □

Therefore it is clear that the coproduct of two crossed P-modules $\mu : M \to P$ and $\nu : N \to P$ is the crossed module associated to the precrossed module $\mu \ltimes \nu : M \ltimes N \to P$, i.e.

$$\mathcal{M} \circ \mathcal{N} = ((\mu \ltimes \nu)^{cr} : (M \ltimes N)^{cr} \to P) = (M \circ N \to P).$$

This has some striking consequences.

Remark 4.2.3 If we have two crossed P-modules such that M and N are finite groups (resp. finite p-groups), then so also is the semidirect product $M \ltimes N$ and hence their coproduct as crossed modules $M \circ N$ is also a finite group (resp. a finite p-group). This result was not clear at all from previous descriptions of the coproduct of crossed P-modules. □

Remark 4.2.4 If $(\mu : M \to P),\ (\nu : N \to P)$ are crossed P-modules such that each of $M,\ N$ act trivially on the other via P, then $M \ltimes N = M \times N$ and $\partial : M \times N \to P$, where $\partial(m, n) = (\mu m)(\nu n)$ is the coproduct where $(m, n)^p = (m^p, n^p)$. □

We now study the Peiffer subgroup $[\![M \ltimes N, M \ltimes N]\!]$ of $M \ltimes N$, which we shall write $\{M, N\}$. As we have seen, it is the subgroup generated by the Peiffer commutators of all elements of $M \ltimes N$. Alternatively, $\{M, N\}$ is generated by the images by φ of $r(m, n)$, i.e. by

$$\{\{n, m\} \mid m \in M, n \in N\}$$

Lemma 4.2.5 *The elements* $\{n, m\}$ *satisfy*

$$\{n, m\} = ([m, n], [n, m]),$$

where $[m, n] = m^{-1}m^n$ *and* $[n, m] = n^{-1}n^m$.

Proof Notice that any $m, m' \in M$ and $n \in N$ satisfy the relation

$$n'^{(m^n)} = ((n'^{n^{-1}})^m)^n = n^{-1}(nn'n^{-1})^m n = n^{-1}n^m n'^m (n^{-1})^m n \qquad (*)$$

Thus,

$$\begin{aligned}
\{n, m\} &= n^{-1}m^{-1}nm^{n} \\
&= (1, n^{-1})(m^{-1}, 1)(1, n)(m^{n}, 1) \\
&= (m^{-1}, (n^{-1})^{m^{-1}})(m^{n}, (n^{m})^{n}) \\
&= (m^{-1}m^{n}, ((n^{-1})^{m^{-1}})^{m^{n}}(n^{m})^{n}) \\
&= (m^{-1}m^{n}, n^{-1}n^{m}) \qquad \text{using } (*).
\end{aligned}$$

Finally, we have

$$\{n, m\} = ([m, n], [n, m]).$$

□

Using the previous result and some well known facts on the semidirect product, we get a presentation of the coproduct of two crossed modules as follows. First, recall that the semidirect product has a presentation with generators the elements $(m, n) \in M \times N$ and relations

$$(m, n)(m', n') = (mm', n^{m'}n')$$

for all $m, m' \in M$ and $n, n' \in N$. The set of relations may equivalently be expressed as

$$(m, n^{m'^{-1}})(m', n') = (mm', nn').$$

To get a presentation of $M \circ N$ we add the relations corresponding to the Peiffer subgroup $\{M, N\}$. By the preceding property the relation $\{m, n\} = 1$ is equivalent to $[m, n] = [n, m]^{-1}$, giving $(m^{n})^{-1}m = n^{-1}n^{m}$, or $n(m^{-1})^{n} = (n^{m^{-1}})^{-1}m^{-1}$. This may be expressed, taking $m' = m^{-1}$,

$$nm'^{n} = (n^{m'})^{-1}m'$$

suggesting the next proposition.

Theorem 4.2.6 *The group* $M \circ N$ *has a presentation with generators* $\{m \circ n \mid m \in M, n \in N\}$, *and relations*

$$mm' \circ nn' = (m \circ n^{m'^{-1}})(m' \circ n') = (m \circ n)(m'^{n} \circ n'),$$

for all $m, m' \in M$ *and* $n, n' \in N$.

Proof Let K be the group with this presentation. Then P acts on K by $(m \circ n)^{p} = m^{p} \circ n^{p}$, and the map

$$\xi : K \to P, \ m \circ n \mapsto (\mu m)(\nu n),$$

is a well defined homomorphism. It is routine to verify the crossed module rules for this structure.

It is also not difficult to check that this crossed module together with the morphisms $i : M \to K, \ m \mapsto m \circ 1$ and $j : N \to K, \ n \mapsto 1 \circ n$ satisfy the universal property of the coproduct. We omit further details. □

We describe some extra facts about $\{M, N\}$. In particular, the expression of the products and inverses of the elements $\{n, m\}$.

Proposition 4.2.7 *For any* $m, m' \in M$ *and* $n, n' \in N$ *we have*

$$\{n, m\}\{n', m'\} = ([m, n][m', n'], [n', m'][n, m]).$$

Proof

$$\begin{aligned}\{n,m\}\{n',m'\} &= ([m,n],[n,m])([m',n'],[n',m'])\\ &= ([m,n][m',n'],[n,m]^{[m',n']}[n',m'],\end{aligned}$$

and

$$\begin{aligned}[n,m]^{[m',n']}[n',m'] &= (n^{-1}n^m)^{m'^{-1}m'^{n'}}n'^{-1}n'^{m'}\\ &= ((n^{-1})^{m'^{-1}}(n^m)^{m'^{-1}})^{m'^{n'}}n'^{-1}n'^{m'}\\ &= n'^{-1}n'^{m'}n^{-1}n^m(n'^{-1})^{m'}n'n'^{-1}n'^{m'} \qquad \text{using } (*) \text{ in Lemma 4.2.5}\\ &= n'^{-1}n'^{m'}n^{-1}n^m\\ &= [n',m'][n,m].\end{aligned}$$

Thus

$$\{n,m\}\{n',m'\} = ([m,n][m',n'],[n',m'][n,m])$$

as indicated. □

Remark 4.2.8 This result extends to any finite product of elements $\{n_i, m_i\}$ with $m_i \in M, n_i \in N$. □

Corollary 4.2.9 *For any $m \in M$ and $n \in N$ we have*

$$\{n,m\}^{-1} = \{n^{-1}, m^n\}.$$

The proof is left to the reader.

4.3 The coproduct and the 2-dimensional van Kampen Theorem

One of the interesting features of the coproduct of crossed P-modules is its topological applications. The 2-dimensional van Kampen Theorem as stated in Theorem 2.3.1 involved a kind of generalised pushout (a coequaliser, in fact) and the coproduct of two crossed P-modules may also be interpreted as a pushout.

Proposition 4.3.1 *If $(\mu : M \to P)$, $(\nu : N \to P)$ are crossed P-modules then the following diagram*

$$\begin{tikzcd}(1 \to P) \arrow[r] \arrow[d] & (N \to P) \arrow[d]\\ (M \to P) \arrow[r] & (M \circ N \to P)\end{tikzcd} \qquad (4.3.1)$$

is a pushout in the category XMod/P *and also in the category* XMod/Groups.

Proof The equivalence of the pushout property in the category XMod/P with the universal property of the coproduct is easy to verify. We defer the proof of the pushout property in the category XMod/Groups until we have introduced in Section 5.2 the pullback functor $f^* : \mathsf{XMod}/Q \to \mathsf{XMod}/P$ for a morphism $f : P \to Q$ of groups. □

One of the simpler cases of the 2-dimensional van Kampen Theorem is the following.

Theorem 4.3.2 *Suppose that the connected space X is the union of the interior of two connected subspaces U_1, U_2, with connected intersection U_{12}. Suppose that the pairs (U_1, U_{12}) and (U_2, U_{12}) are 1-connected. Then the pair (X, U_{12}) is 1-connected and the morphism*

$$\Pi_2(U_1, U_{12}) \circ \Pi_2(U_2, U_{12}) \to \Pi_2(X, U_{12})$$

induced by inclusions is an isomorphism of crossed $\pi_1(U_{12})$-modules.

Proof We apply Theorem 2.3.1 to the cover of X given by U_1 and U_2 with $A = U_{12}$. The connectivity result is immediate. Also by the same theorem the following diagram is a pushout of crossed modules:

$$\begin{tikzcd}
\Pi_2(U_{12}, U_{12}) \arrow[r] \arrow[d] & \Pi_2(U_1, U_{12}) \arrow[d] \\
\Pi_2(U_2, U_{12}) \arrow[r] & \Pi_2(X, U_{12})
\end{tikzcd}$$

Since $\Pi_2(U_{12}, U_{12}) = (1 \to \pi_1(U_{12}))$, the result follows from Proposition 4.3.1. $\Box$

We would like to extract from this result some information on the absolute homotopy group $\pi_2(X)$. Consider the following part of the homotopy exact sequence of the pair (X, U_{12}) stated in 2.1.3,

$$\cdots \to \pi_2(U_{12}) \xrightarrow{i_*} \pi_2(X) \xrightarrow{j_*} \pi_2(X, U_{12}) \xrightarrow{\partial} \pi_1(U_{12}) \to \cdots .$$

It is clear that we have an isomorphism

$$\frac{\pi_2(X)}{i_*(\pi_2(U_{12}))} \cong \operatorname{Ker} \partial = \operatorname{Ker}(\partial_1 \circ \partial_2). \tag{4.3.2}$$

Notice than, in particular, this result gives complete information on $\pi_2(X)$ when $\pi_2(U_{12}) = 0$.

It would be a good thing to be able to identify the kernel of the coproduct of two crossed P-modules in a more workable way. To do this, let us introduce the pull back of crossed P-modules. Given two crossed modules $\mathcal{M} = (\mu : M \to P)$, $\mathcal{N} = (\nu : N \to P)$ we form the pullback square

$$\begin{tikzcd}
M \times_P N \arrow[r, "p_1"] \arrow[d, "p_2"'] & M \arrow[d, "\mu"] \\
N \arrow[r, "\nu"'] & P
\end{tikzcd} \tag{4.3.3}$$

where $M \times_P N = \{(m, n) \in M \times N \mid \mu(m) = \nu(n)\}$, p_1 and p_2 are the projections. Obviously $M \times_P N$ is a P-group (P acts diagonally).

Proposition 4.3.3 $M \times_P N$ *is isomorphic as* P*-group to* $\operatorname{Ker}(\mu \ltimes \nu)$.

Proof Let

$$\phi : M \times_P N \to M \ltimes N$$

be defined as $\phi(m, n) = (m, n^{-1})$.

To check that it is a homomorphism of groups we compute for all $m, m' \in M$ and $n, n' \in N$

$$\begin{aligned}
\phi(m,n)\phi(m',n') &= (m, n^{-1})(m', n'^{-1}) \\
&= (mm', (n^{-1})^{m'} n'^{-1}) \\
&= (mm', (n^{-1})^{n'} n'^{-1}) \\
&= (mm', n'^{-1} n^{-1} n' n'^{-1}) \\
&= (mm', (nn')^{-1}) \\
&= \phi(mm', nn').
\end{aligned}$$

Clearly, ϕ is a bijection onto $\operatorname{Ker}(\mu \ltimes \nu)$ that preserves the P-actions. □

Now, to any $m \in M$ and $n \in N$ we associate an element of $M \times_P N$ defined as

$$\langle m, n \rangle = (m^{-1} m^n, (n^{-1})^m n). \tag{4.3.4}$$

If we write $\langle M, N \rangle$ for the normal subgroup of $M \times_P N$ generated by $\{\langle m, n\rangle | m \in M, n \in N\}$, we have seen that $\phi(\langle M, N\rangle) = \{M, N\}$.

Thus, there is an induced map

$$\overline{\phi} : \frac{M \times_P N}{\langle M, N \rangle} \longrightarrow \frac{M \ltimes N}{\{M, N\}} = M \circ N.$$

We deduce immediately from the proposition

Corollary 4.3.4 *The map $\overline{\phi}$ gives an isomorphism of P-modules*

$$\overline{\phi} : \frac{M \times_P N}{\langle M, N \rangle} \cong \operatorname{Ker}(\mu \circ \nu).$$

Remark 4.3.5 Notice that this result has some purely algebraic consequences. Since $\mathcal{M} \circ \mathcal{N}$ is a crossed module, Ker $(\mu \circ \nu)$ is abelian; so $\langle M, N \rangle$ contains the commutator subgroup of $M \times_P N$. □

Now we can translate this algebraic result into a topological one.

Theorem 4.3.6 *If (U_1, U_{12}) and (U_2, U_{12}) are 1-connected and $\pi_2(U_{12}) = 0$, we have,*

$$\pi_2(X) \cong \frac{\pi_2(U_1, U_{12}) \times_{\pi_1(U_{12})} \pi_2(U_2, U_{12})}{\langle \pi_2(U_1, U_{12}), \pi_2(U_2, U_{12}) \rangle}.$$

Proof Since $\pi_2(U_{12}) = 0$, from the equation (4.3.2), we have $\pi_2(X) \cong \operatorname{Ker}(\partial_1 \circ \partial_2)$ and the result follows from the corollary before. □

Let us study some other algebraic way of computing $\operatorname{Ker}(\mu \circ \nu)$ or, equivalently, the quotient

$$\frac{M \times_P N}{\langle M, N \rangle}.$$

We may also define a homomorphism of groups $k : M \times_P N \to P$ by the formula $k(m, n) = \mu(m) = \nu(n)$. This gives the following result.

Proposition 4.3.7 *There is an exact sequence of P-groups*

$$0 \to \operatorname{Ker} \mu \oplus \operatorname{Ker} \nu \to M \times_P N \xrightarrow{k} \mu(M) \cap \nu(N) \to 1.$$

Proof It is immediate that $k(M \times_P N) = \mu(M) \cap \nu(N)$. It remains to check that $\operatorname{Ker} k = \operatorname{Ker} \mu \oplus \operatorname{Ker} \nu$; but this is clear since

$$\operatorname{Ker} k = \{(m, n) \mid \mu(m) = \nu(n) = 0\}$$

and all $m \in \operatorname{Ker} \mu$ and $n \in \operatorname{Ker} \nu$ commute. □

Bringing the subgroup $\langle M, N \rangle$ into the picture, it is immediate that $k(\langle m, n \rangle) = [\mu(m), \nu(n)]$. Then we have $k(\langle M, N \rangle) = [\mu(M), \nu(N)]$ giving a homomorphism $\overline{k}$ onto the quotient. This gives directly the next result.

Corollary 4.3.8 *There is an exact sequence of* P*-modules*

$$0 \to (\operatorname{Ker} \mu \oplus \operatorname{Ker} \nu) \cap (\langle M, N \rangle) \to \operatorname{Ker} \mu \oplus \operatorname{Ker} \nu \to \frac{M \times_P N}{\langle M, N \rangle} = \operatorname{Ker}(\mu \circ \nu) \xrightarrow{\overline{k}} \frac{\mu(M) \cap \nu(N)}{[\mu(M), \nu(N)]} \to 0.$$

Remark 4.3.9 An easy consequence is that $\mu \circ \nu$ is injective if and only if

i) $\operatorname{Ker} \mu \oplus \operatorname{Ker} \nu \subset \langle M, N \rangle$ and

ii) $[\mu(M), \nu(N)] = \mu(M) \cap \nu(N)$. □

As before, we can apply this result to the topological case, getting a way to compute the second homotopy group of a space in some cases.

Theorem 4.3.10 *If* (U_1, U_{12}) *and* (U_2, U_{12}) *are* 1*-connected and* $\pi_2(U_{12}) = 0$*, the following sequence of groups and homomorphisms is exact*

$$0 \to (\pi_2(U_1) \oplus \pi_2(U_2)) \cap \langle \pi_2(U_1, U_{12}), \pi_2(U_2, U_{12}) \rangle \to \pi_2(U_1) \oplus \pi_2(U_2) \to \pi_2(X) \to \frac{R_1 \cap R_2}{[R_1, R_2]} \to 1,$$

where $R_l = \operatorname{Ker}(\pi_1(U_{12}) \to \pi_1(U_l))$ *for* $l = 1, 2$*.*

If further $\pi_2(U_1) = \pi_2(U_2) = 0$*, then there is an isomorphism*

$$\pi_2(X) \cong \frac{R_1 \cap R_2}{[R_1, R_2]}.$$

Proof Let us consider the crossed modules $\partial_l : \pi_2(U_l, U_{12}) \to \pi_1(U_{12})$. Recall from (2.1.3) that the homotopy exact sequence of the pair (U_l, U_{12}) is

$$\cdots \to \pi_2(U_{12}) \xrightarrow{i_{l*}} \pi_2(U_l) \xrightarrow{j_{l*}} \pi_2(U_l, U_{12}) \xrightarrow{\partial_l} \pi_1(U_{12}) \to \cdots .$$

Directly from this exact sequence, we have

$$\operatorname{Im} \partial_l = R_l.$$

On the other hand,

$$\operatorname{Ker} \partial_l = \pi_2(U_l)$$

using the same homotopy exact sequence and $\pi_2(U_{12}) = 0$.

Thus the result is a translation of Corollary 4.3.8. □

Remark 4.3.11 Whenever U_1, U_2 are based subspaces of X with intersection U_{12} there is always a natural map

$$\sigma : \pi_2(U_1, U_{12}) \circ \pi_2(U_2, U_{12}) \to \pi_2(X, U_{12})$$

determined by the inclusions, but in general σ is not an isomorphism. Bogley and Gutierrez in [BG92] have had some success in describing Ker σ and Coker σ in the case when all the above spaces are connected. □

4.4 Some special cases of the coproduct

We end this chapter by giving a careful description of the coproduct of crossed P-modules in the particular case of two crossed P-modules $\mu : M \to P, \nu : N \to P$ in a useful special case, i.e. when $\nu(N) \subseteq \mu(M)$ and there is a P-equivariant section $\sigma : \mu M \to M$ of μ. Notice that this includes the case when $M = P$ and μ is the identity. These results were first published in [73].

This case is important because of the topological applications and also because it is useful in Section 5.6 for describing as a coproduct the crossed module induced by a monomorphism.

We start with some general results that will be used several times in this book.

Definition 4.4.1 If M acts on the group N we define $[N, M]$ to be the subgroup of N generated by the elements, often called *pseudo-commutators*, $n^{-1}n^m$ for all $n \in N$, $m \in M$. This subgroup is called the *displacement subgroup* and measures how much N is moved under the M-action. □

The following result is analogous to a standard result on the commutator subgroup.

Proposition 4.4.2 *The displacement subgroup* $[N, M]$ *is a normal subgroup of* N.

Proof It is enough to prove that the conjugate of any generator of $[N, M]$ lies also in $[N, M]$.

Let $m \in M, n, n_1 \in N$. We easily check that

$$n_1{}^{-1}(n^{-1}n^m)n_1 = ((nn_1)^{-1}(nn_1)^m)(n_1{}^{-1}n_1{}^m)^{-1}$$

and the product on the right hand side belongs to $[N, M]$ since both factors are generators. So we have proved $n_1{}^{-1}[N, M]n_1 \subseteq [N, M]$, whence $[N, M]$ is a normal subgroup of N. □

Definition 4.4.3 The quotient of N by the displacement subgroup is written $N_M = N/[N, M]$. The class in N_M of an element $n \in N$ is written $[n]$. It is clear that N_M is a trivial M-module since $[n^m] = [n]$. □

Proposition 4.4.4 *Let* $\mu : M \to P,\ \nu : N \to P$ *be crossed* P*-modules, so that* M *acts on* N *via* μ*. Then* P *acts on* N_M *by* $[n]^p = [n^p]$*. Moreover this action is trivial when restricted to* μM*.*

Proof To see that the P-action on N induces one on N_M, we have to check that [N,M] is a P-invariant subgroup and this follows because $(n^{-1}n^m)^p = (n^{-1})^p(n^m)^p = (n^p)^{-1}(n^p)^{m^p}$ for all $n \in N,\ m \in M,\ p \in P$.

The action of μM is trivial since $[n]^{\mu m} = [n^{\mu m}] = [n^m] = [n]$. □

Now we study the homomorphism

$$\xi : M \times N_M \to P, \ (m, [n]) \mapsto \mu m.$$

We have just seen that N_M is a P-group.

Proposition 4.4.5 *With P acting on $M \times N_M$ by the diagonal action, $\xi : M \times N_M \to P$ is a precrossed P-module.*

Proof If $m \in M, n \in N, p \in P$ then

$$\xi((m, [n])^p) = \xi(m^p, [n^p]) = \mu(m^p) = p^{-1}(\mu m)p = p^{-1}(\xi(m, [n])^p))p.$$

$\square$

Remark 4.4.6 In general it is not a crossed module. Nevertheless when N_M is abelian, the actions of both factors on each other are trivial. In this case it follows from Remark 4.2.4 that $\xi : M \times N_M \to P$ is a crossed module. (It is an easy exercise to prove this directly.) $\square$

We shall study now a condition first stated in [GH89] that implies that N_M is abelian.

Proposition 4.4.7 *Let $\mu : M \to P$, $\nu : N \to P$ be crossed P-modules such that $\nu N \subseteq \mu M$. Then N_M is abelian and therefore $\xi : M \times N_M \to P$ is a crossed P-module.*

Proof Let $n, n_1 \in N$. Choose $m \in M$ such that $\nu n_1 = \mu m$. Then by the crossed module rule CM2)

$$n_1{}^{-1} n n_1 = n^{\nu n_1} = n^{\mu m}$$

and so in the quotient $[n_1]^{-1}[n][n_1] = [n^{\mu m}] = [n]$. $\square$

We now study the case where there is also a P-equivariant section $\sigma : \mu M \to M$ of μ defined on μM. We will see that in this case $\xi : M \times N_M \to P$ is isomorphic to the coproduct $\mathcal{M} \circ \mathcal{N}$ of crossed P-modules. We shall follow the later proof given by Brown and Wensley in [BW96]. This contains the main result of [GH89] but it is stronger in the sense that it determines explicitly the coproduct structure. Since we shall use this structure for later results, we give the proof in detail.

Theorem 4.4.8 *Let $\mu : M \to P$, $\nu : N \to P$ be crossed P-modules with $\nu N \subseteq \mu M$ and let $\sigma : \mu M \to M$ be a P-equivariant section of μ. Then the morphisms of crossed P-modules*

$$\begin{aligned} i : M &\to M \times N_M, & j : N &\to M \times N_M, \\ m &\mapsto (m, 1) & n &\mapsto (\sigma\nu n, [n]) \end{aligned}$$

give a coproduct of crossed P-modules. Hence the canonical morphism of crossed P-modules

$$M \circ N \to M \times N_M$$

given by $m \circ n \mapsto (m(\sigma\nu n), [n])$ is an isomorphism.

Proof We need to verify that the pair (i, j) satisfies the universal property of the coproduct of crossed P-modules. Consider an arbitrary crossed P-module $\chi : C \to P$ and morphisms of crossed P-modules $\beta : M \to C$, and $\gamma : N \to C$. We have the following diagram:

$$\begin{tikzcd}
M \arrow[dr, "i"] \arrow[ddr, bend right, "\beta"'] & & N \arrow[dl, "j"'] \arrow[ddl, bend left, "\gamma"] \\
& M \times N_M \arrow[d, dashed, "\phi"] & \\
& C &
\end{tikzcd}$$

and we want to prove that there is a unique $\phi : M \times N_M \to C$ determining a morphism of crossed P-modules closing the diagram i.e. such that $\phi i = \beta$, and $\phi j = \gamma$.

Let us consider uniqueness. For any $m \in M, n \in N$, since ϕ has to be a homomorphism, we have

$$\begin{aligned}\phi(m,[n]) &= \phi((m,0)(\sigma\nu n,0)^{-1}(\sigma\nu n,[n]))\\ &= (\beta m)(\beta\sigma\nu n)^{-1}(\gamma n).\end{aligned}$$

This proves uniqueness of any such a ϕ. We now prove that this formula gives a well-defined morphism.

It is immediate from the formula that $\phi : M \times N_M \to C$ has to be β on the first factor and is defined on the second one by the map $[n] \mapsto (\beta\sigma\nu n)^{-1}(\gamma n)$. We have to check that this latter map is a well defined homomorphism.

We define the function

$$\psi : N \to C$$

by $n \mapsto (\beta\sigma\nu n^{-1})(\gamma n)$ and prove in turn the following statements.

4.4.9 $\psi(N) \subseteq Z(C)$*, the centre of* C*, and* $\chi(C)$ *acts trivially on* $\psi(N)$.

Proof of 4.4.9 Since $\chi\beta = \mu$ and $\chi\gamma = \nu$, it follows that $\chi\psi = 0$ and $\psi(N) \subseteq \operatorname{Ker}\chi$. Since C is a crossed module, $\chi(C)$ acts trivially on $\operatorname{Ker}\chi$ and $\operatorname{Ker}\chi \subseteq Z(C)$. □

4.4.10 ψ *is a morphism of crossed* P*-modules.*

Proof of 4.4.10 We have to prove that ψ is a morphism and is P-equivariant. The latter is clear, since $\beta, \gamma, \sigma, \nu$ are P-equivariant. So let $n, n_1 \in N$. Then

$$\begin{aligned}\psi(nn_1) &= (\beta\sigma\nu n_1^{-1})(\beta\sigma\nu n^{-1})(\gamma n)(\gamma n_1)\\ &= (\beta\sigma\nu n_1^{-1})(\psi n)(\gamma n_1)\\ &= (\psi n)(\beta\sigma\nu n_1^{-1})(\gamma n_1) && \text{by (4.4.9)}\\ &= (\psi n)(\psi n_1).\end{aligned}$$ □

Note that even if σ is not P-equivariant, ψ is still a group homomorphism.

4.4.11 M *acts trivially on* $\psi(N)$.

Proof of 4.4.11 Let $m \in M,\ n \in N$. Note that $(\beta\sigma\mu m)(\beta m^{-1})$ lies in $\operatorname{Ker}\chi$, and so belongs to $Z(C)$. Hence

$$\begin{aligned}(\psi n)^m &= (\beta\sigma\nu n^m)^{-1}(\gamma n)^{\mu m}\\ &= \beta\sigma((\mu m^{-1})(\nu n)(\mu m))^{-1}(\gamma n)^{\chi\beta m}\\ &= (\beta\sigma\mu m^{-1})(\beta\sigma\nu n^{-1})(\beta\sigma\mu m)(\beta m^{-1})(\gamma n)(\beta m)\\ &= (\beta\sigma\mu m^{-1})(\beta\sigma\mu m)(\beta m^{-1})(\beta\sigma\nu n^{-1})(\gamma n)(\beta m)\\ &= (\beta m^{-1})(\psi n)(\beta m)\\ &= \psi n && \text{by (4.4.9)}\end{aligned}$$ □

It follows that ψ induces a morphism $\psi' : N_M \to C,\ [n] \mapsto \psi n$, and so we define

$$\phi = (\beta, \psi') : M \times N_M \to C$$

by $(m,[n]) \mapsto (\beta m)(\psi n)$. Since ψn commutes with βm we easily verify that ϕ is a homomorphism, $\phi i = \beta,\ \phi j = \gamma$ and $\chi\phi = \xi$. Thus the pair of morphisms $i : M \to M \times N_M,\ j : N \to M \times N_M$ satisfies the universal property of a coproduct. This completes the proof of the theorem.

A standard consequence of the existence of a homomorphism $\sigma : \mu M \to M$ which is a section of μ on μM is that M is isomorphic to the semidirect product $\mu M \ltimes \operatorname{Ker}\mu$, where μM acts on $\operatorname{Ker}\mu$ by conjugation, i.e. $m'^{\mu m} = m^{-1}m'm$. Moreover, in the case when μ is a crossed module and σ is P-equivariant, the isomorphism is as crossed P-modules. Thus we have a third expression for the coproduct.

Proposition 4.4.12 *Let $\mu : M \to P$, $\nu : N \to P$ be crossed P-modules with $\nu N \subseteq \mu M$ and let $\sigma : \mu M \to M$ be a P-equivariant section of μ. There is an isomorphism of crossed P-modules*

$$M \circ N \cong (\mu M \ltimes \operatorname{Ker}\mu) \times N_M$$

given by $m \circ n \mapsto (m(\sigma\mu m)^{-1}, (\mu m)(\nu n), [n])$.

We now give a topological application.

Corollary 4.4.13 *Let (Y, A) be a connected based pair of spaces, and let $X = Y \cup CA$ be obtained from Y by attaching a cone on A. Then there is an isomorphism of crossed $\pi_1(A)$-modules*

$$\pi_2(X, A) \cong \pi_1(A) \times \pi_2(Y, A)_{\pi_1(A)}.$$

Proof We apply Theorem 4.3.2 with $U_1 = CA, U_2 = Y$, so that $U_{12} = A$. Then $\pi_2(CA, A) \cong \pi_1(A)$, by the exact sequence of the pair (CA, A), so that we have $\pi_2(X, A) \cong \pi_1(A) \circ \pi_2(Y, A)$. The result now follows from Theorem 4.4.8. □

As another application of Theorem 4.4.8, we analyse the symmetry of the coproduct in a special case.

The symmetry morphism $\tau : M \circ N \to N \circ M$ is, as usual for a coproduct, given by the pair of canonical morphisms $M \to N \circ M$, $N \to N \circ M$. Hence τ is given by $m \circ n \mapsto (1 \circ m)(n \circ 1) = n \circ m^n$.

Proposition 4.4.14 *Let $\mu : M \to P$ be a crossed module where μ is an inclusion of a normal subgroup of the group P. Then the isomorphism of crossed P-modules*

$$\begin{aligned} \theta : M \circ M &\to M \times M^{ab} \\ \theta(m \circ n) &= (mn, [n]) \end{aligned}$$

transforms the twist isomorphism $\tau : M \circ M \to M \circ M$ to the isomorphism

$$\begin{aligned} \theta^{-1}\tau\theta : M \times M^{ab} &\to M \times M^{ab} \\ (m, [n]) &\mapsto (m, [n^{-1}m]). \end{aligned}$$

Proof Notice that in this case $M^{ab} = M_M$. The isomorphism $\theta : M \circ M \to M \times M^{ab}$ is given in theorem 4.4.8. The twist isomorphism is transformed into the composition

$$(m, [n]) \mapsto mn^{-1} \circ n \mapsto n \circ (mn^{-1})^n = n \circ n^{-1}m \mapsto (m, [n^{-1}m]).$$

□

For an application in the next section, we now extend the last results to more general coproducts. We first prove:

Proposition 4.4.15 *Let T be an indexing set, and let $\mu : M \to P$ and $\nu_t : N_t \to P$, $t \in T$, be crossed P-modules. Let*

$$N = \bigcirc_{t \in T} N_t.$$

Suppose that $\nu_t N_t \subseteq \mu M$ *for all* $t \in T$. *Then there is an isomorphism of* P*-modules*

$$N_M \cong \bigoplus_{t \in T} (N_t)_M.$$

Proof Since $N = \bigcirc_{t\in T} N_t$ is the quotient of the free product $*N_t$ by the Peiffer relations, N_M can be presented as the same free product with the Peiffer relations $n_s^{-1} n_t^{-1} n_s n_t^{\nu_s n_s} = 1$ and the relations $n_t^{\mu m} = n_t$ for all $n_s \in N_s,\ n_t \in N_t,\ m \in M$.

These relations are equivalent to the commutator relations $[n_s, n_t] = 1$ together with $n_t^{\mu m} = n_t$ for all $n_s \in N_s,\ n_t \in N_t,\ m \in M$. □

Corollary 4.4.16 *Suppose in addition that the restriction* $\mu| : M \to \mu M$ *of* μ *has a* P*-equivariant section* σ*. Then there are isomorphisms of crossed* P*-modules between*

(i) $M \circ (\bigcirc_{t\in T} N_t)$,

(ii) $\xi : M \times \bigoplus_{t\in T} (N_t)_M \to P,\ \xi(m, n) = \mu m$,

(iii) $\xi\eta^{-1} : \mu M \times \mathrm{Ker}\mu \times \bigoplus_{t\in T} (N_t)_M \to P$.

Under the first isomorphism, the coproduct injections $i : M \to M \circ (\bigcirc_{t\in T} N_t),\ j_t : N_t \to M \circ (\bigcirc_{t\in T} N_t)$ *are given by* $m \mapsto (m, 0),\ n_t \mapsto (\sigma\nu_t n_t, [n_t])$.

When T is well-ordered, we may also obtain explicit isomorphisms by writing a typical element of $\bigcirc_{t\in T} N_t$ as $\bigcirc_{t\in T} n_t$, and by writing a finite product of elements $\nu_t n_t \in P$ as $\prod_{t\in T} \nu_t n_t$.

Corollary 4.4.17 *When* T *is well-ordered, the rules*

$$m \circ (\bigcirc_{t\in T} n_t) \mapsto \Big(m(\prod_{t\in T} (\sigma\nu_t n_t)), \bigoplus_{t\in T} [n_t] \Big) \mapsto \Big(m(\sigma\mu m^{-1}),\ (\mu m)(\prod_{t\in T} \nu_t n_t),\ \bigoplus_{t\in T} [n_t] \Big)$$

define isomorphisms $M \circ (\bigcirc_{t\in T} N_t) \cong M \times \bigoplus_{t\in T} (N_t)_M, \cong \mu M \times \mathrm{Ker}\mu \times \bigoplus_{t\in T} (N_t)_M$.

4.5 Notes

The construction of the coproduct had a precursor in Whitehead's paper [Whi41], and then was taken up in [Bro84], linked with consequences of the 2-dimensional van Kampen theorem.

Chapter 5

Induced crossed modules

Here we give a full account of another construction which allows detailed computations of non abelian homotopical information in dimension 2, namely the induced crossed modules. These arise topologically from a pushout of pairs of spaces of the form

$$\begin{tikzcd} (A, A) \arrow[r] \arrow[d] & (X, A) \arrow[d] \\ (Y, Y) \arrow[r] & (X \cup_f Y, Y) \end{tikzcd}$$

on applying the 2-dimensional van Kampen Theorem. The above diagram in fact gives a format for what is known topologically as *excision*, since if all the maps are closed inclusions then $X \cup_f Y$ with Y cut out, or excised, is the same as X with A excised. In the case of homology, and under suitable conditions, we end up with isomorphisms $H_n(X, A) \to H_n(X \cup_f Y, Y)$.

This is by no means so for relative homotopy groups, and this illustrates the complication of 2-dimensional algebra. The result we give on induced crossed modules shows how crossed modules cope with this complication. There are many implications.

We also find as a consequence of these methods that we obtain the relative Hurewicz theorem in dimension 2 and also a famous formula of Hopf on the second homology of an aspherical space. This formula was one of the starting points of the important theory of the cohomology of groups. These applications give a model for higher dimensional results.

The induced construction illustrates a feature of homotopy theory, that identifications in low dimensions can influence strongly high dimensional homotopy. Applications of Higher Homotopy van Kampen Theorems give information, though in a limited range of dimensions and under restrictive conditions, on how this influence is controlled.

The constructions in this chapter are quite elaborate and in places quite technical. This illustrates the complications of the geometry. We are illustrating the complications of 2-dimensional homotopy theory, and also that the algebra can cope with this.

Also the crossed module "induced" by a homomorphism of groups $f : P \to Q$ may be seen as one of the family of "change of base" functors of algebraic categories that have proved interesting in many fields from algebraic geometry to homological algebra. A general account of induced constructions in the context of cofibred categories is given in Appendix A.8.

The construction of the induced crossed module follows a natural pattern. Given the morphism f as above and a crossed P-module $\mu : M \to P$, we need to construct from M and f a new group N

on which Q acts so as to be a candidate for a crossed Q-module. Therefore we need new elements of the form m^q for $m \in M, q \in Q$. Since these do not for the moment exits, we construct them by taking the free group on pairs (m, q) and then adding appropriate relations. This is done in detail in Section 5.3.

In Section 5.1 we describe the pullback of a crossed module $f^*(\mathcal{M})$. This is quite easy to construct and the existence of the induced crossed module $f_*(\mathcal{M})$ defined in Section 5.2 follows from the existence of an adjoint to the pullback construction. We prove by the universal property that the free crossed module of Section 3.4 is a particular case of the induced crossed module and that an induced crossed module is the pushout of $\mathcal{M}$ and the trivial crossed module $1 \to Q$ over the trivial crossed module $1 \to P$.

That leaves the induced crossed module ready to be used in applications of the 2-dimensional van Kampen Theorem. In Section 5.4 we prove that when X is a topological space having a decomposition in two sets U_1, U_2 such that both pairs (U_2, U_{12}) are 1-connected, then the fundamental crossed module $\Pi_2(X, U_1)$ is the crossed module induced from $\Pi_2(U_2, U_{12})$ by the homomorphism induced by the inclusion (Theorem 5.4.1). As a consequence we get some homotopical results, in particular Whitehead's Theorem.

The second part of the Chapter is devoted to study the construction of the induced crossed module in a more useful guise. Since the direct construction is in general enormous (the first step uses a free group), it is interesting to get a more manageable way of producing induced crossed modules. One fruitful idea is to study separately the case when f is surjective and the case when f is injective and this is done in the next two sections.

The surjective case (Section 5.5) is quite direct and we prove that $f_*(\mathcal{M})$ is the quotient of M by the displacement subgroup $[M, \operatorname{Ker} f]$. This case has some interesting topological applications, in particular the relative Hurewicz's Theorem in dimension 2 and Hopf's formula for the second homology group of a group.

The case when f is injective, i.e. a monomorphism (Section 5.6), is essentially the inclusion of a subgroup. This case is much more intricate and we need the concept of the copower construction M^{*T} where T is a transversal of P in Q. We get a description of the induced crossed module as a quotient of the copower (Corollary 5.6.6). Both the group and the action have alternative descriptions that can be used to develop some examples, so obtaining in particular a bound for the number of generators and relations for an induced crossed module.

It is also proved (in Section 5.7) that the induced crossed module is finite when both M and the index $[P : Q]$ are finite. This suggests the problem of explicit computation, and in the last section of the chapter we explain some computer calculations in the finite case obtained using the package GAP.

The next Section (5.8) is quite technical but contains a detailed description of the induced crossed module in a useful special case, with many interesting examples, namely when P and M are both normal subgroups of Q. We start by studying the induced crossed module when P is a normal subgroup of Q, getting a description in terms of the coproduct $M^{\circ T}$. Then we use the description of the coproduct given in the last Section of the preceding Chapter to derive just from the universal property both the action (Theorem 5.8.6) and the map (Theorem 5.8.7). When M is just another normal subgroup included in P, we get some more concrete formulas.

This leaves many finite examples not covered by the previous theorems: the last section gives some computer calculations. [9]

5.1 Pullbacks of precrossed and crossed modules.

The work of this section can be done both for crossed and for precrossed modules. We shall state only the crossed case but, if nothing is said, it is understood that a similar result is true for precrossed modules. We shall not repeat the statement, but we only shall give indications of the differences.

Let us start by defining the functor that is going to be the adjoint of the induced crossed module, the "pullback". This is an important construction which, given a morphism of groups $f : P \to Q$, enables us to move from crossed Q-modules to crossed P-modules.

Definition 5.1.1 Let $f : P \to Q$ be a homomorphism of groups and let $\mathcal{N} = (\nu : N \to Q)$ be a crossed module. We define the subgroup of $N \times P$

$$f^*N = N \times_Q P = \{(n, p) \in N \times P \mid \nu n = fp\}.$$

This is the usual pullback in the category Groups. There is a commutative diagram

$$\begin{tikzcd} f^*N \arrow[r, "\bar{f}"] \arrow[d, "\bar{\nu}"'] & N \arrow[d, "\nu"] \\ P \arrow[r, "f"'] & Q \end{tikzcd}$$

where $\bar{\nu} : (n, p) \mapsto p$, $\bar{f} : (n, p) \mapsto n$. Then P acts on f^*N via f and the diagonal, i.e. $(n, p)^{p'} = (n^{fp'}, p'^{-1}pp')$. It is easy to see that this gives a P-action. The *pullback crossed module* is

$$f^*\mathcal{N} = (\bar{\nu} : f^*N \to P)$$

It is also called the pullback of $\mathcal{N}$ along f and it is easy to see that $f^*\mathcal{N}$ is a crossed module. □

This construction satisfies a crucial universal property, analogous to that of the pullback of groups. To state it, we use also the morphism of crossed modules

$$(\bar{f}, f) : f^*\mathcal{N} \longrightarrow \mathcal{N}.$$

Theorem 5.1.2 *For any crossed module* $\mathcal{M} = (\mu : M \to P)$ *and any morphism of crossed modules*

$$(h, f) : \mathcal{M} \longrightarrow \mathcal{N}$$

there is a unique morphism of crossed P*-modules* $h' : \mathcal{M} \to f^*\mathcal{N}$ *such that the following diagram commutes*

$$\begin{tikzcd} M \arrow[dr, dashed, "h'"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "\mu"'] & & \\ & f^*N \arrow[r, "\bar{f}"'] \arrow[d, "\bar{\nu}"] & N \arrow[d, "\nu"'] \\ & P \arrow[r, "f"] & Q \end{tikzcd}$$

Proof The existence and uniqueness of the homomorphism h' follows from the fact that f^*N is the pullback in the category of groups. It is defined by $h'(m) = (h(m), \mu(m))$. So we have only to prove that h' is a morphism of crossed P-modules. This can be checked directly. □

Using this universal property, it is not difficult to see that this construction gives a functor

$$f^* : \mathsf{XMod}/Q \to \mathsf{XMod}/P.$$

Moreover, these functors are 'natural' in the sense that there are natural equivalences $f^*f'^* \simeq (f'f)^*$ for any homomorphisms $f : P \to Q$ and $f' : Q \to R$.

In the last chapter, we dealt with the coproduct of crossed P-modules, which satisfied a universal property in the category XMod/P of crossed P-modules. We shall need an extension of this property in Section 5.8. It gives the existence and uniqueness of a morphism of crossed modules associated to a family of morphisms of crossed modules $\{(\beta_t, f)\}$ over the same homomorphism $f : P \to Q$. The standard universal property of the coproduct is just the particular case $f = \mathrm{Id}$. The argument we give uses the above pullback functor f^* and can be seen in a more general categorical light. You may skip this part until the result is needed. The proof takes time to write out but is in essence quite direct.

Proposition 5.1.3 *Let $\mathcal{M}_t,\ t \in T$ be a family of crossed P-modules. Let $f : P \to Q$ be a homomorphism of groups, let $\mathcal{N} = (\nu : N \to Q)$ be an arbitrary crossed Q-module, and for each $u \in T$ let $\beta_u : M_u \to N$ be a homomorphism giving a morphism of crossed modules over f. Then there exists a unique crossed module morphism $\phi : \bigcirc_t M_t \to N$ over f such that $\phi i_u = \beta_u$ for all $u \in T$.*

Proof The proof can be summarised by saying that we use the universal property of the pullback functor to show that the universal property for the coproduct in the category XMod/P extends to the more general case.

This general universal property asks for the existence and uniqueness of the dashed homomorphism ϕ in the diagram

$$\begin{tikzcd}
M_u \arrow[dr, "i_u"] \arrow[drr, bend left, "\beta_u"] \arrow[ddr, bend right, "\mu_u"'] & & \\
& \bigcirc_t M_t \arrow[r, dashed, "\phi"'] \arrow[d, "\mu"] & N \arrow[d, "\nu"'] \\
& P \arrow[r, "f"] & Q
\end{tikzcd}$$

such that the diagram commute and (ϕ, f) is a morphism of crossed modules.

As happens many times, uniqueness is immediate from the fact that $\bigcup i_t(M_t)$ generates $\bigcirc_t M_t$.

By construction of the pullback of groups, if the homomorphism ϕ exists, it has to factor through f^*N giving a commutative diagram

$$\begin{tikzcd}
\bigcirc_t M_t \arrow[dr, dashed, "\phi'"] \arrow[drr, dashed, "\phi"] \arrow[ddr, bend right, "\mu"'] & & \\
& f^*N \arrow[r, "\bar{f}"'] \arrow[d, "\bar{\nu}"] & N \arrow[d, "\nu"'] \\
& P \arrow[r, "f"] & Q.
\end{tikzcd}$$

So we just have to construct a homomorphism ϕ' that gives a morphism of crossed P-modules.

By the universal property of pullbacks, for each u there is a unique homomorphism $\beta'_u : M_u \to f^*N$ such that $\bar{f}\beta'_u = \beta_u$. Moreover, β'_u is a morphism of crossed P-modules and makes the diagram commutative:

$$\begin{tikzcd}
M_u \arrow[dr, "\beta'_u"] \arrow[drr, bend left, "\beta_u"] \arrow[ddr, bend right, "\mu_u"'] & & \\
& f^*N \arrow[r, "\bar{f}"'] \arrow[d, "\bar{\nu}"] & N \arrow[d, "\nu"'] \\
& P \arrow[r, "f"] & Q.
\end{tikzcd}$$

By the universal property of coproducts of crossed modules over P, there is a unique morphism of crossed P-modules $\phi' : \bigcirc_t M_t \to f^*N$ such that for all $u \in T$ the diagrams

$$\begin{tikzcd}
M_u \arrow[dr, "i_u"] \arrow[drr, bend left, "\beta'_u"] \arrow[ddr, bend right, "\mu_u"'] & & \\
& \bigcirc_t M_t \arrow[r, "\phi'"'] \arrow[d, "\mu"] & f^*N \arrow[d, "\bar{\nu}"'] \\
& P \arrow[r, "="] & P
\end{tikzcd}$$

commute.

The composite morphism $\phi = \bar{f}\phi'$ is the unique morphism satisfying $\phi i_u = \beta_u$ for all $u \in T$. $\square$

5.2 Induced precrossed and crossed modules

Now we define a functor f_* left adjoint to the pullback f^* of the previous section. In particular we prove that the free crossed module is a particular case of an induced crossed module. Then we apply this to the topological case to get Whitehead's Theorem (Corollary 5.4.8).

The "induced crossed module" functor is defined by the following universal property, adjoint to that of pullback.

Definition 5.2.1 For any crossed P-module $\mathcal{M} = (\mu : M \to P)$ and any homomorphism $f : P \to Q$ the crossed module *induced* by f from $\mathcal{M}$ should be given by:

i) a crossed Q-module $f_*\mathcal{M} = (f_*\mu : f_*M \to Q)$;

ii) a morphism of crossed modules $(\phi, f) : \mathcal{M} \longrightarrow f_*\mathcal{M}$, satisfying the dual universal property that for any morphism of crossed modules

$$(h, f) : \mathcal{M} \longrightarrow \mathcal{N}$$

there is a unique morphism of Q-crossed modules $h' : f_*M \to N$ such that the diagram

$$\begin{tikzcd}
& & N \arrow[ddl, bend left, "\nu"] \\
M \arrow[r, "\phi"'] \arrow[d, "\mu"] \arrow[urr, bend left, "h"] & f_*M \arrow[d, "f_*\mu"'] \arrow[ur, dashed, "h'"] & \\
P \arrow[r, "f"] & Q &
\end{tikzcd}$$

commutes. $\square$

Now we prove that this functor if it exists, forms an adjoint pair with the pullback functor. Using general categorical considerations, we can deduce the existence of the induced crossed module functor

$$f_* : \mathsf{XMod}/P \to \mathsf{XMod}/Q$$

and, also, that they satisfy the 'naturality condition' that there is a natural equivalence of functors $f'_* f_* \simeq (f'f)_*$.

Theorem 5.2.2 *For any homomorphism of groups $f : P \to Q$, f_* is the left adjoint of f^*.*

Proof From the naturality conditions expressed earlier, it is immediate that for any crossed modules $\mathcal{M} = (\mu : M \to P)$ and $\mathcal{N} = (\nu : N \to Q)$ there are bijections

$$(\mathsf{XMod}/P)(\mathcal{N}, f^*\mathcal{N}) \cong \{h : M \to N \mid (h, f) : \mathcal{M} \to \mathcal{N} \text{ is a morphism of crossed modules}\},$$

as proved in Proposition 5.1.2 , and

$$(\mathsf{XMod}/Q)(f_*\mathcal{M}, \mathcal{N}) \cong \{h : M \to N \mid (h, f) : \mathcal{M} \to \mathcal{N} \text{ is a morphism of crossed modules}\}$$

as given in the definition.

Their composition gives the bijection needed for adjointness. □

We end this section by comparing the universal properties defining the induced crossed module and two other constructions. The first one is the free crossed module on a map. Using the induced crossed module, we get an alternative description of the free crossed module.

Proposition 5.2.3 *Let P be a group and $\{\omega_r \mid r \in R\}$ be an indexed family of elements of P, or, equivalently, a function $\omega : R \to P$. Let F be the free group generated by R and $f : F \to P$ the homomorphism of groups such that $f(r) = \omega_r \in P$. Then the crossed module $f_*(1_F) : f_*F \to P$ induced from $1_F = (\mathrm{Id}_F : F \to F)$ by f is the free crossed P-module on $\{(1, r) \in f_*F \mid r \in R\}$.*

Proof Both universal properties assert the existence of morphisms of crossed P-modules commuting the appropriate diagrams. Let us check that the data in both constructions are equivalent.

The data in the induced crossed module are a crossed module $\mathcal{N}$ and a morphism of crossed modules $(h, f) : 1_F \to \mathcal{N}$. The data in the free crossed module are a crossed module $\mathcal{N}$ and a map $\omega' : R \to N$ lifting ω. Since F is the free group on R, the map ω' is equivalent to a homomorphism of groups $h : F \to N$ lifting ω (i.e. $h(r) = \omega'(r)$). Moreover, h satisfies

$$h(r^{r'}) = h(r'^{-1} r r') = h(r')^{-1} h(r) h(r') = (hr)^{\nu h(r')} = (hr)^{f(r')} \tag{5.2.1}$$

for all $r, r' \in R$. So h preserves the action and (h, f) is a morphism of crossed modules.

Thus the data in both cases are equivalent. □

Remark 5.2.4 It is clear that the proof in Proposition 5.2.3 does not work for precrossed modules since in proving the equality (5.2.1) we have used axiom CM2). It is easy to see that the precrossed module induced from $\mathrm{Id}_F : F \to F$ is not the free precrossed module but its quotient with respect to the normal subgroup generated by all relations

$$(p, r^{r'}) = (p\omega(r), r')$$

when $p \in P$ and $r, r' \in R$.

It is a nice exercise to find a crossed module $L \to F$ such that the free precrossed module is the induced from L. □

We now give an important re-interpretation of induced crossed modules in terms of a pushout of crossed modules. This is how we can show that induced crossed modules arise from a 2-dimensional van Kampen theorem. The proof is obtained by relating the two universal properties. The general situation of which this proof is an example is given in Proposition A.8.7 and Theorem A.9.2, using notions of fibrations of categories.

Proposition 5.2.5 *For any crossed module $\mathcal{M} = (\mu : M \to P)$ and any homomorphism $f : P \to Q$, the induced crossed module $f_*\mathcal{M}$ is such that the commutative diagram of crossed modules*

$$\begin{tikzcd}
(1 \to P) \arrow[r, "(1,f)"] \arrow[d, "(0,\mathrm{Id})"'] & (1 \to Q) \arrow[d, "(0,\mathrm{Id})"] \\
(M \to P) \arrow[r, "(\phi,f)"'] & (f_*M \to Q)
\end{tikzcd}$$

is a pushout of crossed modules.

Proof To check that the diagram satisfies the universal property of the pushout, let $\mathcal{N} = (\nu : N \to R)$ be a crossed module, and $(h, f') : \mathcal{M} \to \mathcal{N}$ and $(1, g) : 1_Q \to \mathcal{N}$ morphisms of crossed modules, such that the diagram of full arrows commutes. We have to construct the dotted morphism of crossed modules (k, g):

$$\begin{tikzcd}
(1 \to P) \arrow[r, "(0,f)"'] \arrow[d, "(0,1)"] & (1 \to Q) \arrow[d, "(0,1)"'] \arrow[ddr, bend left, "(0,g)"] & \\
(M \to P) \arrow[r, "(\phi,f)"] \arrow[drr, bend right, "(h,f')"'] & (f_*M \to Q) \arrow[dr, dotted, "(k,g)"] & \\
& & (N \to R)
\end{tikzcd}$$

It immediate that $f' = gf,\ k\phi = h$. So we can transform morphisms in turn

$$\begin{aligned}
(M \to P) &\xrightarrow{(k\phi,gf)} (N \to R) \\
(M \to P) &\xrightarrow{(\overline{k\phi},1)} ((gf)^*N \to P) \\
(M \to P) &\xrightarrow{(\overline{k\phi},1)} (f^*g^*N \to P) \\
(f_*M \to Q) &\xrightarrow{(\overline{\phi},1)} (g^*N \to Q) \\
(f_*M \to Q) &\xrightarrow{(k,g)} (N \to R)
\end{aligned}$$

as required. □

5.3 Induced crossed modules: Construction in general.

We now give a simple construction of the induced crossed module, thus showing its existence. This construction is not particularly useful for computations, and this problem is dealt with later.

We are going to construct the induced crossed module in two steps, producing first the induced precrossed module and then from this the associated crossed module by quotienting out by its Peiffer subgroup.

Let us start with a homomorphism of groups $f : P \to Q$ and a crossed module $(\mu : M \to P)$. We construct

$$F = F(M \times Q),$$

the free group generated by the elements of $M \times Q$ (to make things easier to remember, we think of (m, q) as "m^q").

There is an obvious Q-action on F given on generators by

$$(m, q)^{q'} = (m, qq')$$

for any $q, q' \in Q$ and $m \in M$. Also, the map

$$\tilde{\mu} : F \to Q$$

given on generators by $\tilde{\mu}(m, q) = q^{-1} f\mu(m) q$ for any $q \in Q$ and $m \in M$ is a precrossed module.

To get the *induced precrossed module* from this map $\tilde{\mu}$, we take into the picture both the multiplication and the P-action on the first factor, and so make a quotient by the appropriate normal subgroup. Let S be the normal subgroup generated by all the relations of the two following types:

$$(m, q)(m', q) = (mm', q) \tag{5.3.1}$$

$$(m^p, q) = (m, f(p)q) \tag{5.3.2}$$

for any $m, m' \in M, p \in P, q \in Q$. We define $E = F/S$. It is easy to see that the action of Q on F induces one on E. Also, $\tilde{\mu}$ induces a precrossed module

$$\hat{\mu} : E \to Q.$$

There is a map

$$\phi : M \to E$$

got by projecting the map on F defined as $\phi(m) = (m, 1)$. This map is a morphism of groups thanks to the relations of type 5.3.1, while (ϕ, f) is a morphism of precrossed modules thanks to the relations of 5.3.2.

Theorem 5.3.1 *The precrossed module* $\hat{\mu} : E \to Q$ *is that induced from* μ *by the homomorphism* f.

Proof We have only to check the universal property.

For any morphism of precrossed modules

$$(h, f) : (\mu : M \to P) \longrightarrow (\nu : N \to Q)$$

there is a unique morphism of precrossed Q-modules $h' : E \to N$ such that $h = h'\phi$ because the only way to define this homomorphism is by $h'(m, q) = (hm)^q$ on generators. It is a very easy exercise to check that this definition maps S to 1, and that the induced homomorphism gives a morphism of crossed modules. □

Remark 5.3.2 If $\mathcal{M} = (\mu : M \to P)$ is a crossed module, there are two equivalent ways to obtain the induced crossed module $f_*\mathcal{M} = (f_*M \to Q)$. One way is to get the associated crossed module to the one above. The second way is to quotient out F, not only by the relations of the above two kinds, but also adding the Peiffer relations

$$(m_1, q_1)^{-1}(m_2, q_2)(m_1, q_1) = (m_2, q_2 q_1^{-1} f\mu(m_1) q_1)$$

for any $q_1,\ q_2 \in Q$ and $m_1,\ m_2 \in M$.

There are much easier descriptions of the induced crossed module in the particular cases that f is either surjective or injective and they go back to [BH78]. They give an alternative way of constructing the induced crossed module since every map decomposes as the product of an injection and a surjection. These are given later in Sections 5.5 and 5.6.

These constructions will be dealt with in a general setting in Section **??**.

5.4 Induced crossed modules and the 2-dimensional van Kampen Theorem

The relation between induced crossed module and the pushout of crossed modules suggests that the induced crossed module may appear in some cases when using the 2-dimensional van Kampen Theorem 2.3.1. After looking to the statement of the theorem for general subspaces $A, U_1, U_2 \subseteq X$ it is easy to see that this case occurs when $A = U_1$, and this situation ia also known as 'excision'. We should give some background to this idea.

In the situation where $X = U_1 \cup U_2$, the inclusion of pairs

$$E : (U_1, U_1 \cap U_2) \to (X, U_2)$$

is known as the 'excision map' because the smaller pair is obtained by cutting out or 'excising' $X \setminus U_2$ from the larger pair. It is a theorem of homology (The Excision Theorem) that if U_1, U_2 are open in X then the excision map induces an isomorphism of relative homology groups. This is one of the basic results which make homology groups readily computable.

Here we get a result that can be interpreted as a limited form of Excision Theorem for homotopy, but it shows that the excision map is in general not an isomorphism even for second relative homotopy groups. Lack of excision is one of the reasons for the difficulty of computing homotopy groups of spaces.

Theorem 5.4.1 (Homotopical excision in dimension 2) *Let X be a space which is the union of the interior of two subspaces U_1 and U_2 and define $U_{12} = U_1 \cap U_2$. If all spaces are connected and (U_2, U_{12}) is 1-connected, then (X, U_1) is also 1-connected and the morphism of crossed modules*

$$\Pi_2(U_2, U_{12}) \to \Pi_2(X, U_1)$$

realises the crossed module $\Pi_2(X, U_1)$ as induced from $\Pi_2(U_2, U_{12})$ by the homomorphism induced by the inclusion $\pi_1(U_{12}) \to \pi_1(U_1)$.

Proof Following the notation of Theorem 2.3.1 with $A = U_1$ we have

$$A_1 = A \cap U_1 = U_1, \ A_2 = A \cap U_2 = U_{12} \text{ and } A_{12} = A \cap U_{12} = U_{12}.$$

It is clear that the hypothesis of Theorem 2.3.1 are satisfied since $(U_1, A_1) = (U_1, U_1)$, $(U_2, A_2) = (U_2, U_{12})$ and $(U_{12}, A_{12}) = (U_{12}, U_{12})$ are 1-connected. The consequence is that the diagram of crossed modules

$$\begin{tikzcd} \Pi_2(U_{12}, U_{12}) \arrow[r] \arrow[d] & \Pi_2(U_2, U_{12}) \arrow[d] \\ \Pi_2(U_1, U_1) \arrow[r] & \Pi_2(X, U_1) \end{tikzcd} \tag{5.4.1}$$

is a pushout.

Proposition 5.2.5 now implies the result. □

As in the case of Theorem 2.3.1, using standard mapping cylinder arguments, we can prove a slightly more general statement.

Corollary 5.4.2 *Let (X, A) be a pair and $f : A \to Y$ a continuous map. If all spaces are connected, the inclusion $i : A \to X$ is a closed cofibration and the pair (X, A) is 1-connected, then the pair $(Y \cup_f X, Y)$ is also 1-connected and $\Pi_2(Y \cup_f X, Y)$ is the crossed module induced from $\Pi_2(X, A)$ by $f_* : \pi_1(A) \to \pi_1(Y)$.*

Proof This can either be deduced from the proceeding theorem by use of mapping cylinder arguments, or can be seen as a particular case of Theorem 2.3.3 when $U_1 = A$ and $Y_1 = Y$. □

This last corollary has as a consequence a homotopical Excision Theorem for closed subsets under weak conditions.

Corollary 5.4.3 *Let X be a space that is the union of two closed subspaces U_1 and U_2 and let $U_{12} = U_1 \cap U_2$. If all spaces are connected, the inclusion $U_1 \to X$ is a cofibration, and the pair (U_2, U_{12}) is connected, then the pair (U_1, X) is also connected and the crossed module $(\pi_2(X, U_1) \to \pi_1(U_1))$ is the one induced from $(\pi_2(U_2, U_{12}) \to \pi_1(U_{12}))$ by the morphism $\pi_1(U_{12}) \to \pi_1(U_1)$ induced by the inclusion.*

Before proceeding any further, we consider the case of a space X given as the homotopy pushout of classifying spaces.

Theorem 5.4.4 *Let $\mathcal{M} = (\mu : M \to P)$ be a crossed module, and let $f : P \to Q$ be a morphism of groups. Let $\beta : BP \to B\mathcal{M}$ be the inclusion. Consider the pushout diagram*

$$\begin{tikzcd} BP \arrow[r, "\beta"] \arrow[d, "Bf"'] & B\mathcal{M} \arrow[d] \\ BQ \arrow[r, "\beta'"'] & X. \end{tikzcd}$$

i.e. $X = BQ \cup_{Bf} B\mathcal{M}$. Then the fundamental crossed module $\Pi_2(X, BQ)$ is isomorphic to the induced crossed module $f_*\mathcal{M}$.

Further, there is a map of spaces $X \to Bf_*\mathcal{M}$ inducing an isomorphism of the corresponding π_1, π_2.

Proof This first part immediate from Corollary 5.4.2.

The last statement requires a generalisation of Proposition 2.4.8, in which the 1-skeleton is replaced by a subcomplex Z with the property that $\pi_2(Z) = 0$ and the induced map $\pi_1(Z) \to \pi_1(X)$ is surjective. (In our case $Z = BQ$.) This result is proved in Chapter 10. □

Remark 5.4.5 The most striking consequence of the last theorem is that we have determined completely a non trivial homotopy 2-type of a space. That is, we have replaced geometric constructions by corresponding algebraic ones. As we shall see, induced crossed modules are computable in many cases, and so we can obtain many explicit computations of homotopy 2-types. The further surprise is that all this theory is needed for just this example. This shows the difficulty of homotopy theory, in that new ranges of algebraic structures are required to explain what is going on.

In the next sections, we will be able to obtain some explicit calculations as a consequence of the last results.

Remark 5.4.6 An interesting special case of the last theorem is when $\mathcal{M}$ is an inclusion of a normal subgroup, since then $B\mathcal{M}$ has the homotopy type of $B(P/M)$ by Proposition 2.4.6. So we have determined the fundamental crossed module of (X, BR) when X is the homotopy pushout

$$\begin{tikzcd} BP \arrow[r, "Bp"] \arrow[d, "Bf"'] & BR \arrow[d] \\ BQ \arrow[r, "p'"'] & X \end{tikzcd}$$

in which $p : P \to R$ is surjective. In this case $\mathcal{M} = (\operatorname{Ker} p \to P)$. □

To end, we consider the case where the space we are attaching is a cone.

Theorem 5.4.7 *Let $f : A \to Y$ be a continuous map between connected spaces. Then the pair $(CA \cup_f Y, Y)$ is 1-connected and $\Pi_2(CA \cup_f Y, Y)$ is the crossed module induced from the identity crossed module $1_{\pi_1(A)}$ by $f_* : \pi_1(A) \to \pi_1(Y)$.*

Proof Using part of the homotopy exact sequence of the pair (CA, A),

$$\pi_2(CA, x) = 0 \to \pi_2(CA, A, x) \to \pi_1(A, x) \to \pi_1(CA, x) = 0$$

we get an isomorphism of $\pi_1(A, x)$ groups transforming the fundamental crossed module $\Pi_2(CA, A)$ in $1_{\pi_1(A,x)}$.

Now, we can use Corollary 5.4.2 and identify the induced crossed module with the free module by Proposition 5.2.3. □

As a consequence we get Whitehead's theorem on free crossed modules [Whi49b].

Corollary 5.4.8 (Whitehead's Theorem) *Let Y be a space constructed from the path-connected space X by attaching cells of dimension two. Then the map $\pi_1(X) \to \pi_1(Y)$ is surjective and $\Pi_2(Y, X)$ is isomorphic to the free crossed module on the characteristic maps of the 2-cells.*

As before, we apply the results just obtained to the case of a space X which is a pushout of classifying spaces.

Theorem 5.4.9 *Let $f : P \to Q$ be a morphism of groups. Then the crossed module $\Pi_2(BQ \cup_{Bf} CBP, BQ)$ is isomorphic to the induced crossed module $f_*(1_P)$.*

Proof Taking in the preceding remark $R = 1$, its classifying space is contractible. Thus, we can take CBP as equivalent to the classifying space BR. □

5.5 Calculation of induced crossed modules: the epimorphism case.

Let us consider now the case where $f : P \to Q$ is an epimorphism. Then $\operatorname{Ker} f$ acts on M via the map f and the induced crossed module f_*M may be seen as M quotiented out by the normal subgroup appropriate for trivialising the action of $\operatorname{Ker} f$ (since Q is isomorphic to $P/\operatorname{Ker} f$), i.e. by quotienting out the displacement subgroup (recall 4.4.1 to 4.4.7).

Proposition 5.5.1 *If* $f : P \to Q$ *is an epimorphism and* $\mu : M \to P$ *is a crossed module, then*

$$f_* M \cong \frac{M}{[M, \operatorname{Ker} f]}$$

where $[M, \operatorname{Ker} f]$ *is the displacement subgroup, i.e. the subgroup of* M *generated by* $\{m^{-1}m^k \mid m \in M, k \in \operatorname{Ker} f\}$.

Proof Let us recall that by Proposition 4.4.7 the quotient $M/[M, \operatorname{Ker} f]$ is a Q-crossed module with the Q-action on $M/[M, \operatorname{Ker} f]$ given by $[m]^q = [m^p]$ for $m \in M, q \in Q, q = f(p), p \in P$, and the homomorphism

$$\overline{f\mu} : \frac{M}{[M, \operatorname{Ker} f]} \to Q,$$

is induced by the composition $\mu f : M \to Q$.

It remains only to prove that this $\overline{f\mu}$ satisfies the universal property. Let

$$(h, f) : (\mu : M \to P) \longrightarrow (\nu : N \to Q)$$

be a morphism of crossed modules. We have to prove that there exists a unique homomorphism of groups

$$h' : \frac{M}{[M, \operatorname{Ker} f]} \longrightarrow N$$

such that

$$(h', f) : (\overline{f\mu} : \frac{M}{[M, \operatorname{Ker} f]} \to P) \longrightarrow (\nu : N \to Q)$$

is a morphism of crossed modules and $h'\phi = h$ where ϕ is the natural projection. Equivalently, we have to prove that h induces a homomorphism of groups h' and that (h', f) is a morphism of crossed modules.

Since $h(m^p) = (hm)^{f(p)}$ for any $m \in M$ and $p \in P$, we have $h[M, \operatorname{Ker} f] = 1$. Then, h induces a homomorphism of groups h' as above such that $h'\phi = h$.

We have only to check that h' is a map of Q-crossed modules. But

$$\nu h'[m] = \nu h(m) = f\mu(m) = \overline{f\mu}[m],$$

so the square commutes, and

$$h'([m]^q) = h'[m^p] = h(m^p) = (hm)^{f(p)} = (h'[m])^q$$

so h' preserves the actions. □

This description gives as a topological consequence a version of the relative Hurewicz Theorem.

Theorem 5.5.2 (Relative Hurewicz Theorem in dimension 2) *Consider a 1-connected pair of spaces* (Y, A) *such that the inclusion* $i : A \to Y$ *is a closed cofibration. Then the space* $Y \cup C(A)$ *is simply connected and its second homotopy group* $\pi_2(Y \cup C(A))$ *and the singular homology group* $H_2(Y \cup C(A))$ *are each isomorphic to* $\pi_2(Y, A)$ *factored by the action of* $\pi_1(A)$.

Proof It is clear from the classical van Kampen Theorem that the space $Y \cup C(A)$ is 1-connected.

Using the homotopy exact sequence of the pair $(Y \cup C(A), C(A))$,

$$\cdots \to 0 = \pi_2(C(A)) \to \pi_2(Y \cup C(A)) \to \pi_2(Y \cup C(A), C(A)) \to 0 = \pi_1(C(A)) \to \cdots$$

we have

$$\pi_2(Y \cup C(A)) \cong \pi_2(Y \cup C(A), C(A)).$$

Now we can apply Corollary 5.4.2 to show that the crossed module

$$\pi_2(Y \cup C(A), C(A)) \to \pi_1(C(A)) = 1$$

is induced from $\pi_2(Y, A) \to \pi_1(A)$ by the map given by the morphism $\pi_1(A) \to 1$ induced by the inclusion $A \to CA$.

Moreover, since the map $i_* : \pi_1(A) \to \pi_1(Y)$ is onto, by Proposition 5.5.1 we have

$$\pi_2(Y \cup C(A), C(A)) \cong \pi_2(Y, A)/[\pi_2(Y, A), \pi_1(A)].$$

This yields the result on the second homotopy group.

The absolute Hurewicz theorem for $Y \cup C(A)$ (which we prove in Theorem 14.7.9) yields the result on the second homology group. □

Corollary 5.5.3 *The first two homotopy groups of S^2 are given by $\pi_1(S^2) = 0, \pi_2(S^2) \cong \mathbb{Z}$.*

Proof This is the case of Theorem 5.5.2 when $A = S^1, Y = E^2_+$, where E^2_+ denotes the top hemisphere of the 2-sphere S^2. Then $\pi_2(Y, A) \cong \mathbb{Z}$ with trivial action by $\pi_1(A) \cong \mathbb{Z}$. □

Actually we have a more general result.

Corollary 5.5.4 *If A is a path connected space, and $SA = CA \cup_A CA$ denotes the suspension of A, then SA is simply connected and*

$$\pi_2(SA) \cong \pi_1(A)^{ab}.$$

Proof This is simply the result that $\pi_1(A)^{ab} = \pi_1(A)/[\pi_1(A), \pi_1(A)]$. □

One interest in this result is the method, which extends to other situations where the notion of abelianisation is not so clear, [BL87a].

Example 5.5.5 Let $f : A \to Y$ be as in Theorem 5.4.7, let $Z = Y \cup_f CA$, and suppose that $f_* : \pi_1(A) \to \pi_1(Y)$ is surjective with kernel K. An application of Proposition 5.5.1 to the conclusion of Theorem 5.4.7 gives $\pi_2(Z) = \pi_1(A)/[\pi_1(A), K]$, and it follows from the homotopy exact sequence of the pair (Z, Y) that there is an exact sequence

$$\pi_2(Y) \to \pi_2(Z) \to K/[\pi_1(A), K] \to 0. \tag{5.5.1}$$

It follows from this exact sequence that if $A = BP$ and $Y = BQ$, so that we have an exact sequence $1 \to K \to P \to Q \to 1$ of groups, then $\pi_2(Z) \cong K/[P, K]$. Now we assume some knowledge of homology of spaces. In particular, the homology $H_i(P)$ of a group P is defined to be the homology $H_i(BP)$ of the space BP, $i \geqslant 0$. Since Z is simply connected, we get the same value for $H_2(Z)$, by the absolute Hurewicz theorem. Now the homology exact sequence of the cofibre sequence $A \to Y \to Z$ gives an exact sequence

$$H_2(P) \to H_2(Q) \to K/[P, K] \to H_1(P) \to H_1(Q) \to 0$$

(originally due to Stallings). In particular if $P = F$, a free group, or one with $H_2(F) = 0$, then we obtain an exact sequence

$$0 \to H_2(Q) \to K/[F, K] \to F^{ab} \to Q^{ab} \to 0.$$

This gives the famous Hopf formula

$$H_2(Q) \cong \frac{K \cap [F, F]}{[K, F]}$$

which was one of the starting points of homological algebra. □

5.6 The monomorphism case. Inducing from crossed modules over a subgroup

In Section 5.3 we have considered the construction of an induced crossed module for a general homomorphism, and in Section 5.5 we have got a simpler expression for the case when f is an epimorphism. Now we study the case of a monomorphism. This is essentially the same as studying the case of an inclusion in a subgroup. So in all this section we shall consider the inclusion $\iota : P \to Q$ of a subgroup P of Q.

As we shall see this case is rather involved and we get an expression of the induced crossed module that is quite complicated and in some cases very much related to the coproduct. Let us introduce some concepts that shall be helpful.

Definition 5.6.1 Let M be a group and let T be a set, we define the *copower* M^{*T} to be the free product of the groups $M_t = M \times \{t\}$ for all $t \in T$. Notice that all M_t are naturally isomorphic to M under the map $(m, t) \mapsto m$. So M^{*T} can be seen as the free product of copies of M indexed by T. □

This construction satisfies the adjointness condition that for any group N there is a bijection

$$\mathsf{Sets}(T, \mathsf{Groups}(M, N)) \cong \mathsf{Groups}(M^{*T}, N)$$

natural in M, N, T. Notice also that the precrossed module induced from $\mathcal{M} : (\mu : M \to P)$ by $f : P \to Q$ is a quotient of M^{*UQ} where UQ is the underlying set of Q.

In the case where we have the inclusion of a subgroup $\iota : P \to Q$, we choose T to be a *right transversal* of P in Q, by which is meant a subset of Q including the identity 1 and such that any $q \in Q$ has a unique representation as $q = pt$ where $p \in P, t \in T$. For any crossed P-module $\mathcal{M} = (\mu : M \to P)$, the precrossed Q-module induced by ι will have the form $\hat{\mu} : M^{*T} \to Q$. Let us describe the Q-action.

Proposition 5.6.2 *Let $\iota : P \to Q$, $\mathcal{M}$, and T be as above. Then there is a Q-action on M^{*T} defined on generators using the coset decomposition by*

$$(m, t)^q = (m^p, u)$$

for any $q \in Q,\ m \in M,\ t \in T$, where p, u are the unique $p \in P,\ u \in T$, such that $tq = pu$.

Proof Let $m \in M$, $t, u, u' \in T$, $p, p' \in P$ and $q, q' \in Q$ be elements such that $tq = pu$ and $uq' = p'u'$. We have $t(qq') = puq' = pp'u'$. Therefore,

$$((m, t)^q)^{q'} = (m^p, u)^{q'} = (m^{pp'}, u') = (m, t)^{qq'}$$

and Q acts on M^{*T}. □

Remark 5.6.3 We can think of (m, t) as m^t, so the action is $(m^t)^q = (m^p)^u$ where $tq = pu$. Notice that if P is normal in Q then the Q-action induces an action of P on M_t given by $(m, t)^p = (m^{tpt^{-1}}, t)$. We shall exploit this later. □

Now we define the boundary homomorphism by specifying the images of the generators

$$\hat{\mu} : M^{*T} \to Q,\ (m, t) \mapsto t^{-1}\mu(m)t.$$

Proposition 5.6.4 *Let* $\iota : P \to Q$, $\mathcal{M}$ *and* T *be as above. Then* $(\hat{\mu} : M^{*T} \to Q)$ *is a precrossed* Q*-module with the above action.*

Proof We verify axiom CM1). For any $m \in M$, $t \in T$, and $q \in Q$, we have

$$\begin{aligned}
\hat{\mu}((m,t)^q) &= \hat{\mu}(m^p, u) && \text{when } tq = pu \\
&= u^{-1}\mu(m^p)u && \text{by definition of } \hat{\mu} \\
&= u^{-1}(p)^{-1}\mu(m)pu && \text{because } \mu \text{ is a crossed module} \\
&= q^{-1}(t)^{-1}\mu(m)tq && \text{because } tq = pu \\
&= q^{-1}\hat{\mu}(m,t)q && \text{because } \mu \text{ is a crossed module.}
\end{aligned}$$

□

To complete the characterisation we now prove that in this case this precrossed module is the induced one.

Theorem 5.6.5 *If* $\iota : P \to Q$ *is a monomorphism, and* $\mathcal{M} = (\mu : M \to P)$ *is a crossed P-module then* $\hat{\mu} : M^{*T} \to Q$ *is the precrossed module induced by* ι *from* μ*.*

Proof We check the universal property. There is a homomorphism of groups $\phi : M \to M^{*T}$ defined by $\phi(m) = (m, 1)$ that makes commutative the square

$$\begin{tikzcd}
M \arrow[r, "\phi"] \arrow[d, "\mu"'] & M^{*T} \arrow[d, "\hat{\mu}"] \\
P \arrow[r, "\iota"'] & Q
\end{tikzcd}$$

and so that (ϕ, ι) is a morphism of precrossed modules.

For any morphism of precrossed modules

$$(h, \iota) : (\mu : M \to P) \longrightarrow (\nu : N \to Q)$$

the only possible definition of a homomorphism of groups $h' : M^{*T} \to N$ such that $h'\phi = h$ is the one given by $h'(m,t) = (hm)^t$ on generators. It is easy to see that it is a morphism of Q-precrossed modules. □

It is immediate that the induced crossed module is the one associated to the precrossed module $\hat{\mu}$, i.e. the quotient with respect to the Peiffer subgroup.

Corollary 5.6.6 *If* $\iota : P \to Q$ *is a monomorphism, and* $(\mu : M \to P)$ *is a crossed P-module, then the crossed module induced by* ι *from* μ *is the homomorphism induced by* $\hat{\mu}$ *on the quotient*

$$\hat{\mu} : \frac{M^{*T}}{[\![M^{*T}, M^{*T}]\!]} \to Q$$

together with the induced action of Q*.*

It is useful to have a smaller number of generators of the Peiffer subgroup $[\![M^{*T}, M^{*T}]\!]$.

Proposition 5.6.7 *Let* $\iota : P \to Q$ *be a monomorphism,* $\mathcal{M} = (\mu : M \to P)$ *be a crossed P-module and* M^{*T} *as before. Let S be a set of generators of M as a group, and let us define* $S^P = \{s^p \mid s \in S,\ p \in P\}$*. Then there is an isomorphism of the induced crossed module* $\iota_*\mathcal{M} = (\iota_* M \to Q)$ *to a quotient*

$$\iota_* M \cong \frac{(M^{*T})}{R}$$

where R is the normal closure in M^{*T} *of the elements*

$$[\![(r,t),(s,u)]\!] = (r,t)^{-1}(s,u)^{-1}(r,t)(s,u)^{\hat{\mu}(r,t)}$$

for all $r, s \in S^P$ *and* $t, u \in T$.

Proof By Corollary 5.6.6 we just have to prove that R is the Peiffer subgroup $[\![M^{*T}, M^{*T}]\!]$ of M^{*T}.

Now, M^{*T} is generated by the set

$$(S^P, T) = \{(s^p, t) \mid s \in S,\ p \in P,\ t \in T\}$$

and this set is Q-invariant since $(s^p, t)^q = (s^{pp'}, u)$ where $u \in T$, $p' \in P$ satisfying $tq = p'u$. Then by Proposition 3.3.5 $\{M^{*T}, M^{*T}\}$ is the normal closure of the set $\{(S^P,T),(S^P,T)\}$ of basic Peiffer commutators and this is just R. □

The next corollary gives a bound on the number of generators and relations of a presentation for the induced crossed module in terms of those of a presentation of M and the index of $\iota\mu(M)$ in Q.

Corollary 5.6.8 *Suppose* $\iota : P \to Q$ *is injective,* M *has a presentation as a group with* g *generators and* r *relations, the set of generators of* M *is* P*-invariant, and* $n = [Q : \iota\mu(M)]$*. Then* $\iota_* M$ *has a presentation with* gn *generators and* $rn + g^2 n(n-1)$ *relations.*

Proof This is just a process of counting. The transversal T has n elements, so M^{*T} has gn generators and rn relations. To get a presentation of $\iota_* M$ we just add as relations the basic Peiffer commutators of the generators and those are $g^2 n(n-1)$ relations more. □

We show how this construction works out in the case of the dihedral crossed module, which exhibits a number of typical features.

Example 5.6.9 Recall that the dihedral group D_{2n} of order $2n$ has presentation

$$\langle x, y \mid x^n, y^2, xyxy \rangle.$$

We consider another copy $\tilde{D}_{2n}$ of D_{2n} with presentation $\langle u, v \mid u^n, v^2, uvuv \rangle$ and the homomorphism

$$\partial : \tilde{D}_{2n} \to D_{2n},\ u \mapsto x^2,\ v \mapsto y.$$

With this boundary and action of D_{2n} on $\tilde{D}_{2n}$ given on generators by the equations

$$u^y = vuv^{-1},\ v^y = v,\ u^x = u,\ v^x = vu,$$

this becomes the *dihedral crossed module*. As an exercise, check this result and also that $\partial : \tilde{D}_{2n} \to D_{2n}$ is an isomorphism if n is odd, and has kernel and cokernel isomorphic to C_2 if n is even. □

Example 5.6.10 We let $Q = D_{2n}$ be the dihedral group with generators x, y, and let $M = P = C_2$ be the cyclic subgroup of order 2 generated by y. Let us denote by $\iota : C_2 \hookrightarrow D_{2n}$ the inclusion.

We have that $\mathrm{Id} : C_2 \to C_2$ is a crossed module and we are going to identify the induced crossed module

$$\hat{\mu} = \iota_*(\mathrm{Id}) : \iota_*(C_2) \longrightarrow D_{2n}.$$

A right transversal of C_2 in D_{2n} is given by the elements $T = \{x^i \mid i = 0, 1, 2, \dots, n-1\}$.

If we apply the Proposition 5.6.7 we have that $\iota_* C_2$ has a presentation with generators $a_i = (y, x^i)$, $i = 0, 1, 2, \dots, n-1$ and relations $a_i^2 = 1$, $i = 0, 1, 2, \dots, n-1$, together with the Peiffer relations associated to these generators.

Since the D_{2n}-action on C_2^{*T} is given by

$$a_i^x = a_{i+1} \text{ and } a_i^y = a_{n-i},$$

and

$$\hat{\mu}(a_i) = x^{-i}yx^i = yx^{2i},$$

we have $(a_i)^{\hat{\mu}a_j} = a_{2j-i}$, so that the Peiffer relations become

$$a_j^{-1}a_ia_j = a_{2j-i}.$$

In this group, we define $u = a_0a_1,\ v = a_0$. As a consequence, we have $u = a_ia_{i+1}$ and $u^i = a_0a_i$ and it is now easy to check that $(C_2^{*T})^{\mathrm{cr}} \cong \tilde{D}_{2n}$. Also the map $\hat{\mu}$ satisfies

$$\hat{\mu}u = \hat{\mu}(a_0a_1) = yyx^2 = x^2,\ \hat{\mu}v = y.$$

Thus y acts on ι_*C_2 by conjugation by v. However x acts by $u^x = u,\ v^x = vu$.

This crossed module is the dihedral crossed module of the previous Example 5.6.9.

It is worth pointing out that this induced crossed module is finite while the corresponding pre-crossed module M^{*T} is clearly infinite. We shall insist on these points in the next section. □

Our next and last proposition in this area determines induced crossed modules under some abelian conditions, and has useful applications. If M is a P-module, i.e. an abelian P-group, and T is a set we define the *copower* of M with T, written $M^{\oplus T}$, to be the P-module which is the sum of copies of M one for each element of T.

Proposition 5.6.11 *Let $\iota : P \to Q$ and $(\mu : M \to P)$ be as before. Moreover assume that M is abelian and $\iota\mu(M)$ is normal in Q. Then ι_*M is abelian and as a Q-module is just the induced Q-module in the usual sense.*

Proof We use the result and notation of Proposition 5.6.7. Note that if $u, t \in T$ and $r \in S$ then

$$u\hat{\mu}(r,t) = ut^{-1}\mu(r)t = \iota\mu(m)ut^{-1}t = \mu(m)u$$

for some $m \in M$, by the normality condition.

The Peiffer commutator given in Proposition 5.6.7 can therefore be rewritten as

$$(r,t)^{-1}(s,u)^{-1}(r,t)(s,u)^{\hat{\mu}(r,t)} = (r^{-1},t)(s,u)^{-1}(r,t)(s^m,u).$$

Since M is abelian, $s^m = s$. Thus the basic Peiffer commutators reduce to ordinary commutators. Hence ι_*M is the copower $M^{\oplus T}$, and this, with the given action, is the usual presentation of the induced Q-module. □

Example 5.6.12 Let $M = P = Q$ be the infinite cyclic group, which we write $\mathbb{Z}$, and let $\iota : P \to Q$ be multiplication by 2. Then

$$\iota_*M \cong \mathbb{Z} \oplus \mathbb{Z},$$

and the action of a generator of Q on ι_*M is to switch the two copies of $\mathbb{Z}$. This result could also be deduced from well known results on free crossed modules. However, our results show that we get a similar conclusion simply by replacing each $\mathbb{Z}$ in the above by for example C_4, and this fact is new. □

5.7 On the finiteness of some induced crossed modules

With the results of the previous section, we have an alternative way of constructing the induced crossed module associated to a homomorphism f. We can factor f in an epimorphism and a monomorphism and then apply the constructions. As pointed out before it is always a good thing to have as many equivalent ways as possible since then we can choose the most appropriate to some particular situation.

As we have seen in the previous section, if we have a (pre)crossed module $\mathcal{M} = (M \to P)$ in which M is generated by a finite P-set of a generators, and a group homomorphism $P \to Q$ with finite cokernel, the induced (pre)crossed module is also generated by a finite set. In this section we give an algebraic proof that a crossed module induced from a finite crossed module by a morphism with finite cokernel is also finite. The result is false for precrossed modules.

Theorem 5.7.1 *Let $\mu : M \to P$ be a crossed module and let $f : P \to Q$ be a morphism of groups. Suppose that M and the index of $f(P)$ in Q are finite. Then the induced crossed module f_*M is finite.*

Proof Factor the morphism $f : P \to Q$ as $\tau\sigma$ where τ is injective and σ is surjective. Then f_*M is isomorphic to $\tau_*\sigma_*M$. It is immediate from Proposition 5.5.1 that if M is finite then so also is σ_*M. So it is enough to assume that f is injective.

Let T be a right transversal of $f(P)$ in Q. Then there are maps

$$(\xi, \eta) : T \times Q \to f(P) \times T$$

defined by $(\xi, \eta)(t, q) = (p, u)$ where $p \in P$, $u \in T$ are elements such that $tq = f(p)u$. With this notation, the form of a basic Peiffer relation got in Corollary 5.6.6 is then of the form

$$(m, t)(n, u) = (n, u)(m^{\xi(t, u^{-1}f\mu(n)u)}, \eta(t, u^{-1}f\mu(n)u)) \tag{5.7.1}$$

where $m, n \in M$, $t, u \in T$.

We now assume that the finite set T has l elements and has been given the total order $t_1 < t_2 < \cdots < t_l$. An element of M^{*T} may be represented as a word

$$(m_1, u_1)(m_2, u_2)\ldots(m_e, u_e). \tag{5.7.2}$$

Such a word is said to be *reduced* when $u_i \neq u_{i+1}$, $1 \leqslant i < e$, and to be *ordered* if $u_1 < u_2 < \cdots < u_e$ in the given order on T. This yields a partial ordering of $M * T$ where $(m_i, u_i) \leqslant (m_j, u_j)$ whenever $u_i \leqslant u_j$.

A *twist* uses the Peiffer relation (5.7.1) to replace a reduced word $w = w_1(m, t)(n, v)w_2$, with $v < t$, by $w' = w_1(n, v)(m^p, u)w_2$. If the resulting word is not reduced, multiplication in M_v and M_u may be used to reduce it. In order to show that any word may be ordered by a finite sequence of twists and reductions, we define an integer weight function on the set W_n of non-empty words of length at most n by

$$\begin{array}{rcl} \Omega_n : W_n & \longrightarrow & \mathbb{Z}^+ \\ (m_1, t_{j_1})(m_2, t_{j_2})\ldots(m_e, t_{j_e}) & \mapsto & l^e \sum_{i=1}^{e} l^{n-i} j_i. \end{array}$$

It is easy to see that $\Omega_n(w') < \Omega_n(w)$ when $w \to w'$ is a reduction. Similarly, for a twist

$$w = w_1(m_i, t_{j_i})(m_{i+1}, t_{j_{i+1}})w_2 \;\to\; w' = w_1(m_{i+1}, t_{j_{i+1}})(n, t_k)w_2$$

the weight reduction is

$$\Omega_n(w) - \Omega_n(w') = l^{n+e-i-1}(\, l(j_i - j_{i+1}) + j_{i+1} - j_k\,) \geqslant l^{n+e-i-1},$$

so the process terminates in a finite number of moves.

We now specify an algorithm for converting a reduced word to an ordered word. Various algorithms are possible, some presumably more efficient than others, but we are not interested in efficiency here. We call a reduced word k-*ordered* if the subword consisting of the first k elements is ordered and the remaining elements are greater than these. Every reduced word is at least 0-ordered. Given a k-ordered, reduced word, find the rightmost minimal element to the right of the k-th position. Move this element one place to the left with a twist, and reduce if necessary. The resulting word may only be j-ordered, with $j < k$, but its weight will be less than that of the original word. Repeat until an ordered word is obtained.

Let $Z = M_{t_1} \times M_{t_2} \times \ldots \times M_{t_l}$ be the product of the sets $M_{t_i} = M \times \{t_i\}$. Then the algorithm yields a function $\phi : Y \to Z$ such that the quotient morphism $Y \to f_*M$ factors through ϕ. Since Z is finite, it follows that f_*M is finite. □

Remark 5.7.2 In this last proof, it is in general not possible to give a group structure on the set Z such that the quotient morphism $Y \to f_*M$ factors through a morphism to Z. For example, in the dihedral crossed module of Example 5.6.9, with $n = 3$, the set Z will have 8 elements, and so has no group structure admitting a morphism onto D_6.

So the proof of the main theorem of this section does not extend to a proof that the induced crossed module construction is closed also in the category of p-groups. Nevertheless, the result is true and there is a topological proof in [BW95]. □

5.8 Inducing crossed modules by a normal inclusion

We continue the study of Section 5.6 of the crossed modules induced by the inclusion $P \to Q$ of a subgroup, by considering the case when P is normal in Q. We shall show in Theorem 5.8.4 that the coproduct of crossed P-modules described in Section 4.1 may be used to give a presentation of crossed Q-modules induced by the inclusion $\iota : P \to Q$ analogous to known presentations of induced modules.

Let us start by digressing a bit about crossed modules constructed from a given one using an isomorphism.

Definition 5.8.1 Let $\mu : M \to P$ be a crossed P-module and let α be an automorphism of P. The crossed module $\mu_\alpha : M_\alpha \to P$ *associated to* α is defined as follows. The group M_α is just $M \times \{\alpha\}$, the morphism μ_α is given by $(m, \alpha) \mapsto \alpha\mu m$ and the action of P is given by $(m, \alpha)^p = (m^{\alpha^{-1}p}, \alpha)$. □

Proposition 5.8.2 *The map $\mu_\alpha : M_\alpha \to P$ is a crossed module. Moreover this crossed module is isomorphic to μ since the map $k_\alpha : M \to M_\alpha$ given by $k_\alpha m = (m, \alpha)$ produces an isomorphism over α.*

Proof Let us check both properties of crossed module

$$\mu_\alpha(m^{\alpha^{-1}p}, \alpha) = \alpha(\mu m^{\alpha^{-1}p}) = \alpha(\alpha^{-1}(p)^{-1}\mu(m)\alpha^{-1}(p)) = p^{-1}\alpha\mu(m)p = p^{-1}\mu^\alpha(m)p$$

and

$$(m, \alpha)^{\mu_\alpha(m', \alpha)} = (m, \alpha)^{\alpha\mu(m')} = (m^{\alpha^{-1}\alpha\mu(m')}, \alpha) = (m^{\mu(m')}, \alpha) = (m', \alpha)^{-1}(m, \alpha)(m', \alpha).$$

It is immediate that the map $k_\alpha : M \to M_\alpha$ is an isomorphism. Also, the diagram

$$\begin{tikzcd} M \arrow[r, "k_\alpha"] \arrow[d, "\mu"'] & M_\alpha \arrow[d, "\mu_\alpha"] \\ P \arrow[r, "\alpha"'] & P. \end{tikzcd}$$

commutes and the map k_α preserves the P-action over α. □

Remark 5.8.3 Notice that if $\alpha = \mathrm{Id}$, there is a natural identification $M_{\mathrm{Id}} = M$. □

We continue to assume that P is a normal subgroup of Q. In this case, for any $t \in Q$, there is an inner automorphism $\alpha_t : P \to P$ defined by $\alpha_t(p) = t^{-1}pt$. Let us write $(\mu_t : M_t \to P)$ instead of $(\mu_{\alpha_t} : M_{\alpha_t} \to P)$.

Let recall that this crossed P-module is the same $(\mu_t : M_t \to P)$ that we have used to construct $\iota_* M$ in Section 5.6, namely $M_t = M \times \{t\}$, the P-action was given by $(m,t)^p = (m^{tpt^{-1}}, t)$ and the homomorphism μ_t was defined by $\mu_t(m,t) = t^{-1}\mu m t$. We have just seen that it is a crossed P-module isomorphic to $\mathcal{M}$.

Now let T be a right transversal of P in Q. We can form the precrossed Q-module $\mathcal{M}' = (\partial' : M^{*T} \to Q)$ as in Proposition 5.6.2. Recall that the Q-action is defined on generators as follows. For any $q \in Q,\ m \in M,\ t \in T$ we define

$$(m,t)^q = (m^p, u),$$

where $p \in P$ and $u \in T$ are the only ones satisfying $tq = pu$. Also the homomorphism ∂' is defined by $\partial'(m,t) = t^{-1}pt$

We had seen in Theorem 5.6.5 that the induced crossed Q-module $\iota_*\mathcal{M}$ is the quotient of M^{*T} by the Peiffer subgroup associated to the Q-action. On the other hand, we have seen in Corollary 4.1.2 that the coproduct as crossed P-modules

$$\partial : M^{\circ T} \to P$$

is the quotient of M^{*T} with respect to the Peiffer subgroup associated to the P-action. We are going to check that they are the same.

Theorem 5.8.4 *In the situation we have just described, the homomorphism*

$$M^{\circ T} \xrightarrow{\partial} P \overset{\iota}{\hookrightarrow} Q$$

with the morphism of crossed modules

$$(i_1, \iota) \ : \ \mathcal{M} \to (\iota\partial : M^{\circ T} \to Q)$$

is the induced crossed Q-module.

Proof It is immediately checked in this case that the Peiffer subgroup is the same whether M^{*T} is considered as a precrossed P-module $M^{*T} \to P$ or as a precrossed Q-module $M^{*T} \to Q$. It can also be directly checked. We leave that as an exercise. □

We remark that the result of Theorem 5.8.4 is analogous to well known descriptions of induced modules, except that here we have replaced the direct sum which is used in the module case by the coproduct of crossed modules. Corresponding descriptions in the non-normal case look to be considerably harder.

As a consequence we obtain easily a result on p-finiteness that can be strengthened by topological means ([BW95]). We prove it here for normal subgroups.

Proposition 5.8.5 *If M is a finite p-group and P is a normal subgroup of finite index in Q, then the induced crossed module $\iota_* M$ is a finite p-group.*

Proof This follows immediately from the discussion in Section 4.1. □

Now the induced module $(\iota\partial : M^{\circ T} \to Q)$ in Theorem 5.8.4 may be described using Corollary 4.4.16, if the hypotheses there are satisfied. So let P be a normal subgroup of Q and T a transversal as before, and let $(\mu : M \to P)$ be a crossed P-module.

We can divide the construction of the group $M^{\circ T}$ into two steps. We define $W = M^{\circ T'}$ the coproduct of all but $M_1 = M$. Then there is an isomorphism of crossed Q-modules

$$\iota_* \mathcal{M} \cong M \circ W.$$

To apply Corollary 4.4.16 we have to assume that for all $t \in T$ we have $\mu_t(M) \subseteq \mu(M)$, i.e. that for all $t \in T$ we have $t^{-1}\mu(M)t \subseteq \mu(M)$ (notice that this is immediately satisfied if μM is normal in Q), and that there is a section $\sigma : \mu M \to M$ of μ defined on μM. Most of the time we shall require also that σ is P-equivariant.

Then there is an isomorphism

$$\iota_* \mathcal{M} \cong M \times \bigoplus_{t \in T'} (M_t)_M$$

through which the morphisms giving the coproduct structure become

$$(i, \iota) : (\mu : M \to P) \longrightarrow (\xi = \iota\mu\,\mathrm{pr}_1 : M \times \bigoplus_{t \in T'} (M_t)_M \to Q)$$

where $i = i_1 : (m, 1) \mapsto (m, 0)$ and

$$(i_t, \iota) : (\mu : M_t \to P) \longrightarrow (\xi = \iota\mu\,\mathrm{pr}_1 : M \times \bigoplus_{t \in T'} (M_t)_M \to Q)$$

where for $t \neq 1$, $i_t(m, t) = (\sigma((\mu m)^t), [m, t])$.

Let us describe first how the Q-action is defined on this last crossed Q-module. Later we shall check the universal property.

The result we give is quite complicated, technical and non memorable. It is given principally because it illustrates the method, and also shows that these methods give control over quite complex actions in a way which seems to be unobtainable by traditional methods, which do not allow control of nonabelian structures.

Theorem 5.8.6 *The Q-action on the group $M \times \bigoplus_{t \in T'} (M_t)_M$ is given as follows,*

(i) For any $m \in M,\ q \in Q$

$$(m, 0)^q = \begin{cases} (m^q, 0) & \text{if } \nu = 1, \\ (\sigma((\mu m)^q), [m^r, \nu]) & \text{if } \nu \neq 1; \end{cases}$$

where $r \in P$ *and* $v, \in T$, *satisfy* $q = rv$ *and* $[m, v]$ *denotes the class of* (m, v) *in* $(M_v)_M$

(ii) If $m \in M, t \in T',\ q \in Q$ *then*

$$(1, [m, t])^q = \begin{cases} (1, [m^p, t]) & \textit{if } v = 1, \\ (\sigma(\mu m^p)^{-1} m^p, -[\sigma((\mu m^p)^{v^{-1}}), v]) & \textit{if } v \neq 1,\ u = 1, \\ (1, -[\sigma((\mu m^p)^{uv^{-1}}), v] + [m^p, u]) & \textit{if } v \neq 1,\ u \neq 1, \end{cases}$$

where $p \in P,\ u \in T$ *are the unique elements satisfying* $tq = pu$.

Proof We use the description of the morphisms associated to the coproduct structure given above to calculate the action given by Theorem 5.8.4.

The formulae (i) and (ii) for the case $v = 1$ follow from the description of the action of P on M_t given at the beginning of this section.

The remaining cases will be deduced from the formula for the action of Q given in Theorem 5.8.4, namely if $m \in M,\ t \in T,\ q \in Q$ then

$$(i_t(m, t))^q = \begin{cases} i_1(m^p, 1) = (m^p, 0), & \text{if } tq = p \in P, \\ i_u(m^p, u) = (\sigma((\mu m^p)^u), [m^p, u]), & \text{if } tq = pu,\ p \in P,\ u \in T'. \end{cases}$$

We first prove (i) for $v \neq 1$. We have since $q = rv,\ v \in T'$,

$$\begin{aligned} (m, 0)^q &= (i_1(m, 1))^{rv} \\ &= i_v(m^r, v) \\ &= (\sigma((\mu m^r)^v), [m^r, v]). \end{aligned}$$

To prove (ii) with $v \neq 1$, first note that

$$\begin{aligned} (1, [m, t]) &= (\sigma((\mu m)^t), 0)^{-1} (\sigma((\mu m)^t), [m, t]) \\ &= (\sigma((\mu m)^t), 0)^{-1} i_t(m, t). \end{aligned}$$

But

$$\begin{aligned} (\sigma((\mu m)^t), 0)^q &= (\sigma((\mu\sigma((\mu m)^t))^q), [(\sigma((\mu m)^t))^r, v]) \text{by (i)} \\ &= (\sigma((\mu m)^{tq}), [\sigma((\mu m)^{tr}), v]) \qquad \text{since } \mu\sigma = 1, \end{aligned}$$

and, from the definition of the Q-action,

$$(i_t(m, t))^q = \begin{cases} (m^p, 0) & \text{if } u = 1, \\ (\sigma((\mu m)^{tq}), [m^p, u]) & \text{if } u \neq 1. \end{cases}$$

It follows that

$$(1, [m, t])^q = \begin{cases} (\sigma(\mu m^p)^{-1} m^p, -[\sigma((\mu m^p)^{v^{-1}}), v]) & \text{if } u = 1, \\ (1, -[\sigma((\mu m^p)^{uv^{-1}}), v] + [m^p, u]) & \text{if } u \neq 1. \end{cases}$$

□

Now we check that the universal property is satisfied.

Theorem 5.8.7 *For any crossed module* $\mathcal{N} = (\nu : N \to Q)$ *and any morphism of crossed modules* $(\beta, \iota) : \mathcal{M} \to \mathcal{N}$, *the induced morphism* $\phi : M \times \bigoplus_{t \in T'} (M_t)_M \to N$ *is given by*

$$\phi(m, 0) = \beta m, \quad \phi(m, [n, v]) = (\beta m)\, \beta(\sigma((\mu n)^v))^{-1} (\beta n)^v.$$

Proof The formula for ϕ is obtained as follows:

$$\begin{aligned}\phi(m,[n,\nu]) &= \phi(m,0)\,\phi(\sigma((\mu n)^{\nu}),0)^{-1}\,\phi(i_{\nu}(n,\nu))\\ &= (\beta m)\,(\beta(\sigma((\mu n)^{\nu}))^{-1})\,(\beta n)^{\nu}\end{aligned}$$

where the definition of ϕ is taken from Theorem 5.8.4 □

We now include an example for Theorem 5.8.6 showing the action in the case $\nu \neq 1,\ u = 1$.

Example 5.8.8 Let n be an odd integer and let $Q = D_{8n}$ be the dihedral group of order $8n$ generated by elements $\{t, y\}$ with relators $\{t^{4n}, y^2, (ty)^2\}$. Let $P = D_{4n}$ be generated by $\{x, y\}$, and let $\iota : P \to Q$ be the monomorphism given by $x \mapsto t^2,\ y \mapsto y$. Then let $M = C_{2n}$ be generated by $\{m\}$. Define $\mathcal{M} = (\mu : M \to P)$ where $\mu m = x^2,\ m^x = m$ and $m^y = m^{-1}$. This crossed module is isomorphic to a sub-crossed module of $(D_{4n} \to \mathrm{Aut}(D_{4n}))$ and has kernel $\{1, m^n\}$.

The image μM is the cyclic group of order n generated by x^2, and there is an equivariant section $\sigma : \mu M \to M,\ x^2 \mapsto m^{n+1}$ since $(x^2)^{(n+1)} = x^2$ and $\gcd(n+1, 2n) = 2$. Then $Q = P \cup Pt$, $T = \{1, t\}$ is a transversal, M_t is generated by (m, t) and $\mu_t(m, t) = x^2$. The action of P on M_t is given by

$$(m,t)^x = (m,t), \quad (m,t)^y = (m^{-1},t).$$

Since M acts trivially on M_t,

$$\iota_* M \;\cong\; M \times M_t \;\cong\; C_{2n} \times C_{2n}.$$

Using the section σ given above, Q acts on $\iota_* M$ by

$$\begin{aligned}(m,0)^t &= (m^{n+1},[m,t]),\\ (m,0)^y &= (m^{-1},0),\\ (1,[m,t])^t &= (m^n,(n-1)[m,t]),\\ (1,[m,t])^y &= (1,-[m,t]).\end{aligned}$$

□

It is worth recalling that our objective was not only to get an easier expression of the induced crossed module, but also to have some information about the kernel of its boundary map. We can obtain some information on the later in the case where P is of index 2 in Q, even without the assumption that μM is normal in Q following [Bro80].

Suppose then that $T = \{1, t\}$ is a right transversal of P in Q. Let the morphism $M \ltimes M_t \to P$ be given as usual by $(m, (n, t)) \mapsto (\mu m)(\mu_t(n, t)) = mt^{-1}nt$.

Write $\langle M, M_t \rangle$ for the subgroup of $M \times_P M_t$ generated by the elements

$$\langle m,(n,t)\rangle = (m^{-1}m^{t^{-1}(\mu n)t}, ((n,t)^{-1})^m (n,t)),$$

for all $m \in M,\ (n, t) \in M_t$.

Proposition 5.8.9 *Let $\mu : M \to P$ and $\iota : P \to Q$ be inclusions of normal subgroups. Suppose that P is of index 2 in Q, and $t \in Q \setminus P$. Then the kernel of the induced crossed module $(\partial : \iota_* M \to Q)$ is isomorphic to*

$$(M \cap t^{-1}Mt)\,/\,[M, t^{-1}Mt].$$

In particular, if M is also normal in Q, then this kernel is isomorphic to $M/[M, M]$, i.e. to M made abelian.

Proof By previous results $\iota_* M$ is isomorphic to the coproduct crossed P-module $M \circ M_t$ with a further action of Q. The result follows from Corollary 4.3.8 □

We now give some topological applications of the last result.

Example 5.8.10 Let $\iota : P = D_{4n} \to Q = D_{8n}$ be as in Example 5.8.8, and let $M = D_{2n}$ be the subgroup of P generated by $\{x^2, y\}$, so that $\iota M \lhd \iota P \lhd Q$ and $t^{-1}Mt$ is isomorphic to a second D_{2n} generated by $\{x^2, yx\}$. Then
$$M \cap t^{-1}Mt = [M, t^{-1}Mt]$$
(since $[y, yx] = x^2$), and both are isomorphic to C_n generated by $\{x^2\}$.

It follows from Proposition 5.8.9 that if X is the homotopy pushout of the maps

$$\begin{tikzcd} BD_{4n} \arrow[r] \arrow[d] & BC_2 \arrow[d] \\ BD_{8n} \arrow[r] & X \end{tikzcd}$$

where the horizontal map is induced by $D_{4n} \to D_{4n}/D_{2n} \cong C_2$, then $\pi_2(X) = 0$. □

Example 5.8.11 Let M, N be normal subgroups of the group G, and let Q be the wreath product
$$Q = G \wr C_2 = (G \times G) \rtimes C_2.$$
Take $P = G \times G$, and consider the crossed module $(\partial : Z \to Q)$ induced from $M \times N \to P$ by the inclusion $P \to Q$. If t is the generator of C_2 which interchanges the two factors of $G \times G$, then $Q = P \cup Pt$ and $t^{-1}(M \times N)t = N \times M$. So
$$(M \times N) \cap t^{-1}(M \times N)t = (M \cap N) \times (N \cap M)$$
and
$$[M \times N, N \times M] = [M, N] \times [N, M].$$

It follows that if X is the homotopy pushout of

$$\begin{tikzcd} BG \times BG \arrow[r] \arrow[d] & B(G/M) \times B(G/N) \arrow[d] \\ B(G \wr C_2) \arrow[r] & X \end{tikzcd}$$

then
$$\pi_2(X) \cong ((M \cap N)/[M, N])^2.$$
If $([m], [n])$ denotes the class of $(m, n) \in (M \cap N)^2$ in $\pi_2(X)$, the action of Q is determined by
$$([m], [n])^{(g,h)} = ([m^g], [n^h]), \ (g, h) \in P, \quad ([m], [n])^t = ([n], [m]).$$
□

We end this section by giving a very concrete description of the induced crossed module in the case that both M and P are normal subgroups of Q and $M \subseteq P$. It is proved by a direct verification of the universal property for an induced crossed module.

There are two construction used in the description. The first one is the abelianisation M^{ab} of a group M. If $n \in M$, then the class of n in M^{ab} is written $[n]$.

The second construction is the augmentation ideal IQ of a group Q, which we further develop later on. For now let us say that the augmentation ideal $I(Q/P)$ of a quotient group Q/P has basis $\{\bar{t} - 1 \mid t \in T'\}$ where T is a transversal of P in Q, $T' = T \setminus \{1\}$ and $\bar{q}$ denotes the image of q in Q/P.

Theorem 5.8.12 *Let $M \subseteq P$ be normal subgroups of Q, so that Q acts on P and M by conjugation. Let $\mu : M \to P,\ \iota : P \to Q$ be the inclusions and let $\mathcal{M} = (\mu : M \to P)$. Then the induced crossed Q-module $\iota_* \mathcal{M}$ is isomorphic as a crossed Q-module to*

$$(\zeta : M \times (M^{ab} \otimes I(Q/P)) \to Q)$$

where for $m, n \in M,\ x \in I(Q/P)$:

(i) $\zeta(m, [n] \otimes x) = m$;

(ii) the action of Q is given by

$$(m, [n] \otimes x)^q = (m^q, [m^q] \otimes (\bar{q} - 1) + [n^q] \otimes x\bar{q}).$$

The universal map $i : M \to M \times (M^{ab} \otimes I(Q/P))$ is given by $m \mapsto (m, 0)$.

Proof This could be proved directly (see [BW96]) but instead, in view of what has already been set up, we will deduce it from Theorem 5.8.6. Specialising this theorem to the current situation, in which $\sigma\mu = 1$ and $i_t(m, t) = (m^t, [m, t])$, yields an isomorphism of crossed Q-modules

$$\iota_* \mathcal{M} \to \mathcal{X} = (\xi = \iota\mu\, \mathrm{pr}_1 : M \times \bigoplus_{t \in T'} (M^{ab}) \to Q).$$

In $\mathcal{X}$ the action of Q is given as follows, where $m \in M,\ r \in P,\ q = rv$ and $v \in T$:

(i)

$$(m, 0)^q = \begin{cases} (m^q, 0) & \text{if } v = 1, \\ (m^q, [m^r, v]) & \text{if } v \neq 1. \end{cases}$$

(ii) if $tq = pu,\ t \in T',\ p \in P$ and $u \in T$, then

$$(1, [m, t])^q = \begin{cases} (1, [m^p, t]) & \text{if } \ v = 1, \\ (1, -[m^{pv^{-1}}, v]) & \text{if } \ v \neq 1,\ u = 1, \\ (1, -[m^{puv^{-1}}, v] + [m^p, u]) & \text{if } \ v \neq 1,\ u \neq 1. \end{cases}$$

Now we construct an isomorphism

$$\omega : M \times \bigoplus_{t \in T'} (M^{ab}) \to M \times (M^{ab} \otimes I(Q/P))$$

where for $m, n \in M,\ t \in T'$,

$$\omega(m, 0) = (m, 0), \quad \omega(m, [n, t]) = (m, [n^t] \otimes (\bar{t} - 1)).$$

Clearly ω is an isomorphism of groups, since it is an isomorphism on the part determined by a fixed $t \in T'$, and $I(Q/P)$ has a basis $\{\bar{t} - 1 : t \in T'\}$ when considered as an abelian group. Now we prove that ω preserves the action of Q. Let $m, n \in M,\ t \in T',\ q \in Q$. Let $q = rv,\ tq = pu,\ p, r \in P,\ u, v \in T$. When $v = 1$ we have $tqt^{-1} \in P$ and so $u = t$. Then

$$\begin{aligned} \omega((m, 0)^q) &= \begin{cases} \omega(m^q, 0) & \text{if } \ v = 1, \\ \omega(m^q, [m^r, v]) & \text{if } \ v \neq 1. \end{cases} \\ &= \begin{cases} (m^q, 0) & \text{if } \ v = 1, \\ (m^q, [m^q] \otimes (\bar{v} - 1) & \text{if } \ v \neq 1, \end{cases} \\ &= (\omega(m, 0))^q. \end{aligned}$$

Further,

$$\begin{aligned}
\omega((1,[m,t])^q) &= \begin{cases} \omega(1,[m^p,t]) & \text{if} \quad v=1, \\ \omega(1,-[m^{pv^{-1}},v]) & \text{if} \quad v\neq 1,\ u=1, \\ \omega(1,-[m^{puv^{-1}},v]+[m^p,u]) & \text{if} \quad v\neq 1,\ u\neq 1, \end{cases} \\
&= \begin{cases} (1,[m^{pt}]\otimes(\bar{t}-1)) & \text{if} \quad v=1, \\ (1,-[m^p]\otimes(\bar{v}-1)) & \text{if} \quad v\neq 1,\ u=1, \\ (1,-[m^{pu}]\otimes(\bar{v}-1)+[m^{pu}]\otimes(\bar{u}-1)) & \text{if} \quad v\neq 1,\ u\neq 1, \end{cases} \\
&= (1,-[m^{pu}]\otimes(\bar{v}-1)+[m^{pu}]\otimes(\bar{u}-1) \qquad \text{in every case}, \\
&= (1,[m^{tq}]\otimes(\bar{t}-1)\bar{q}), \\
&= (\omega(1,[m,t]))^q
\end{aligned}$$

since, in $I(Q/P)$,

$$(\bar{t}-1)\bar{q} = \overline{pu} - \overline{rv} = \bar{u} - \bar{v} = (\bar{u}-1) - (\bar{v}-1).$$

Finally, we have to compute the universal extension ϕ of β. For this, it is sufficient to determine

$$\begin{aligned}
\phi(1,[n]\otimes(\bar{q}-1)) &= \phi\omega(1,[n^{v^{-1}},v]) \\
&= \phi\omega((n^{-1},0)\, i_v(n^{v^{-1}},v)) \\
&= \beta(n^{-1})\,\beta(n^{v^{-1}})^v \\
&= \beta(n^{-1})\,\beta(n^{q^{-1}})^q
\end{aligned}$$

since β is a P-morphism and $\bar{q} = \overline{rv} = \bar{v}$. □

With this description, we can get new results on the fundamental crossed module of a space which is the pushout of classifying spaces. The following corollary is immediate.

Corollary 5.8.13 *Under the assumptions of the theorem, let us consider the space $X = BQ \cup_{BP} B(P/M)$. Its fundamental crossed module $\Pi_2(X, BQ)$ is isomorphic to the above crossed Q-module*

$$(\zeta : M \times (M^{ab} \otimes I(Q/P)) \to Q).$$

In particular, the second homotopy group $\pi_2(X)$ is isomorphic to $M^{ab} \otimes I(Q/P)$ as Q/M-module.

Proof The proof is immediate. □

Note one of our major arguments: in order to compute an abelian second homotopy group, we may have to use nonabelian algebraic methods which better reflect the structure of the problem than the usual abelian methods.

Corollary 5.8.14 *In particular, if the index $[Q : P]$ is finite, and $\mathcal{P}$ is the crossed module $(1 : P \to P)$, then $\iota_* \mathcal{P}$ is isomorphic to the crossed module $(\mathrm{pr}_1 : P \times (P^{ab})^{[Q:P]-1} \to Q)$ with action as above.*

Remark 5.8.15 In this case, $X = BQ \cup_{BP} B(P/P)$ may be interpreted either as the space obtained from BQ by collapsing BP to a point, or, better, as $X = BQ \cup_{BP} CB(P)$ the space got by attaching a cone. This is a consequence of the gluing theorem for homotopy equivalences proved in [Bro06].

This crossed module is not equivalent to the trivial one. At first sight, it seems that the projection

$$\mathrm{pr}_2 : P \times (P^{ab} \otimes I(Q/P)) \to (P^{ab} \otimes I(Q/P))$$

determines a morphism of crossed modules to the trivial one $0 : (P^{ab} \otimes I(Q/P)) \to I(Q/P))$, but this is not so because the map pr_2 is not a Q-morphism.

We are going to show later that this crossed module is not equivalent in a certain sense to the projection crossed module. □

We have now completed the applications of the 2-dimensional van Kampen Theorem which we will give in this book. In the next chapter we give the proof of the theorem, using the algebraic concepts of double groupoids. In the next section, we explain how the computer algebra system GAP has been used to give further computations of induced crossed modules, and of course these have topological applications according to the results of this chapter.

5.9 Computational issues for induced crossed modules

The following discusses significant aspects of the computation of induced crossed modules. Let us consider the description of the induced module from a computational point of view. It involves the copower, i.e. a free product of groups. This usually gives infinite groups, but let us consider how to get a finite presentation in the case $M \subseteq P \subseteq Q$.

If $M = \langle X \mid R\rangle$ is a finite presentation of M, there is a finite presentation of M^{*T} with $|X||T|$ generators and $|R||T|$ relations.

Let X^P be the closure of X under the action of P. Then $\iota_*(M) = (M^{*T})/N$ where N is the normal closure in M^{*T} of the elements

$$\langle (m,t),(n,u)\rangle = (m,t)^{-1}(n,u)^{-1}(m,t)(n,u)^{\delta(m,t)} \quad (m,n \in \Sigma^P,\ t,u \in T). \tag{5.9.1}$$

The homomorphism ι_* is induced by the projection $\mathrm{pr}_{()}m = (m, ()\ )$ onto the first factor, and the boundary δ of $\iota_*\mathcal{M}$ is induced from δ' as shown in the following diagram:

$$\begin{tikzcd} M \arrow[r, "\iota_*"] \arrow[d, "\mu"'] & (M^{*T})/N \arrow[d, "\delta"] \\ P \arrow[r, "\iota"'] & Q \end{tikzcd}$$

When Σ is a set and $\sigma : \Sigma \to Q$ any map, take $M = P = F(\Sigma)$ to be the free group on Σ and let $\mathcal{F}_\Sigma = (\mathrm{id}_{F(\Sigma)} : F(\Sigma) \to F(\Sigma)\ )$. Then σ extends uniquely to a homomorphism $\sigma' : F(\Sigma) \to Q$ and $\sigma'_*\mathcal{F}_\Sigma$ is the free crossed module $\mathcal{F}_\sigma$ described in section 3.4. However, computation in free crossed modules is in general difficult since the groups are usually infinite.

So, in order to compute the induced crossed module $\iota_*\mathcal{M}$ for $\mathcal{M} = (\mu : M \to P)$ a conjugation crossed module and $\iota : P \to Q$ an inclusion, we construct finitely presented groups FM, FP, FQ isomorphic to the permutation groups M, P, Q and monomorphisms $FM \to FP \to FQ$ mimicking the inclusions $M \to P \to Q$.

As well as returning an induced crossed module, the construction should return a morphism of crossed modules $(\iota_*, \iota) : \mathcal{M} \to \iota_*\mathcal{M}$.

A finitely presented form FC for the copower M^{*T} is constructed with $|X||T|$ generators. The relators of FC comprise $|T|$ copies of the relators of FM, suitably renumbered.

The inclusion δ' maps the generators of FM to the first $|X|$ generators of FC. A finitely presented form FI for ι_*M is then obtained by adding to the relators of FC further relators corresponding to the list of elements in equation (5.9.1).

Then we can apply some Tietze transformations to the resulting presentation. During the resulting simplification, some of the first $|X|$ generators may be eliminated, so the projection $\mathrm{pr}_{()}$ may be lost. In order to preserve this projection, and so obtain the morphism ι_*, it is necessary to record for each eliminated generator g a relator gw^{-1} where w is the word in the remaining generators by which g was eliminated.

The Tietze transformation code in GAP was modified so that the resulting presentation **presI** contained an additional field **presI.remember**, namely a list of (at least) $|X||T|$ relators expressing the original generators in terms of the final ones. (In the recent release 3.4.4 of GAP this facility has been made generally available using the **TzInitGeneratorImages** function).

Let us see how this process works in some examples, and note some of the limitations of the process.

Recall that a *polycyclic group* is a group G with power-conjugate presentation having generators $\{g_1, \ldots, g_n\}$ and relations

$$\{g_i^{o_i} = w_{ii}(g_{i+1}, \ldots, g_n), \quad g_i^{g_j} = w'_{ij}(g_{j+1}, \ldots, g_n) \quad \forall\, 1 \leqslant j < i \leqslant n\}. \tag{5.9.2}$$

(These are implemented in GAP as **AgGroups** (see [Gro02], Chapters 24, 25)). Since subgroups $M \leqslant P \leqslant G$ have induced power-conjugate presentations, if T is a transversal for the right cosets of P in G, then the relators of M^{*T} are all of the form in (5.9.2).

Furthermore, all the Peiffer relations in equation (5.9.1) are of the form $g_i^{g_j} = g_k^p$, so one might hope that a power conjugate presentation would result. Consideration of the cyclic-by-cyclic case in the following example shows that this does not happen in general.

Example 5.9.1 Let C_n be cyclic of order n and let $\alpha : x \mapsto x^a$ be an automorphism of C_n of order p. Take $G = \langle g, h \mid g^p, h^n, h^g h^{-a}\rangle \cong C_p \ltimes C_n$. It follows from these relators that $h^i g = gh^{ai}$, $0 < i < n$ and that $h^{-1}(gh^{i(1-a)})h = gh^{(i+1)(1-a)}$. So if we put $g_i = gh^{i(1-a)}$, $0 \leqslant i < n$ then $g_i^{g_j} = g_{[j+a(i-j)]}$. When $M = P = C_n \lhd G$ Theorem 5.8.12 apply, and $\iota_* P \cong C_n^m$. Now take $M = P = C_p$, with power-conjugate form $\langle g \mid g^p\rangle$, and $\iota : C_p \to G$. We may choose as transversal $T = \{\lambda, h, h^2, \ldots, h^{n-1}\}$, where λ is the empty word. Then M^{*T} has generators $\{(g, h^i) \mid 0 \leqslant i < n\}$, all of order p, and relators $\{(g, h^i)^p \mid 0 \leqslant i < n\}$. The additional Peiffer relators in equation (5.9.1) have the form

$$(g, h^i)(g, h^j) = (g, h^j)(g^k, h^l) \;\text{ where }\; h^i h^{-j} g h^j = g^k h^l$$

so $k = 1$ and $l = [j + a(i-j)]$. Hence $\theta : \iota_* M \to Q,\ \ (g, h^i) \mapsto g_i$ is an isomorphism, and $\iota_*\mathcal{M}$ is isomorphic to the identity crossed module on Q. Furthermore, if we take M to be a cyclic subgroup C_m of C_p then $\iota_*\mathcal{M}$ is the conjugation crossed module $(\partial : C_m \ltimes C_n \to C_p \ltimes C_n)$. □

Also, we know that many of the induced groups $\iota_* M$ are direct products. However the generating sets in the presentations that arise following the Tietze transformation do not in general split into generating sets for direct summands. This is clearly illustrated by the following simple example.

Example 5.9.2 Let $Q = S_4$, the symmetric group of degree 4, and $M = P = A_4$, the alternating subgroup of Q of index 2. Since the abelianisation of A_4 is cyclic of order 3, Theorem 5.8.12 shows that $\iota_* M \cong A_4 \times C_3$. However a typical presentation for $A_4 \times C_3$ obtained from the program is

$$\langle x, y, z \mid x^3, y^3, z^3, (xy)^2, zy^{-1}z^{-1}x^{-1}, yzyx^{-1}z^{-1}, y^{-1}x^2y^2x^{-1}\rangle,$$

and one generator for the C_3 summand is yzx^2. Converting to an isomorphic permutation group H gives a degree 12 representation with generating set

$$\{(2,9,4)(3,5,6)(8,12,10),\ (1,4,2)(3,5,7)(10,11,12),\ (1,8,3)(2,10,5)(7,9,12)\}.$$

Converting H to an AgGroup produces a 4-generator group with subnormal series $A_4 \times C_3 > A_4 > C_2^2 > C_2 > I$, and $g_1 g_2 g_4$ is a generator for the normal C_3. After conversion of this AgGroup to a SpecialAgGroup, the corresponding generator is g_1g_2. In all these representations, the cyclic summand remains hidden, and an explicit search among the normal subgroups must be undertaken to find it. □

We finish the results obtained in our computation by listing all the induced crossed modules coming from subgroups of groups of order at most 23 (excluding 16) which are not covered by the special cases mentioned earlier. This enables us to exclude abelian and dihedral groups, cases $P \lhd Q$ and $Q \cong C_m \ltimes C_n$.

In the first table, we assume given an inclusion $\iota : P \to Q$ of a subgroup P of a group Q, and a normal subgroup M of P. We list the crossed module $\iota_* M$ induced from $(\mu : M \to P)$ by the inclusion ι. The kernel of $\partial : \iota_* M \to Q$ is written $\nu_2(\iota)$. This kernel is related to the second homotopy group in the topological application (in some cases like Theorems 5.4.4 and 5.4.7 it is exactly the second homotopy group).

In this table the labels I, C_n, D_{2n}, A_n, S_n denote the identity, cyclic, dihedral, alternating and symmetric groups of order $1, n, 2n, n!/2$ and $n!$ respectively. The group H_n is the holomorph of C_n and H_n^+ is its positive subgroup in degree n. $SL(2,3)$ and $GL(2,3)$ are the special and general linear groups of order $24, 48$ respectively. Labels of the form **m.n** refer to the nth group of order m according to the GAP numbering.

Table 1

$\|Q\|$	M	P	Q	$\iota_* M$	$\nu_2(\iota)$
12	C_2	C_2	A_4	H_8^+	C_4
	C_3	C_3	A_4	$SL(2,3)$	C_2
18	C_2	C_2	$C_2 \ltimes C_3^2$	54.10	C_3
	S_3	S_3	$C_2 \ltimes C_3^2$	54.10	C_3
20	C_2	C_2	H_5	D_{10}	C_2
	C_2	C_2^2	D_{20}	D_{10}	I
	C_2^2	C_2^2	D_{20}	D_{20}	I
21	C_3	C_3	H_7^+	H_7+	I

The second table contains the results of calculations with $Q = S_4$, where $C_2 = \langle (1,2) \rangle$, $C_2' = \langle (1,2)(3,4) \rangle$, and $C_2^2 = \langle (1,2), (3,4) \rangle$. The final column contains the automorphism group $\mathrm{Aut}(\iota_* M)$ (where known).

Table 2

M	P	$\iota_* M$	$\nu_2(\iota)$	$\mathsf{Aut}(\iota_* M)$
C_2	C_2	$GL(2,3)$	C_2	S_4C_2
C_3	C_3	$C_3\ SL(2,3)$	C_6	$144.?$
C_3	S_3	$SL(2,3)$	C_2	S_4
S_3	S_3	$GL(2,3)$	C_2	S_4C_2
C_2'	C_2'	$128.?$	$C_4C_2^3$	
C_2'	C_2^2, C_4	H_8^+	C_4	S_4C_2
C_2'	D_8	C_2^3	C_2	$SL(3,2)$
C_2^2	C_2^2	S_4C_2	C_2	S_4C_2
C_2^2	D_8	S_4	I	S_4
C_4	C_4	96.219	C_4	96.227
C_4	D_8	S_4	I	S_4
D_8	D_8	S_4C_2	C_2	S_4C_2

Notes

[9]p. 86 The results of this chapter are taken mainly from [BH78, BW95, BW96, BW03].

General

There are a number of papers which prove Whitehead's theorem 5.4.8, e.g. [Rat80], but it is not generally acknowledged that the theorem is a consequence of a van Kampen type theorem in dimension 2, a theorem which is not mentioned in, say, [HAM93], although other deductions from it are given. A modern account of Whitehead's proof is given in [Bro80].

Two recent papers using crossed modules are [Far08, FK08].

Remark 5.9.3 Note that in Theorem 5.5.2 we obtain immediately a result on the second absolute homotopy group of $Y \cup C(A)$ without using any homology arguments. This is significant because the setting up of singular homology, proving all its basic properties, and proving the absolute Hurewicz theorem takes a considerable time. An exposition of the Hurewicz theorems occurs on pages 166-180 of G. Whitehead's text [Whi78], assuming the properties of singular homology. The cubical account of singular homology in [Mas80] fits best with our story.

Again, one of the reason for emphasising these kinds of results is that they arise from a uniform procedure, which involves first establishing a Higher Homotopy van Kampen Theorem. This theorem has analogues for algebraic models of homotopy types which are more elaborate than just groups or crossed modules; it has led to new results, such as a higher order Hopf formula [BE88], which is deduced from an $(n+1)$-adic Hurewicz Theorem [BL87a]. The only proof known of the last result is as a deduction from a van Kampen Theorem for n-cubes of spaces [BL87b]. It has also stimulated research into related areas.

Chapter 6

Double groupoids and the 2-dimensional van Kampen Theorem.

In Chapter 2 we saw that an important topological example of crossed module was provided by the fundamental crossed module of a based pair of spaces

$$\Pi_2(X, A)(x) = (\partial : \pi_2(X, A, x) \to \pi_1(A, x)).$$

As in the case of the fundamental group, to prove the 2-dimensional van Kampen Theorem for crossed modules, it is interesting, even necessary, to include in the same structure all the fundamental crossed modules when varying the base point $x \in A$. In the 1-dimensional case, we generalised the fundamental group to the fundamental groupoid. To prove a 2-dimensional van Kampen Theorem the idea was to use double groupoids but it took some time to find the required 2-dimensional analogue of the fundamental group. After a good deal of trying a structure that gives the 2-dimensional van Kampen Theorem happens to be the double groupoids with connection or, equivalently, the crossed module over a groupoid.

Now the question can be fairly put: Why introduce a new version? The answer is the usual kind of answer, that sometimes the new version is useful for proving theorems. In particular, we are unable to prove directly in terms of crossed modules the version of the 2-dimensional van Kampen Theorem which gives a result in terms of the classical crossed modules. One reason for conceiving of the homotopy double groupoid was to find an algebraic gadget more appropriate than groups for giving an

algebraic inverse to subdivision.

This is the slogan underlying the work on Higher Homotopy van Kampen Theorems. Subdividing a square into little squares has a convenient expression in terms of double groupoids, and much more inconvenient expressions, if they exist at all, in terms of crossed modules. The 2-dimensional van Kampen Theorem was conceived first in terms of double groupoids, and it was only gradually that the link with crossed modules was realised. In the end, the aim of obtaining Whitehead's Theorem on free crossed modules (Corollary 5.4.8) as a corollary was an important impetus to forming a definition of a homotopy double groupoid for a pointed *pair* of spaces, since that theorem involved a crossed module defined for such a pair of spaces.

Further, the structure of double groupoids that we use was expressly sought in order to make valid Lemma 6.8.4 in the proof of our 2-dimensional van Kampen Theorem in the last section. This lemma shows that a construction of an element of a double groupoid is independent of all the choices made. This makes use of the notion of commutative cube in a crucial way.

This theory gives also in a sense an *algebraic formulation* of different ways which have been classically used and found necessary in considering properties of second relative homotopy groups. We find that the 2-dimensional double groupoid viewpoint is useful both for understanding the theory and for proving theorems, while the crossed module viewpoint is useful both for specific calculations, and because of its closer relation to chain complexes. The importance of the algebraic formulation of the equivalence between crossed modules and double groupoids is the equivalence between colimits, and in particular pushouts, in the two categories.

Since this is a longish chapter, it seems a good idea to include a more detailed sketch of the way that all this material is presented here.

The first part describes the step up one dimension from groupoids to double groupoids. Since these are double categories where all structures are groupoids and have either a connection pair or a thin structure the first few sections are devoted to defining first double categories and then connections. In parallel another algebraic category is described, that of crossed modules over groupoids, which is equivalent to that of double groupoids. The equivalence is finally proved in Section 6.6

The first Section gives the definition and properties of double categories. Some notions to be used later are also presented here, e.g. the double category of commutative squares or 2-shells in a groupoid.

With this model in mind, we can think of the elements of a double category D as squares. Also, we can restrict our attention to the subspace γD of "squares" having all faces trivial but the top one.

If we restrict ourselves to double categories G that have all three structures groupoids, the space γG is algebraically a crossed module over a groupoid. These algebraic structures are studied in Section 6.2 and they are an easy step away from that of a crossed module over a group.

A direct topological example is the fundamental crossed module of a triple of topological spaces (X, A, C) formed by all the crossed modules $\partial : \pi_2(X, A, x) \to \pi_1(A, x)$ for varying $x \in C$. We denote this crossed module by $\Pi_2(X, A)$ and we shall prove that it is a crossed module in an indirect way by showing in Proposition 6.3.7 that $\Pi_2(X, A)$ is the crossed module associated to the fundamental double groupoid of a triple $\rho(X, A, C)$ defined in Section 6.3.

Both the fundamental crossed module of a triple and the double category of commutative 2-shells on a groupoid have some extra structure that can be defined in two equivalent ways: as a *thin structure* (as in Section 6.4) and as a *connection pair* (in Section 6.5). In this way we define the objects in the category of double groupoids.

Using 2-shells that 'commute up to some element', in Section 6.6 we associate to each crossed module $\mathcal{M}$ a double groupoid $\lambda\mathcal{M}$ in such a way that it is clear that $\gamma\lambda\mathcal{M}$ is naturally isomorphic to $\mathcal{M}$. It is a bit more challenging to prove that for any double groupoid G, $\lambda\gamma G$ is also naturally isomorphic to G. In order to do this we use the folding operation $\Phi : G_2 \to G_2$ which has the effect of folding all faces of an element of G_2 into the top face.

With all the algebra in place, we turn to the topological part. As seen in Chapter 1, the proof of the 2-dimensional van Kampen Theorem uses the homotopy commutativity of squares. Thanks to the algebraic machinery developed earlier, we can talk about commutative 3-cubes and prove that any composition of commutative cubes is commutative. This commutativity of the boundary of a cube in $\rho(X, A, C)$ has a homotopy meaning stated in Section 6.7 which is analogous to the 1-dimensional case.

We finish this chapter by giving in Section 6.8 a proof of the 2-dimensional van Kampen Theorem for the fundamental double groupoid and the main consequences.

The whole chapter can be seen as an introduction to the generalisation to all dimensions which is carried out in Part III. Chapter 13 generalises the algebraic part by giving an equivalence between crossed complexes and cubical ω-groupoids with connections, while Chapter 14 covers the topological part, including the statement, proof and applications of the HHvKT.

6.1 Double categories

Let us start by pointing out that there are several candidates for the name "double groupoids". We are going to keep that name for the structures which are defined in Section 6.4 and are then used to prove the 2-dimensional van Kampen Theorems. We start by investigating what a double category should be.

It is interesting to think of a category in a different way that lends itself better to the generalisation to higher dimensions. As seen in the Appendix A, a category C is given by two sets: the set of objects that we denote C_0 and the set of morphisms that we call C_1; three maps among them: the source $\partial^- : C_1 \to C_0$, target $\partial^+ : C_1 \to C_0$ and identity $1 = \varepsilon : C_0 \to C_1$, satisfying

$$\partial^\sigma \varepsilon = \mathrm{Id}, \sigma = \pm$$

and a partial composition $C_1 \times_{C_0} C_1 \to C_1$ that is associative and has $1_x = \varepsilon(x)$ as right and left identity.

Thus we can think of the elements of C_0 as 0-dimensional, called points, and the elements of C_1 as 1-dimensional and oriented, called arrows. An element $a \in C_1$ is represented by

$$\begin{array}{ccc} & a & \\ \bullet & \text{———————} & \bullet \\ \partial^- a & & \partial^+ a \end{array}$$

and for any $x \in C_0$ its identity $1_x = \varepsilon(x)$ is

$$\begin{array}{ccc} & 1_x & \\ \bullet & \text{———————} & \bullet \\ x & & x \end{array}$$

The composition ab of two elements $a, b \in C_1$ is described by juxtaposition:

$$\begin{array}{ccccc} & a & & b & \\ \bullet & \text{———————} & \bullet & \text{———————} & \bullet \\ \partial^- a & & \partial^+ a = \partial^- b & & \partial^+ b \end{array} \qquad = \qquad \begin{array}{ccc} & ab & \\ \bullet & \text{——————————} & \bullet \\ \partial^-(ab) & & \partial^+(ab) \end{array}$$

This gives a 1-dimensional pictorial description of a category.

For a 2-dimensional generalisation, namely a double category D, apart from the sets of "points", D_0 and of "arrows", D_1, we need a set of "squares", D_2. We shall also have two categories associated to the "horizontal" and "vertical" structures on squares, with their faces and compositions. Also, we should have all the appropriate compatibility conditions dictated by the geometry. In some sense these categories are special since the objects of the horizontal and the vertical category structures on squares are the same; in other words, the horizontal and vertical edges of the squares come from the same category. This is the case we need in this book.

Thus we think of an element $u \in D_2$ as a square

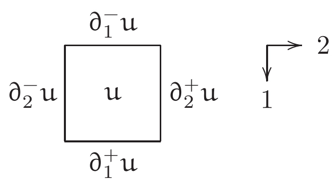
$$\begin{array}{c|c|c} \multicolumn{3}{c}{c} \\ \hline a & \quad u \quad & d \\ \hline \multicolumn{3}{c}{b} \end{array} \qquad \begin{tikzcd} \cdot \arrow[r] \arrow[d] & 2 \\ 1 & \end{tikzcd}$$

where the directions are labeled as indicated, and we call a, b, c, d the *edges, or faces* of u.

Let us make it formal.

Definition 6.1.1 A *double category* is given by three sets D_0, D_1 and D_2 and three structures of category. The first one on (D_1, D_0) has maps ∂^-, ∂^+ and ε and composition denoted as multiplication. The other two are defined on (D_2, D_1), a "vertical" one with maps $\partial_1^-, \partial_1^+$ and ε_1 and composition denoted by $u +_1 w$ and the "horizontal" one with maps $\partial_2^-, \partial_2^+$ and ε_2 and composition denoted by $u +_2 v$, satisfying some conditions.

Before describing the compatibility conditions it is worth getting used to the diagrammatic expression of the elements in a double category. Thus an element $u \in D_2$ is represented using a matrix like convention

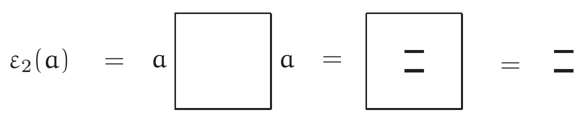


where the labels on the sides are given as indicated.

From this representation it seems indicated, and we assume, that the sources and targets have to satisfy

$$\partial^\tau \partial_1^\sigma = \partial^\sigma \partial_2^\tau \quad \text{for} \quad \sigma, \tau = \pm, \tag{DC 1}$$

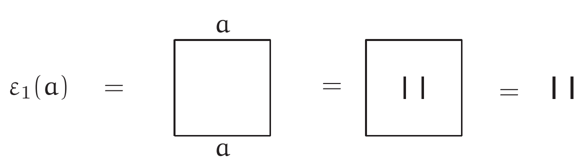
since they represent the same vertex. We shall find it convenient to represent the horizontal identity in several ways, i.e.



In the first representation the unlabeled sides are identities:

$$\partial_1^\sigma \varepsilon_2 = \varepsilon \partial^\sigma \quad \text{for} \quad \sigma = \pm. \tag{DC 2.1}$$

In the other two, the sides corresponding to those drawn in the middle are identities. Similarly, the vertical identity is represented by



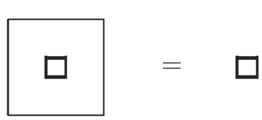
with the same conventions as before. It has also the expected faces in the horizontal direction:

$$\partial_2^\sigma \varepsilon_1 = \varepsilon \partial^\sigma \quad \text{for} \quad \sigma = \pm. \tag{DC 2.2}$$

There are also some relations between the identities. The two double degenerate maps are the same and are denoted by 0:

$$\varepsilon_2 \varepsilon = \varepsilon_1 \varepsilon = 0. \tag{DC 3}$$

So $0_x = 0(x)$ is both a horizontal and a vertical identity and is represented as



All elements $\varepsilon(x)$, $\varepsilon_1(a)$, $\varepsilon_2(a)$ are called *degeneracies*.

The vertical and horizontal compositions can be represented by "juxtaposition" in the corresponding direction, and are indicated by:

$$u +_1 w = \begin{array}{|c|}\hline u \\ \hline w \\ \hline\end{array} \qquad\qquad u +_2 v = \begin{array}{|c|c|}\hline u & v \\ \hline\end{array}$$

They satisfy all the usual rules of a category, and may be given a diagrammatic representation. For example, the fact that ε_2 is the horizontal identity may be represented as

$$\begin{array}{|c|c|}\hline u & = \\ \hline\end{array} \quad = \quad \begin{array}{|c|c|}\hline = & u \\ \hline\end{array} \quad = \quad \begin{array}{|c|}\hline u \\ \hline\end{array}$$

The composition in one direction satisfies compatibility conditions with respect to the faces and degeneracies in the other direction, i.e. these functions are homomorphisms. This can be read from the representation. Thus the horizontal faces of a vertical composition are

$$\partial_2^\sigma(u +_1 v) = (\partial_2^\sigma u)(\partial_2^\sigma v) \quad \text{for} \quad \sigma = \pm. \tag{DC 4.1}$$

and the vertical faces of the horizontal composition are

$$\partial_1^\sigma(u +_2 v) = (\partial_1^\sigma u)(\partial_1^\sigma v) \quad \text{for} \quad \sigma = \pm. \tag{DC 4.2}$$

The same applies to the vertical and horizontal identities, i.e.

$$\varepsilon_2(ab) = \varepsilon_2(a) +_1 \varepsilon_2(b), \tag{DC 5.1}$$

$$\varepsilon_1(ab) = \varepsilon_1(a) +_2 \varepsilon_1(b). \tag{DC 5.2}$$

Our final compatibility condition is known as the "interchange law" and says that, when composing 4 elements in a square, it is irrelevant if we compose first in the horizontal direction and then in the vertical one, or the other way around, i.e.

$$(u +_2 v) +_1 (w +_2 x) = (u +_1 w) +_2 (v +_1 x) \tag{DC 6}$$

when both sides are defined. This can be represented as giving only one way of evaluating the double composition

$$\begin{array}{|c|c|}\hline u & v \\ \hline w & x \\ \hline\end{array}$$

To complete the description of the category of double categories, a *double functor* between two double categories D and D' is given by three maps $F_i : D_i \to D'_i$ for $i = 0, 1, 2$ which commute with all structure maps (faces, degeneracies, composition, etc.). In particular, the pair (F_1, F_0) gives a functor from (D_1, D_0) to (D'_1, D'_0).

With these objects and morphisms, we get the category DCat of double categories. $\square$

Remark 6.1.2 Thus a double category has a structure which is called a *2-truncated cubical set with compositions*. Properties (DC 1-3) give the 2-truncated cube structure and (DC 4-6) the compatibility with compositions. □

Remark 6.1.3 On matrix notation. There is also a matrix notation for the compositions which will be useful later on and is as follows:

$$u +_1 w = \begin{bmatrix} u \\ w \end{bmatrix} \qquad u +_2 v = [u, v].$$

With this notation we can represent all the rules in the definition of double categories. For instance, we have

$$\begin{bmatrix} u \\ \| \end{bmatrix} = [\; u, \;\; = \;] = u.$$

Choosing the matrix description, the 'interchange law' (DC 6) may be written

$$\left[\begin{bmatrix} u \\ w \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix} \right] = \left[\begin{matrix} [\, u \;\; w \,] \\ [\, w \;\; x \,] \end{matrix} \right]$$

This common value is represented by

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix}.$$

Here is a caution about using this interchange law. Let u, v be squares in a double category such that

$$w = \begin{bmatrix} u & v \end{bmatrix} = u +_2 v$$

is defined. Suppose further that

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1 +_1 v_1 \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_2 +_1 v_2.$$

Then we can assert

$$w = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

only when $u_1 +_2 v_1$, and $u_2 +_2 v_2$ are defined. Thus care is needed in 2-dimensional rewriting. □

This matrix notation has a generalisation that we are going to use in proving several equalities.

Definition 6.1.4 Let D be a double category. A *composable array* (u_{ij}) in D, is given by elements $u_{ij} \in D_2$ $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$ satisfying

$$\begin{cases} \partial_2^+ u_{i,j-1} = \partial_2^- u_{i,j} & (1 \leqslant i \leqslant m, 2 \leqslant j \leqslant n), \\ \partial_1^+ u_{i-1,j} = \partial_1^- u_{i,j} & (2 \leqslant i \leqslant m, 1 \leqslant j \leqslant n). \end{cases}$$

It follows from the interchange law that a composable array (u_{ij}) in D can be composed both ways, getting the same result which is denoted by $[u_{ij}]$.

If $u \in D_2$, and (u_{ij}) is a composable array in D satisfying $[u_{ij}] = u$, we say that the array (u_{ij}) is a *subdivision* of u. We also say that u is the *composite* of the array (u_{ij}). □

Remark 6.1.5 Subdivisions and their use. The interchange law implies that if in the composable array (u_{ij}) we partition the rows and columns into blocks which produce another composable array and compute the composite v_{kl} of each block, then $[u_{ij}] = [v_{kl}]$. We call the (u_{ij}) a *refinement* of (v_{kl}) in this case.

This observation is used in several ways to prove equalities. The method consists usually in starting from the definition of one side of the equation, then change the array using this subdivision technique and compose the new array getting the other side of the equation.

Changes in a composable array that are clearly possible using this subdivision technique are

1. Select a block of an array and change it for another block having the same composition and the same boundary (see Proposition 6.6.4)

2. Substitute some adjacent columns by another set of adjacent columns having the same boundary and such that each row has the same horizontal composition in both cases. The same can done with rows (see Proposition 6.4.4 and Theorem 6.4.6)

□

Example 6.1.6 Let us give a couple of examples of double categories associated to a category C. The first one is the double category of "squares" or, better still, "2-shells" in a category C, denoted by $\square' C$.

The points and arrows of $\square' C$ and the category structure on $((\square' C)_1, (\square' C)_0)$ are the same as those of C. The set of squares $(\square' C)_2$ is defined by

$$(\square' C)_2 = \{(a, d, b, c) \in C_1^4 \;:\; \partial^- b = \partial^+ a, \partial^- d = \partial^+ c, \partial^+ b = \partial^+ d, \text{and } \partial^- a = \partial^- c\}.$$

Its elements may be represented by "brackets"

$$\left(a \begin{matrix} c \\ b \end{matrix} d \right)$$

and the horizontal and vertical face and degeneracy maps are obvious from the representation. The compositions are defined by

$$\left(a \begin{matrix} c \\ b \end{matrix} d \right) +_1 \left(f \begin{matrix} b \\ g \end{matrix} h \right) = \left(af \begin{matrix} c \\ g \end{matrix} dh \right)$$

and

$$\left(a \begin{matrix} c \\ b \end{matrix} d \right) +_2 \left(d \begin{matrix} u \\ v \end{matrix} w \right) = \left(a \begin{matrix} cu \\ bv \end{matrix} w \right)$$

It is easy to see that $\square' C$ is a double category and that $\square'$ is a functorial construction. Moreover this functor is right adjoint to the truncation functor which sends each double category D to the category D_1. We leave the proof of adjointness as an exercise.

There are several sub-double-categories of $\square' C$ that can be obtained taking the same 0 and 1-dimensional part and restricting the 2-dimensional part by putting some commutativity condition on the 2-shells.

Let us consider $\square\, C$, the category of "commutative squares" or "commutative 2-shells" . Its squares are

$$\square\, C_2 = \{(a, d, b, c) \in C_1^4 \;:\; ab = cd\}.$$

The horizontal and vertical face and degeneracy maps and the compositions are the restriction of those in $\square' C$.

There are quite a few categories that can be defined in a similar manner, but requiring that the compositions ab and cd differ in some way by the action of an element of some fixed subset of C. It is a good exercise to investigate which conditions C and the action have to satisfy to obtain a double category. We shall come back to this in Example 6.1.8. □

As we have stated above, our main objects of interest are double groupoids. These are double categories where all the categories involved are groupoids and which also have an extra structure. Let us start by studying double categories where all category structures are groupoids.

Definition 6.1.7 The category DCatG is the full subcategory of DCat that has as objects double categories in which all three structures are groupoids. □

First, recall that a groupoid is a category G which has a map $(\ \ )^{-1} : G_1 \to G_1$ such that

$$aa^{-1} = 1_{\partial^- a} \text{ and } a^{-1}a = 1_{\partial^+ a}.$$

Thus in a double category G where all three category structures are groupoids, there are three "inverse" maps

$$(\ \ )^{-1} : G_1 \to G_1, \quad -_1 : G_2 \to G_2 \quad \text{and} \quad -_2 : G_2 \to G_2,$$

where

$$(\varepsilon_i a) +_j (\varepsilon_i a^{-1}) = 0_{\partial^- a}, \ \ (\varepsilon_i a^{-1}) +_j (\varepsilon_i a) = 0_{\partial^+ a}, \text{ for } i \neq j.$$

From the compatibility conditions (DC 4.1, 4.2), we see that the boundary maps preserve inverses in the other direction since they are homomorphisms, i.e.

$$\partial_1^\sigma(-_2 u) = (\partial_1^\sigma(u))^{-1}, \quad \partial_2^\sigma(-_1 u) = (\partial_2^\sigma(u))^{-1}. \tag{DCG 4}$$

From the compatibility conditions (DC 5.1, 5.2), we get that the identity maps also preserve inverses, i.e.

$$\varepsilon_1(a^{-1}) = -_2(\varepsilon_1(a)), \quad \varepsilon_2(a^{-1}) = -_1(\varepsilon_2(a)). \tag{DCG 5}$$

We also easily check from the interchange law that for $u \in G_2$

$$-_1 -_2 u = -_2 -_1 u \tag{DCG 6}$$

and we denote the "rotation" $-_1 -_2$ by $-_{12}$.

Example 6.1.8 In the case G is a groupoid, the double categories $\square G$ of commutative 2-shells and $\square' G$ of 2-shells in G defined in Example 6.1.6 are all double groupoids, the inverses of the first element in the following array being as follows:

$$u = \begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix}, \ -_1 u = \begin{pmatrix} & b & \\ a^{-1} & & d^{-1} \\ & c & \end{pmatrix}, \ -_2 u = \begin{pmatrix} & c^{-1} & \\ d & & a \\ & b^{-1} & \end{pmatrix}, \ -_1 -_2 u = \begin{pmatrix} & b^{-1} & \\ d^{-1} & & a^{-1} \\ & c^{-1} & \end{pmatrix}.$$

□

There are interesting differences between the category and groupoid cases with regard to commutative 2-shells. If G is a groupoid, the commutativity condition of a 2-shell can also be stated as $c = abd^{-1}$ or even as $abd^{-1}c^{-1} = 0$.

Thus when searching for new examples of double categories an obvious generalisation of $\square\, C$ comes by considering 2-shells that are commutative up to an element lying in some subcategory $C' \subseteq C$. That is, instead of $ab = cd$ we require $abd^{-1}c^{-1} \in C'$ which works well in the groupoid case.

It is a nice exercise that you should try at this stage, to check that this works if C is a group and C' is a normal subgroup.

This leads to a possible extension of the notion of normal subgroups to 'normal subgroupoids' (It is also a good exercise for you to think how this extension can be made). At a further stage, the concept of normal subgroupoid can be 'externalised' as a crossed module of groupoids, analogously to what has been done for groups. We shall define this concept and prove that it works in Section 6.2.

6.2 The category XMod of crossed modules of groupoids

We have explained that there was an early hint that crossed modules (of groupoids) were related to double categories where all structures are groupoids. Since crossed modules appear quite naturally in algebraic topology, that was a suggestion of strong links between higher order groupoids and classical objects of algebraic topology.

Crossed modules of groupoids are an easy step away from crossed modules of groups and mimic the structure of the family of fundamental crossed modules $\Pi_2(X, A, x)$ when $x \in A \subseteq X$. Also, for any double category which has all three structures of groupoid, we get an associated crossed module over a groupoid.

It is natural to define a crossed module of groupoids to be a groupoid morphism $(\mu : M \to P)$ with an action of P on M such that axioms equivalent to CM1) and CM2) are satisfied. Thus, we start with a groupoid P where P_0 its set of vertices, ∂^-, ∂^+ its initial and final maps. We write $P_1(p, q)$ for the set of arrows from p to q $(p, q \in P)$ and $P_1(p)$ for the group $P_1(p, p)$.

Definition 6.2.1 A *crossed module* over the groupoid $P = (P_1, P_0)$ is given by a groupoid $M = (M_2, P_0)$ and a morphism of groupoids which is the identity on objects

$$M \xrightarrow{\mu} P$$

satisfying

- M is a totally disconnected groupoid with the same objects as P. Equivalently, it can be seen as a family of groups $\{M_2(p)\}_{p \in P_0}$.

We shall use additive notation for all groups $M_2(p)$ and we shall use the same symbol 0 for all their identity elements.

Also, μ is given by a family of homomorphisms $\{\mu_p : M_2(p) \to P_1(p)\}_{p \in P_0}$.

- The groupoid P operates on the right on M. The action is denoted $(x, a) \mapsto x^a$. If $x \in M_2(p)$ and $a \in P_1(p, q)$ then $x^a \in M_2(q)$. It satisfies the usual two axioms of an action:

i) $(x^{ab}) = (x^a)^b$,

ii) $(xy)^a = x^a y^a$.

(Thus $M_2(p) \cong M_2(q)$ if p and q lie in the same component of the groupoid P.)

- These data satisfy two properties

CM1) μ preserves the actions, i.e. $\mu(x^a) = (\mu x)^a$

CM2) For all $c \in M_2(p)$, μc acts on M by conjugation by c, i.e. for any $x \in M_2(p)$, $x^{\mu c} = -c + x + c$.

Notice that $M_2(p)$ is a crossed module over $P_1(p)$ for all $p \in P_0$. In the case when P_0 is a single point we call μ a *crossed module over a group*, or a *reduced* crossed module.

A *morphism* of crossed modules $f : (\mu : M \to P) \to (\nu : N \to Q)$ is a pair of morphisms of groupoids $f_2 : M \to N$, $f_1 : P \to Q$ inducing the same map of vertices and compatible with the boundary maps and the actions of both crossed modules. We denote by XMod the resulting category of crossed modules over groupoids. Notice that the category $\mathsf{XMod/Groups}$ studied in the preceding chapters can be seen as the full subcategory of XMod whose objects are reduced crossed modules of groupoids. □

Example 6.2.2 As we have pointed out, there is an immediate topological example. For any topological pair (X, A) and $C \subseteq A$, we consider $P = \pi_1(A, C)$, the fundamental groupoid of (A, C). Recall that the objects of $\pi_1(A, C)$ are the points of C and for any $x, y \in C$, the elements of $\pi_1(A, C)(x, y)$ are the homotopy classes rel $\{0, 1\}$ of maps

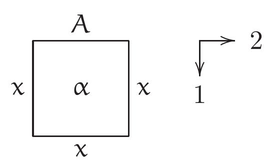
$$\omega : (I, 0, 1) \to (A, x, y).$$

The *fundamental crossed module* $\Pi_2(X, A, C)$ of the triple (X, A, C) is given by the family of groups $\{\pi_2(X, A, x)\}_{x \in C}$. These groups have been defined already in Section 2.1.

Recall that any $[\alpha] \in \pi_2(X, A, x)$ is a homotopy class rel J^+ of maps

$$\alpha : (I^2, \partial I^2, J^+) \to (X, A, x),$$

that can be represented as a square



that is the usual convention for $\mathbb{R}^2$ rotated clockwise through $\pi/2$ to make it equal to the algebraic convention. We shall keep the axes drawn beside the square to make this easier to remember.

The action

$$\pi_2(X, A, x) \times \pi_1(A, C)(x, y) \to \pi_2(X, A, y)$$

was also described in Section 2.1.

The morphism of groupoids $\partial : \pi_2(X, A, C) \to \pi_1(A, C)$ is given, for each $x \in C$, by the restriction to the top face $0 \times I$, so giving

$$\partial(x) : \pi_2(X, A, x) \to \pi_1(A, x).$$

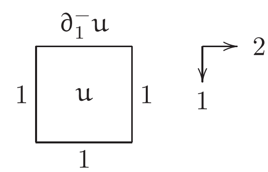
As before, it could be proved directly that these maps satisfy the properties of a crossed module over a groupoid, but we prefer the roundabout way of proving that this crossed module is the one associated to a double groupoid called the *fundamental double groupoid* which will be defined in Section 6.3. □

Let us go back to the general theory and see how to associate to any object $G \in \mathsf{DCatG}$ a crossed module of groupoids which we denote by $\gamma G = (\partial : \gamma G \to P)$. To make it a crossed module we need: groupoid structures on γG and P, a map of groupoids ∂ and an action satisfying CM1) and CM2).

We start by defining P as the groupoid (G_1, G_0). Thus the objects of γG are $(\gamma G)_0 = G_0$ and as morphisms we choose all $u \in G_2$ that have all faces degenerate except $\partial_1^- u$, i.e.

$$(\gamma G)_2 = \{u \in G_2 : \partial_2^+ u = \partial_2^- u = \partial_1^+ u = \varepsilon\partial^-\partial_1^- u = \varepsilon\partial^+\partial_1^- u\}.$$

The reason we chose to use the subindex 2 in the set of morphisms M_2 of M is now apparent: because in this very important example they have "dimension" two. The elements in γG_2, when represented with a matrix like convention, are



With the obvious source, target, and identity, and the composition $u + v$ defined to be $u +_2 v$, we get a totally disconnected groupoid γG.

The next element we need to get a crossed P-module, is a morphism of groupoids. It is defined by

$$\partial = \partial_1^- : \gamma G_2 \to P_1. \tag{6.2.1}$$

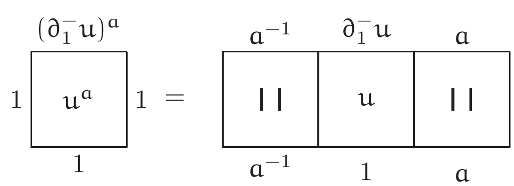
The last ingredient is an action

$$\gamma G_2(x) \times G_1(x, y) \to \gamma G_2(y)$$

for all $x, y \in G_0$. It is given by degeneration and conjugation: i.e. for any $u \in \gamma G_2(x)$ and $a \in G_1(x, y)$,

$$u^a = [-_2\varepsilon_1 a,\ u,\ \varepsilon_1 a], \tag{6.2.2}$$

or, in the usual representation,



Now we have to check that this gives an action which satisfies both properties in the definition of crossed module.

Proposition 6.2.3 *The definition in* (6.2.2) *gives a right action of* G_1 *on* γG_2.

Proof From the diagram, it is clear that $u^a \in \gamma G_2$. It is also not difficult to prove all properties of an action:

$$u^{ab} = (u^a)^b, \qquad (u +_2 v)^a = u^a +_2 v^a \qquad \text{and} \qquad u^+ = u.$$

□

It remains to check the two axioms CM1) and CM2).

Proposition 6.2.4 $\gamma G = (\partial_1^- : \gamma G_2 \to G_1)$ *is a crossed module with the action defined by* (6.2.2).

Proof For CM1) is clear from the diagram that the top face is the conjugate:

$$\partial(u^a) = \partial_1^-(u^a) = \partial_1^-(-_2\varepsilon_1 a)\partial_1^- u \partial_1^-(\varepsilon_1 a) = a^{-1}\partial_1^- u a = (\partial u)^a.$$

Also, for any $a = \partial v, v \in \gamma G_2$, we may construct an array such that when computing both ways gives the equality. In this case the array is

$$\begin{array}{|c|c|c|} \multicolumn{1}{c}{a^{-1}} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{a} \\ \hline \| & u & \| \\ \hline -_2 v & \square & v \\ \hline \end{array}$$

Composing first in the horizontal direction and then in the vertical one, the first row gives u^a and the second one a degenerate square, so we get u^a.

On the other hand, composing first vertically, we get

$$[-_2 v, u, v] = u^v.$$

□

It is important to notice that this construction is functorial, thus giving a functor

$$\gamma : \mathsf{DCatG} \to \mathsf{XMod}.$$

Remark 6.2.5 We finish this section by pointing out that for a double category which has all three structures groupoids we have not only one associated crossed module of groupoids but four, since we may chose any of the sides to be the unique one not equal to the identity. Let us call γG_j^i the crossed module structure on the set of all elements of G_2 having all faces degenerate but the i-face in the j-direction defined by the map ∂_j^i. Then γG_j^- and γG_j^+ are isomorphic. In general, γG_1^j and γG_2^j are not isomorphic but we shall see that they are isomorphic in the case of interest here, namely Example 6.2.2. □

6.3 The fundamental double groupoid of a triple of spaces.

Granted the success of the fundamental groupoid and the known definition of double groupoid, perhaps it was natural in 1966 to attempt to define a fundamental or homotopy double groupoid of a space by considering maps $I^2 \to X$ of a square. Nevertheless, it was not until 1974 that Brown and Higgins realised that a successful theory could be obtained by considering a triple (X, A, C), i.e. a space X and two subspaces $C \subseteq A \subseteq X$.

We shall start by describing the space of maps and some structure over it before taking homotopy classes. We consider a triple (X, A, C). We shall use the triple $(I^2, \partial I^2, \partial^2 I^2)$ given by the square, its boundary and the four vertices, respectively. We consider three sets

$$\begin{array}{rcl} R_0(X, A, C) & = & C \\ R_1(X, A, C) & = & \{\sigma : (I, \{0, 1\}) \to (A, C)\} \\ R_2(X, A, C) & = & \{\alpha : (I^2, \partial I^2, \partial^2 I^2) \to (X, A, C)\}. \end{array}$$

and call the elements of $R_2(X, A, C)$ *filtered maps*

$$\alpha : (I^2, \partial I^2, \partial^2 I^2) \to (X, A, C).$$

Remark 6.3.1 The elements of R_2 can be represented by squares as follows.

$$\begin{tikzcd} C \arrow[r, dash, "A"] \arrow[d, dash, "A"'] \arrow[dr, phantom, "\alpha"] & C \arrow[d, dash, "A"] \\ C \arrow[r, dash, "A"'] & C \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r] \arrow[d] & 2 \\ 1 & \end{tikzcd}$$

□

There is an obvious definition of the source and target maps given by restriction to the appropriate faces of I^2. More formally they are composition with the maps

$$\partial_1^\sigma(x) = (i, x) \quad \text{and} \quad \partial_2^\sigma(x) = (x, i) \quad \text{for } \ \sigma = \pm$$

and they can be seen in the diagram

$$\begin{tikzcd} \cdot \arrow[r, dash, "\partial_1^-"] \arrow[d, dash, "\partial_2^-"'] & \cdot \arrow[d, dash, "\partial_2^+"] \\ \cdot \arrow[r, dash, "\partial_1^+"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r] \arrow[d] & 2 \\ 1 & \end{tikzcd}$$

The identities are given by composing with the projection in the appropriate direction, i.e.

$$p_1(x, y) = x \quad \text{and} \quad p_2(x, y) = y$$

and we use the same notation for degenerate squares as in the previous section.

Also, there are several compositions on R given by juxtaposition. The one in R_1 has been defined when talking about the fundamental groupoid. The set R_2 has two similar compositions given by

$$(\alpha +_1 \beta)(x, y) = \begin{cases} \alpha(2x, y) & \text{if } 0 \leqslant x \leqslant 1/2 \\ \beta(2x - 1, y) & \text{if } 1/2 \leqslant x \leqslant 1 \end{cases}$$

and

$$(\alpha +_2 \beta)(x, y) = \begin{cases} \alpha(x, 2y) & \text{if } 0 \leqslant y \leqslant 1/2 \\ \beta(x, 2y - 1) & \text{if } 1/2 \leqslant y \leqslant 1. \end{cases}$$

We leave the reader to check that the interchange law holds for these two compositions. The reverse of an element $\alpha \in R_2$, with respect these two directions are written $-_1\alpha$, $-_2\alpha$ and are defined respectively by $(x, y) \mapsto \alpha(1 - x, y)$, $\ (x, y) \mapsto \alpha(x, 1 - y)$.

All this structure means in particular that $R(X, A, C)$ is a 2-truncated cubical set with compositions. It is not a double category (no associativity, etc.). Nevertheless, it is useful to fix the meaning of composition of arrays. We study this in the next remark.

Remark 6.3.2 For positive integers m, n let $\varphi_{m,n} : I^2 \to [0, m] \times [0, n]$ be the map $(x, y) \mapsto (mx, ny)$. An $m \times n$ subdivision of a square $\alpha : I^2 \to X$ is a factorisation $\alpha = \alpha' \circ \varphi_{m,n}$; its *parts* are the squares $\alpha_{ij} : I^2 \to X$ defined by

$$\alpha_{ij}(x, y) = \alpha'(x + i - 1, y + j - 1).$$

We then say that α is the *composite* of the squares α_{ij}, and we write $\alpha = [\alpha_{ij}]$. Similar definitions apply to paths and cubes.

Such a subdivision determines a cell-structure on I^2 as follows. The intervals $[0, m], [0, n]$ have cell-structures with integral points as 0-cells and the intervals $[i, i+1]$ as closed 1-cells. Then $[0, m] \times [0, n]$ has the product cell-structure which is transferred to I^2 by $\varphi^{-1}_{m,n}$. We call the 2-cell $\varphi^{-1}_{m,n}([i-1, i] \times [j-1])$ the *domain* of α_{ij}. □

Definition 6.3.3 To define the fundamental double groupoid associated to a triple of spaces (X, A, C) we shall use the three sets

$$\begin{aligned} \rho_0(X, A, C) &= C \\ \rho_1(X, A, C) &= R_1(X, A, C)/\equiv \\ \rho_2(X, A, C) &= R_2(X, A, C)/\equiv . \end{aligned}$$

where $\equiv$ is the relation of homotopy rel vertices on R_1 and of homotopy of pairs rel vertices on R_2. That is, for such a homotopy $H_t : I^2 \to X$, we have $H_t(c) = H_0(c)$ for all $t \in I$ and $c \in \partial^2 I^2$. We call this relation *f-homotopy* (or filter homotopy), to distinguish it from homotopy of maps $I \to A$ or $I^2 \to X$ which we shall write $\simeq$. It is important that f-homotopy is rel vertices, that is that the vertices of I and of I^2 are fixed in the homotopies. This allows us to obtain the groupoid structures on the filtered homotopy classes without adding any condition on the spaces.

The *f-homotopy class* of an element α is written $\langle\langle\alpha\rangle\rangle$. □

We expect all the structure maps in $\rho(X, A, C)$ to be those induced by the corresponding structure maps of $R(X, A, C)$. So we have to prove that they are compatible with the homotopies. In the case of the structure maps for (ρ_1, ρ_0) this is clear, since they form the relative fundamental groupoid of the pair (A, C).

Let us try the maps for the horizontal and vertical structure on (ρ_2, ρ_1). There is no problem with the source and target since the homotopies are filtered. Also a homotopy between elements of $R_1(X, A, C)$ gives easily a homotopy between the associated identities. The only problems appear to be with the compositions.

We develop only the horizontal case; the other follows by symmetry. So, let us consider two elements $\langle\langle\alpha\rangle\rangle, \langle\langle\beta\rangle\rangle \in \rho_2(X, A, C)$ such that $\langle\langle\partial^+_2 \alpha\rangle\rangle = \langle\langle\partial^-_2 \beta\rangle\rangle$, i.e. we have continuous maps

$$\alpha, \beta : (I^2, \partial I^2, \partial^2 I^2) \to (X, A, C)$$

and a homotopy

$$h : (I, \partial(I)) \times I \to (A, C)$$

from $\alpha|_{\{1\}\times I}$ to $\beta|_{\{0\}\times I}$ rel vertices, i.e. $h(0 \times I) = y$ and $h(1 \times I) = x$. We define now the composition by

$$\langle\langle\alpha\rangle\rangle +_2 \langle\langle\beta\rangle\rangle = \langle\langle\alpha +_2 h +_2 \beta\rangle\rangle = \langle\langle[\alpha, h, \beta]\rangle\rangle.$$

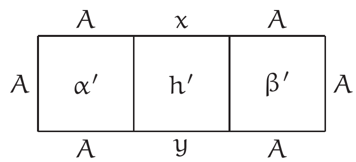
This is given in a diagram by

$$\begin{array}{c|c|c|c|c} & A & x & A & \\ \hline A & \alpha & h & \beta & A \\ \hline & A & y & A & \end{array} \quad . \tag{6.3.1}$$

Our first important step is that these compositions are well defined.

Proposition 6.3.4 *The compositions are well defined in* $\rho_2(X, A, C) = R_2(X, A, C)/\equiv$.

Proof To prove this we chose two other representatives $\alpha' \in \langle\!\langle \alpha \rangle\!\rangle$ and $\beta' \in \langle\!\langle \beta \rangle\!\rangle$ and a homotopy h' from $\alpha'|_{\{1\}\times I}$ to $\beta'|_{\{0\}\times I}$. Using them, we get



which should give the same composition in ρ_2 as (6.3.1).

Since $\langle\!\langle \alpha \rangle\!\rangle = \langle\!\langle \alpha' \rangle\!\rangle, \langle\!\langle \beta \rangle\!\rangle = \langle\!\langle \beta' \rangle\!\rangle$ there are the f-homotopies $\phi : \alpha \equiv \alpha', \psi : \beta \equiv \beta'$ which can be seen in the next figure, in which the thin lines denote edges on which the maps are constant.

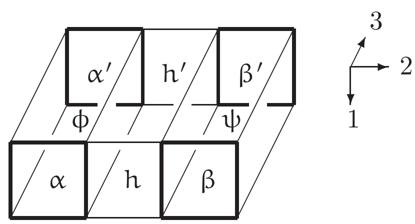


Figure 6.1: Filling the hole in the middle

To complete this to an f-homotopy

$$\alpha +_2 h +_2 \beta \equiv \alpha' +_2 h' +_2 \beta'$$

we need to "fill" appropriately the hole in the middle (see Figure 6.1).

Let $k : I \times \partial I^2 \to A$ be given by $(r, s, 0) \mapsto h(r, s), (r, s, 1) \mapsto h'(r, s), (r, 0, t) \mapsto \phi_t(r, 1), (r, 1, t) \mapsto \psi_t(r, 0)$. In terms of Figure 6.1, k is the map defined on the four side faces of the central hole. But k is constant on the edges of the bottom face, since all the homotopies are rel vertices. So k extends over $\{1\} \times I^2 \to A$ extending k to five faces of I^3.

Now we can retract I^3 onto any five faces by projecting from a point above the centre of the remaining face. Composing with this retraction, we obtain a further extension $k : I^3 \to A$. The composite cube $\phi +_2 k +_2 \psi$ is an f-homotopy $\gamma \equiv \gamma'$ as required: the key point is that the extension maps the top face of the middle cube into A, since that is true for all the other faces of this middle cube. $\square$

Once we have proved that compositions in ρ_2 are well defined, we can easily prove that they are groupoids, with $\langle\!\langle -_i \alpha \rangle\!\rangle$ being the inverse of $\langle\!\langle \alpha \rangle\!\rangle$ for the composition $+_i, i = 1, 2$. We also need to prove the interchange law.

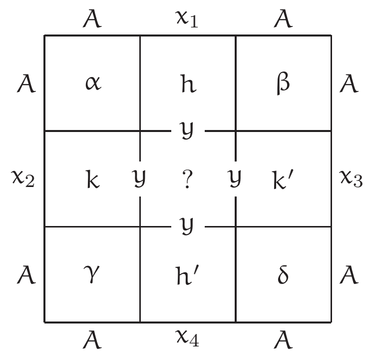
Proposition 6.3.5 *The compositions* $+_1, +_2$ *in* $\rho_2(X, A, C) = R_2(X, A, C)/\equiv$ *satisfy the interchange law.*

Proof The argument involves "filling a hole". We start with four elements $\langle\!\langle \alpha \rangle\!\rangle, \langle\!\langle \beta \rangle\!\rangle, \langle\!\langle \gamma \rangle\!\rangle, \langle\!\langle \delta \rangle\!\rangle \in \rho_2(X, A, C)$ such that $\langle\!\langle \partial_2^+ \alpha \rangle\!\rangle = \langle\!\langle \partial_2^- \beta \rangle\!\rangle$, $\langle\!\langle \partial_2^+ \gamma \rangle\!\rangle = \langle\!\langle \partial_2^- \delta \rangle\!\rangle$, $\langle\!\langle \partial_1^+ \alpha \rangle\!\rangle = \langle\!\langle \partial_1^- \gamma \rangle\!\rangle$ and $\langle\!\langle \partial_1^+ \beta \rangle\!\rangle = \langle\!\langle \partial_1^- \delta \rangle\!\rangle$. To prove that

$$(\langle\!\langle \alpha \rangle\!\rangle +_2 \langle\!\langle \beta \rangle\!\rangle) +_1 (\langle\!\langle \gamma \rangle\!\rangle +_2 \langle\!\langle \delta \rangle\!\rangle) = (\langle\!\langle \alpha \rangle\!\rangle +_1 \langle\!\langle \gamma \rangle\!\rangle) +_2 (\langle\!\langle \beta \rangle\!\rangle +_1 \langle\!\langle \delta \rangle\!\rangle)$$

we construct an element of $R_2(X, A, C)$ that represents both compositions.

Using f-homotopies $h : \partial_2^+ \alpha \equiv \partial_2^- \beta$, $h' : \partial_2^+ \gamma \equiv \partial_2^- \delta$, $k : \partial_1^+ \alpha \equiv \partial_1^- \gamma$ and $k' : \partial_1^+ \beta \equiv \partial_1^- \delta$ given because the compositions are defined we have a map defined on the whole square except on a hole in the middle:



We only need to fill appropriately the hole. But all homotopies are rel vertices, so the map is constant on the boundary of the hole. So we extend with the constant map, and evaluate the resulting composition in two ways to prove the interchange law. □

Thus we have proved that $\rho(X, A, C)$ is a double category where all three structures are groupoids. We call this *the fundamental double groupoid of the triple* (X, A, C) and leave the study of its extra structure which justifies its name till Section 6.4.

A map $f : (X, A, C) \to (X', A', C')$ of triples clearly defines a morphism $\rho(f) : \rho(X, A, C) \to \rho(X', A', C')$ of double categories.

Proposition 6.3.6 *If* $f : (X, A, C) \to (X', A', C')$ *is a map of triples such that each of* $f : X \to X', f_1 : A \to A'$ *are homotopy equivalences, and* $f_0 : C \to C'$ *is a bijection, then* $\rho(f) : \rho(X, A, C) \to \rho(X', A', C')$ *is an isomorphism.*

Proof This is an easy consequence of a cogluing theorem for homotopy equivalences. We give the details for the analogous result for filtered spaces in an Appendix. □

Now let us check the not quite so straightforward fact that the crossed module associated to the fundamental double groupoid $\rho(X, A, C)$ is the fundamental crossed module $\Pi_2(X, A, C)$, i.e. $\gamma(\rho(X, A, C))_2 = \Pi_2(X, A, C)$. Recall that $\gamma(\rho(X, A, C))_2(x)$ is formed by f-homotopy classes of filtered maps $\alpha : (I^2, \partial I^2, \partial^2 I^2) \to (X, A, x)$ such that the restriction to all sides but the last vertical one are homotopically trivial. On the other hand, $\pi_2(X, A, x)$ consists of homotopy classes of maps $\alpha : (I^2, \partial I^2, J^+) \to (X, A, x)$. Let us check that they are the same.

Proposition 6.3.7 *If* $x \in C$*, then the group* $\gamma(\rho(X, A, C))_2(x)$ *may be identified with the group* $\pi_2(X, A, x)$.

Proof Recall from the definitions that in both cases the elements are homotopy classes of maps

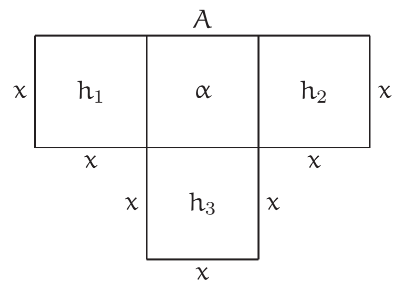
$$\alpha : (I^2, \partial I^2) \to (X, A).$$

For α to define an element in $\pi_2(X, A, x)$, which we are going to denote also by $\langle\!\langle \alpha \rangle\!\rangle$, the maps send all J^+ to x and the same is true for homotopies in this case. In the case $\langle\!\langle \alpha \rangle\!\rangle \in \rho(X, A, C)_2(x)$ the map sends only the vertices to x and the homotopy is rel vertices. Clearly the map

$$\phi : \pi_2(X, A, x) \to \gamma(\rho(X, A, C))_2(x)$$

defined by $\phi(\langle\!\langle \alpha \rangle\!\rangle) = \langle\!\langle \alpha \rangle\!\rangle$ is well defined, is a group homomorphism and preserves action. We only have to prove that ϕ is bijective. We shall use a couple of filling arguments.

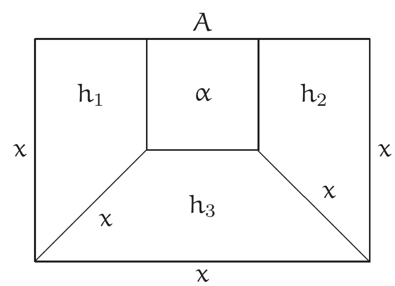
To see that ϕ is onto, let $\langle\!\langle \alpha \rangle\!\rangle \in \gamma(\rho(X, A, C))_2(x)$, i.e. we have a map $\alpha : (I^2, \partial I^2, \partial^2 I^2) \to (X, A, C)$ such that its restrictions to all faces of the square but the top one are homotopic rel vertices to the constant map. Putting all these three homotopies in one diagram we get



We want to get a map $\beta : (I^2, \partial I^2, J^+) \to (X, A, x)$ such that $\phi\langle\!\langle \beta \rangle\!\rangle = \langle\!\langle \alpha \rangle\!\rangle$ i.e filter homotopic rel vertices to α.

We can fold the above diagram, getting a map defined on four of the six faces of a cube I^3. Thus, composing with the retraction of I^3 onto such four faces, as seen in Figure 2.3, we get both the desired β (the restriction to the top face) and the homotopy (the cube).

Intuitively, the map β is



and the homotopy is got by shrinking the bigger square into the smaller one.

It remains to prove that ϕ is injective, i.e. that Ker ϕ contains only the homotopy class of the constant map.

Thus we start with a map $\alpha : (I^2, \{0\} \times I, J^+) \to (X, A, x)$ so that there is an f-homotopy $h : (I^2, \partial I^2, \partial^2 I^2) \times I \to (X, A, x)$ from α to 0. This h can be represented by a cube that is the constant x

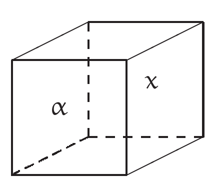
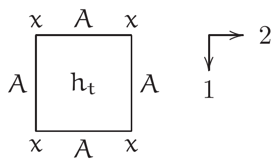


Figure 6.2: f-homotopy from α to the constant map

both in the back face and in the four slanted lines.

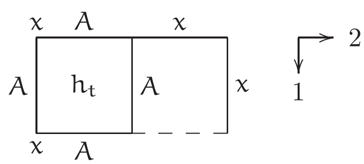
We have to get a homotopy of maps of triples $h' : \alpha \simeq 0$ rel J^+. This $h' : (I^2, \{0\} \times I, J^+) \times I \to (X, A, x)$ is α on the front 2-face and has to be constant not only on $\partial^2 I^2 \times I$ as was h, but also on $(\{1\} \times I \cup I \times \partial I) \times I$.

We will do that by changing h to h' in a similar way to the one used in the first part of this proof. Instead of working in four dimensions, we are going to explain what to do in each section for a fixed third coordinate with the 3-cube given by h. We have the following situation



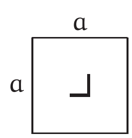
and we want to change this h_t to an h'_t sending all J^+ to x.

So, using a filling argument like the one in 1.3 we extend h_t

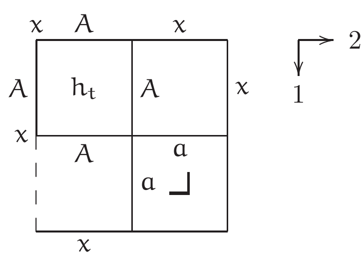


to an square sending one side (the right one) to x. Also, the edge represented by the discontinuous line goes in A, let us call it a.

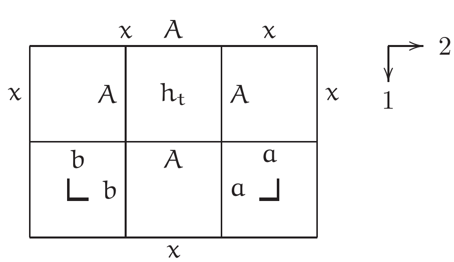
To change another side, we need some way of 'turning right'. This is produced by a degenerate square got by composing a with the map $\sigma : I^2 \to I$ given by $\sigma(s,t) = \max(s,t)$ that is represented by



where the unlabeled sides are constant. Adding this square, we can use a similar filling argument and extend to

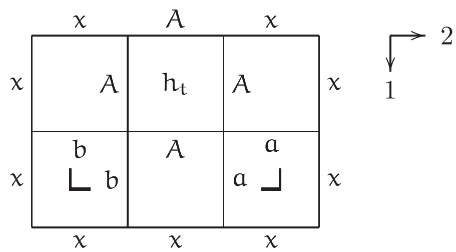


let us call by b the edge without label and repeat the filling argument to get



where the square 'turning left' in the bottom left corner is defined in a similar way as 'turning right'. It is clear that the edges without label goes in A.

Therefore, the above constructions all fit together to obtain h'_t as in the diagram below



Since we could do the above construction for any section t and all of them fit together, we get a homotopy

$$h' : (I^2, \{0\} \times I, J^+) \to (X, A, x)$$

from α to the constant map that is clearly continuous. □

The reader will have noticed the widespread use of filling arguments in the above proofs. These arguments become the key to the proof of corresponding results for higher dimensions which are developed in Chapter 14.

6.4 Thin structures on a double category. The category DGpds of double groupoids.

We have examples of double categories coming from two sources: first, the 2-shells commutative up to an element of a crossed modules over groupoids hinted at the end of Section 6.1 and which will be properly developed in Section 6.6, and second, the fundamental double groupoid of a topological pair seen in Section 6.3. In both cases not only are all three structures groupoids but they have also some extra structure. Let us see one way of introducing this structure.

We have already introduced in Example 6.1.6 the double category $\square' C$ of 2-*shells* in the category C and its sub double category $\square\, C$ of *commuting* 2-*shells*.

For any double category D there is a morphism of double categories $D \to \square' D_1$ which is the identity in dimensions 0,1 and in dimension 2 gives the bounding shell of any element. On the other hand, there is no natural morphism the other way, from either $\square' D_1$ or $\square D_1$, which is the identity on D_1.

In this Section, we are going to study double categories endowed with such a morphism, i.e. for any given commuting shell in D_1, there is a chosen 'filler' in D_2. Next, in Section 6.5, we develop an alternative approach using some extra degeneracies called *connections*.

Definition 6.4.1 We therefore define a *thin structure* on a double category D to be a morphism of double categories

$$\Theta : \square D_1 \to D$$

which is the identity on D_1, D_0. The 2-dimensional elements of the form $\Theta\alpha$ for $\alpha \in (\square D_1)_2$ will be called *thin* squares in (D, Θ) or simply in D if Θ is given. □

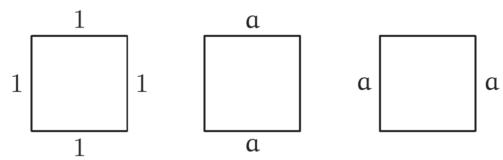
Equivalently, the axioms for thin squares are:

T0) Any identity square in D is thin.

T1) Each commuting shell in D has a unique thin filler.

T2) Any composite of thin squares is thin.

By T0), particular thin squares represent the degenerate squares, namely those of the form

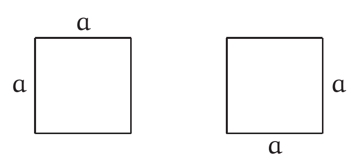


(6.4.1)

which we write in short as

Notice that identity edges are those drawn with a solid line. The notation is ambiguous, since for example the second element is the same as the first if $a = 1$. Also we have not named the vertices. Nevertheless, it is clear that they represent the degenerate squares since Θ is a morphism of double categories.

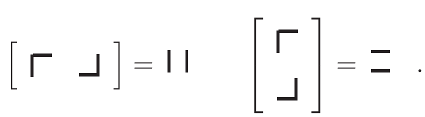
We also have two new 'degenerate' squares



(6.4.2)

which we write in short as

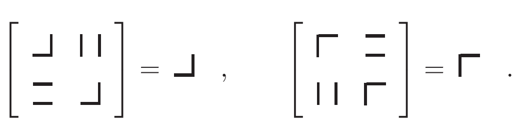
The fact that Θ is a morphism of double categories leads immediately to some equations for compositions of such elements, i.e.



(6.4.3)

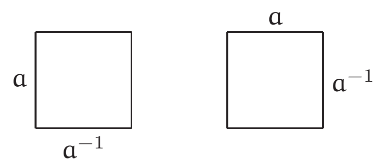
In writing such matrix compositions, of course we always assume that the compositions are defined. The reason why these equations hold is that the composites are certainly thin, by T2), and since they are determined by their shell, by T1), they are by T0) of the form given.

Here are some more consequences which are known as "transport laws":



(6.4.4)

If in addition the category D_1 is a groupoid then we have two further thin elements namely



(6.4.5)

which we write

$$\mathrm{tr}\qquad \llcorner \qquad .$$

Those elements give rise to new equations, for example

$$\begin{bmatrix} \ulcorner & \mathrm{tr} \\ | \; | & | \; | \end{bmatrix} = \square \quad .$$

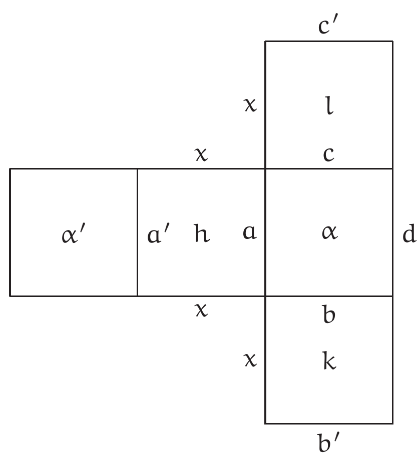
Note here that three of the sides are identities, and hence so also is the fourth, by commutativity.

Now we apply these ideas to the fundamental double groupoid $\rho(X, A, C)$.

Proposition 6.4.2 *The fundamental double groupoid $\rho(X, A, C)$ has a natural thin structure in which a class $\langle\langle\alpha\rangle\rangle$ is thin if and only it has a representative α such that $\alpha(I^2) \subseteq A$.*

Proof Let $a, b, c, d : I \to A$ be paths in A such that $ab \simeq cd$ in A. It is a standard property of the fundamental groupoid that the given paths can then be represented by the sides of a square $\alpha : I^2 \to A$. We have to prove that such a square is unique in ρ_2.

Let $\alpha' : I^2 \to A$ be another square whose edges a', b', c', d' are equivalent in $\pi_1(A, C)$ to a, b, c, d respectively. If we choose maps $h, k, l : I^2 \to A$ giving homotopies rel end points $a \simeq a', b \simeq b', c \simeq c'$. These homotopies, with α and α' can be represented as



folding the diagram they give a map H from five 2-faces of I^3 to A.

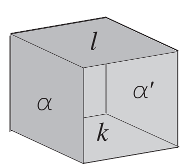


Figure 6.3: Box without a lateral face

Now, using the retraction from I^3, we can extend this to a map $I^3 \to A$. This gives an f-homotopy as required.

Note that this is where we use the fact that an f-homotopy is allowed to move the edges of the square within A. $\square$

Since this important example has this structure, it is reasonable to call them double groupoids. This leads to:

Definition 6.4.3 A *double groupoid* is a double category such that all three structures are groupoids, together with a thin structure. We write DGpds for the category of double groupoids taking as morphisms the double functors that preserve the given thin structures.

We are interested in the restriction to this category of the functor defined in Section 6.2. It is still denoted

$$\gamma : \mathsf{DGpds} \to \mathsf{Crs}.$$

$\square$

Notice that the thin elements $\ulcorner$, $\lrcorner$ in $\rho(X, A, C)$ are, like $=$, $\vert\,\vert$, determined by specific maps, namely in the first two cases are composition of a path $I \to A$ with the maps $\max, \min : I^2 \to I$. We will say more on this in the next section.

An important consequence of the existence of a thin structure in a double groupoid is that the vertical and horizontal groupoid structures in dimension 2 are isomorphic. The isomorphism is given by "rotation" maps $\sigma, \tau : G_2 \to G_2$ which correspond to a clockwise and an anticlockwise rotation through $\pi/2$.

Let G be a double groupoid. We define σ, τ for any $u \in G_2$ by

$$\sigma(u) = \begin{bmatrix} \vert\,\vert & \ulcorner & = \\ \llcorner & u & \mathrm{tr} \\ = & \lrcorner & \vert\,\vert \end{bmatrix} \quad \text{and} \quad \tau(u) = \begin{bmatrix} = & \mathrm{tr} & \vert\,\vert \\ \ulcorner & u & \lrcorner \\ \vert\,\vert & \llcorner & = \end{bmatrix}.$$

To prove the main properties of these operations is a diversion from our main aims, but one which illustrates some points in higher dimensional algebra.

Let us start by proving that σ is a homomorphism from the horizontal to the vertical composition, while τ is a homomorphism from the vertical to the horizontal composition. We next prove that τ is an inverse to σ. It follows that in the case of a double groupoid the horizontal and the vertical groupoid structures in dimension 2 are isomorphic.

Proposition 6.4.4 *For any* $u, v, w \in G_2$,

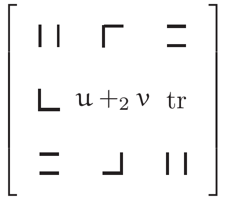
$$\sigma([u, v]) = \begin{bmatrix} \sigma u \\ \sigma v \end{bmatrix} \quad \textit{and} \quad \sigma(\begin{bmatrix} u \\ w \end{bmatrix}) = [\sigma w, \sigma u]$$

$$\tau([u, v]) = \begin{bmatrix} \tau v \\ \tau u \end{bmatrix} \quad \textit{and} \quad \tau(\begin{bmatrix} u \\ w \end{bmatrix}) = [\tau u, \tau w]$$

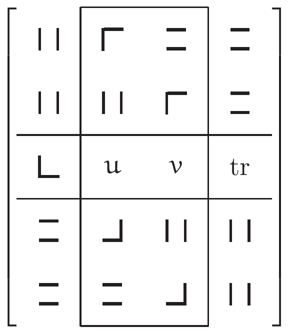
whenever the compositions are defined.

Proof We prove only the first rule and leave the others to the reader.

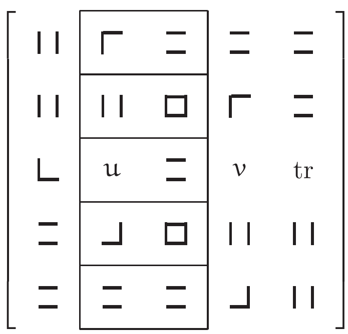
By definition, the element $\sigma([u, v])$ is the composition of the array



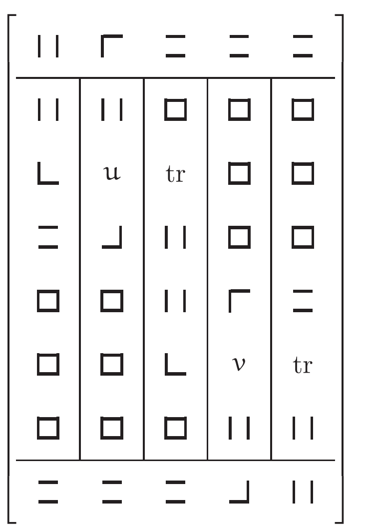
We get a refinement of this array by substituting each element for a box which has the initial element as its composition as follows:



By Remark 6.1.5 this new array has the same composition as the initial one. We now subdivide the second column horizontally in two, getting a new refinement



which still has the same composition. Finally, we expand the three middle rows into six in such a way that we do not change the vertical composition of each column getting

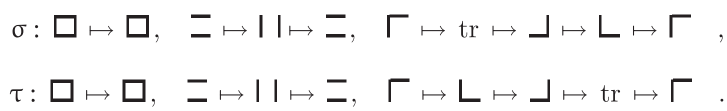


The composition of this array still is $\sigma([u,v])$ by Remark 6.1.5. To get the result, we now see that the composition of the block given by the first four rows is σu and the composition of the other four is σv. □

It is a nice exercise to extend this result to any rectangular array using associativity.

Since thin elements are determined by their boundaries, the next result follows immediately.

Proposition 6.4.5 *The images of thin elements under σ and τ are as follows*



A key fact is that σ is a bijection with inverse τ and that these maps together with the inverse maps $-_1$ and $-_2$, generate all symmetries of a square.

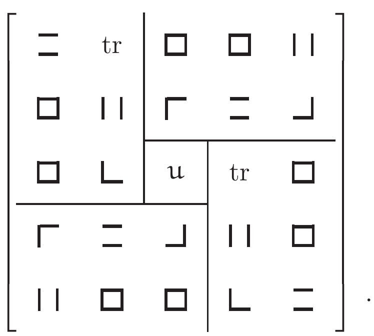
Theorem 6.4.6 *The isomorphisms $-_1$, $-_2$, σ, τ and their composites form a group of transformations of G_2 which is isomorphic to the group D_8 of symmetries of a square.*

Proof We choose a presentation of D_8 and verify that the relations are satisfied:

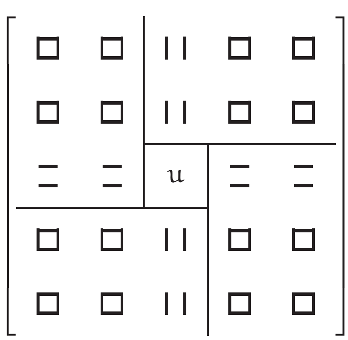
$$D_8 = \langle -_1, -_2, \sigma, \tau : (-_1)^2 = (-_2)^2 = \sigma\tau = (-_{12})^2 = \mathrm{Id}\,,\ -_1\sigma = \tau -_1\,,\ \sigma^2 = -_{12}\rangle .$$

We already know that $\{\mathrm{Id}, -_1, -_2, -_{12}\}$ form a Klein 4-group.

To verify the fourth relation, we show that for any $u \in G_2$, we have $\tau\sigma(u) = u$. It is easily seen that $\tau\sigma(u)$ is the composition of the array



Using Remark 6.1.5 four times, we can change the four blocks one by one and substitute them for another four having the same boundary and composition, getting that $\tau\sigma(u)$ is also the composition of the array



whose composition reduces to u.

We next show that, for any $u \in G_2$, we have

$$-_1\sigma(u) \quad = \quad -_1\begin{bmatrix} \| & \ulcorner & = \\ \llcorner & u & \mathrm{tr} \\ = & \lrcorner & \| \end{bmatrix} \quad = \quad \begin{bmatrix} = & \mathrm{tr} & \| \\ \ulcorner & -_1u & \lrcorner \\ \| & \llcorner & = \end{bmatrix} \quad = \quad \tau(-_1u)$$

For the final relation we note that

$$\sigma^2 = (\sigma -_1)(-_1\sigma) = (-_1\tau)(\tau -_2) = -_{12}$$

□

Remark 6.4.7 When these results are applied to the fundamental double groupoid $\rho(X, A, C)$, they imply the existence of specific f-homotopies. Indeed one of the aims of higher order groupoid theory is to give an algebraic framework for calculating with homotopies and higher homotopies.

6.5 Connections in a double category: equivalence with thin structure.

The extension of the notion of thin structure to higher dimensions is not straightforward since it would require the notion of commutative n-cube and this notion is not easy even for a 3-cube. We shall return to this at the end of this section.

So, we look for an alternative which generalises more easily to higher dimensions. We take as basic the two maps $\Gamma^-, \Gamma^+ : D_1 \to D_2$, that correspond to the thin elements $\lrcorner$, $\ulcorner$, satisfying the properties we have seen in (6.4.3) and (6.4.4). We make this concept clear and develop the equivalence between the two notions in this section.

Definition 6.5.1 A *connection pair* on a double category D is a pair of maps

$$\Gamma^-, \Gamma^+ : D_1 \to D_2$$

satisfying the four properties below.

The first one is that the shells are what one expects, i.e., if $a : x \to y$ in D_1 then $\Gamma^-(a), \Gamma^+(a)$ shells are

$$\Gamma^-(a) \;=\; \begin{array}{ccc} & a & \\ a & \boxed{\lrcorner} & 1_y \\ & 1_y & \end{array} \qquad \Gamma^+(a) \;= \begin{array}{ccc} & 1_x & \\ 1_x & \boxed{\lrcorner} & a \\ & a & \end{array}$$

which can be more formally stated as

$$\partial_2^-\Gamma^-(a) = \partial_1^-\Gamma^-(a) = a \quad \text{and} \quad \partial_2^+\Gamma^-(a) = \partial_1^+\Gamma^-(a) = \varepsilon\partial^+a \tag{CON 1}$$

$$\partial_2^+\Gamma^+(a) = \partial_1^+\Gamma^+(a) = a \quad \text{and} \quad \partial_2^-\Gamma^+(a) = \partial_1^-\Gamma^+(a) = \varepsilon\partial^-a. \tag{CON$'$ 1}$$

We also assume that the connections associate to a degenerate element a double degenerate one:

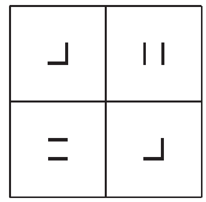
$$\Gamma^-\varepsilon(x) = 0_x \tag{CON 2}$$

$$\Gamma^+ \varepsilon(x) = 0_x. \tag{CON$'$ 2}$$

The relation with composition is given by the "transport laws" (see (6.4.4)):

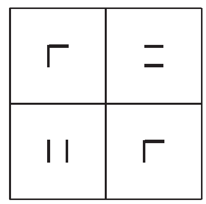
(CON 3)

$$\Gamma^-(ab) \;=\; \begin{bmatrix} \Gamma^- a & | \, | \\ = & \Gamma^- b \end{bmatrix} \;=$$



(CON$'$ 3)

$$\Gamma^+(ab) \;=\; \begin{bmatrix} \Gamma^+ a & = \\ | \, | & \Gamma^+ b \end{bmatrix} \;=$$



Intuitively, a feature that 2-dimensional movements can have extra to 1-dimensional movements is the possibility of turning left or right. The transport laws state intuitively that turning left with one's arm outstretched is the same as turning left, and similarly for turning right.

A final condition deduced from the same idea is that they are "inverse" to each other in both directions (corresponding to (6.4.3)), i.e.

$$\Gamma^+(a) +_2 \Gamma^-(a) = \varepsilon_1(a) \tag{CON 4}$$

$$\Gamma^+(a) +_1 \Gamma^-(a) = \varepsilon_2(a). \tag{CON$'$ 4}$$

□

It is interesting to notice that for double categories where all structures are groupoids we need only a map Γ^- satisfying the conditions CON 1-3 since Γ^+ can be defined using (CON 4).

Proposition 6.5.2 *For a double category in which all structures are groupoids, Γ^- and Γ^+ may be obtained from each other by the formula*

$$\Gamma^+(a) = -_2 -_1 \Gamma^-(a^{-1}).$$

Proof Let us define $\Gamma''(a) = -_2 -_1 \Gamma^-(a^{-1})$.

Since $\Gamma^-(aa^{-1}) = \Gamma^-(1) = \square$, we obtain from the transport law (CON 3.1) that $\Gamma^-(a^{-1}) = -_1[\Gamma^- a, \;\; (\varepsilon_1 a^{-1})]$. Hence $\Gamma''(a) = [(\varepsilon_1 a), \;\; -_2\Gamma^- a]$.

This implies that $\Gamma''(a) +_2 \Gamma^-(a) = \varepsilon_1(a)$, and so by (CON 4) $\Gamma''(a) = \Gamma^+(a)$. □

If we use an analogue of our previous notations $\lrcorner$, $\ulcorner$ for Γ^-, Γ^+ respectively then of course we see that all these laws are the ones we have given before for thin elements. So it is not very difficult to see that any thin structure has associated a unique connection, and that the given thin structure is determined by this connection.

Proposition 6.5.3 *If there is a thin structure Θ on D we have an associated connection defined by*

$$\Gamma^- a = \Theta \begin{pmatrix} & a & \\ a & & 1 \\ & 1 & \end{pmatrix} \quad \textit{and} \quad \Gamma^+ a = \Theta \begin{pmatrix} & 1 & \\ 1 & & a \\ & a & \end{pmatrix}.$$

Moreover, the morphism Θ can be recovered from the connection, since

$$\Theta\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix} = (\varepsilon_2 a +_1 \Gamma^+ b) +_2 (\Gamma^- c +_1 \varepsilon_2 d) = (\varepsilon_1 c +_2 \Gamma^+ d) +_1 (\Gamma^- a +_2 \varepsilon_1 b). \qquad \text{(CON 5)}$$

Proof The results on the behaviour of Γ^- and Γ^+ with respect to boundaries and degeneracies are immediate.

Before proving the relation with the compositions, it is worth mentioning that the values of Θ on degenerate elements are determined by the fact that Θ is a morphism of double categories, so, $\Theta\varepsilon_1(b) = \varepsilon_1(b)$ and $\Theta\varepsilon_2(b) = \varepsilon_2(b)$.

Applying Θ to the equation

$$\begin{pmatrix} & ab & \\ ab & & 1 \\ & 1 & \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} & a & \\ a & & 1 \\ & 1 & \end{pmatrix} & \begin{pmatrix} & b & \\ 1 & & 1 \\ & b & \end{pmatrix} \\ \begin{pmatrix} & 1 & \\ b & & b \\ & 1 & \end{pmatrix} & \begin{pmatrix} & b & \\ b & & 1 \\ & 1 & \end{pmatrix} \end{pmatrix}$$

we get the transport law

$$\Gamma^-(ab) = \begin{bmatrix} \Gamma^- a & \varepsilon_1 b \\ \varepsilon_2 b & \Gamma^- b \end{bmatrix}.$$

and the one for Γ^+ is obtained along the same lines.

Moreover, it is easy to see that on $\square D$, the element

$$\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix}$$

may by decomposed as the product of any of the two arrays

$$\begin{pmatrix} \begin{pmatrix} & 1 & \\ a & & a \\ & 1 & \end{pmatrix} & \begin{pmatrix} & c & \\ c & & 1 \\ & 1 & \end{pmatrix} \\ \begin{pmatrix} & 1 & \\ 1 & & b \\ & b & \end{pmatrix} & \begin{pmatrix} & 1 & \\ d & & d \\ & 1 & \end{pmatrix} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \begin{pmatrix} & c & \\ 1 & & 1 \\ & c & \end{pmatrix} & \begin{pmatrix} & 1 & \\ 1 & & d \\ & d & \end{pmatrix} \\ \begin{pmatrix} & a & \\ a & & 1 \\ & 1 & \end{pmatrix} & \begin{pmatrix} & b & \\ 1 & & 1 \\ & b & \end{pmatrix} \end{pmatrix}$$

where in the first one we have to compose first columns then rows and in the second one the other way about.

Applying Θ to these expressions, we get both formulae. □

Remark 6.5.4 As we have seen in the proof of the preceding property, the thin elements are composition of degenerate elements and connections. Conversely, all degeneracies and connections lie in the image of Θ, so any composition of such elements is a thin element. Thus we have an easy characterisation of the thin elements. □

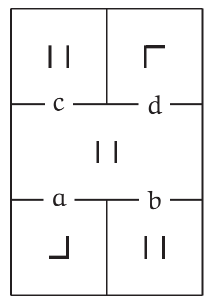
There is more work in obtaining the other implication, i.e. getting the thin structure from the connection maps.

Proposition 6.5.5 *If there is a connection on* D*, we have an associated thin structure* Θ *defined by the formula* (CON 5) *in Proposition 6.5.3. Moreover, the connection can be recovered from* Θ*, since*

$$\Gamma^-(a) = \Theta\begin{pmatrix} & a & \\ a & & 1 \\ & 1 & \end{pmatrix} \quad \textit{and} \quad \Gamma^+(a) = \Theta\begin{pmatrix} & 1 & \\ 1 & & a \\ & a & \end{pmatrix}.$$

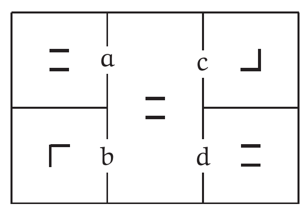
Proof Let us first prove that either formulae gives the same function. This will make it easier to prove the morphism property. We write

$$\Theta_1\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix} = (\varepsilon_1 c +_2 \Gamma^+ d) +_1 (\Gamma^- a +_2 \varepsilon_1 b) =$$

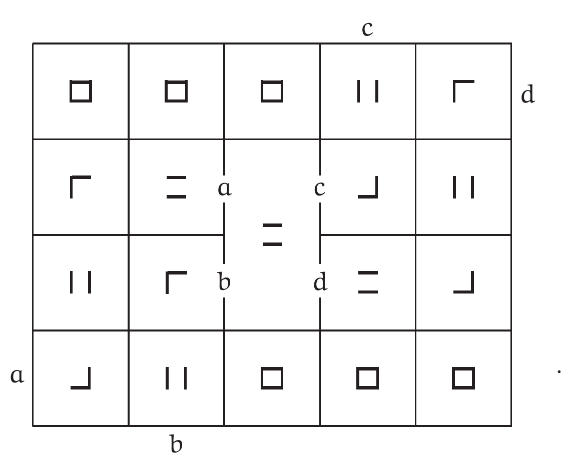


where the last diagram is obtained adding the degenerate middle row, and

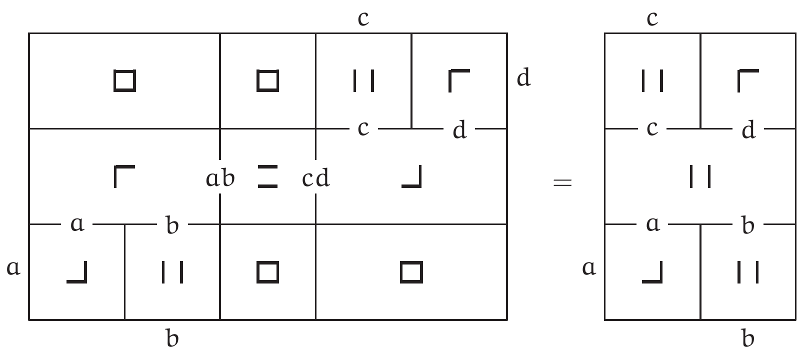
$$\Theta_2\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix} = (\varepsilon_2 a +_1 \Gamma^+ b) +_2 (\Gamma^- c +_1 \varepsilon_2 b) =$$



Then we want to prove $\Theta_1 = \Theta_2$. A usual way of proving that two compositions of arrays produce the same result is to construct a common subdivision. One that is appropriate for this case is

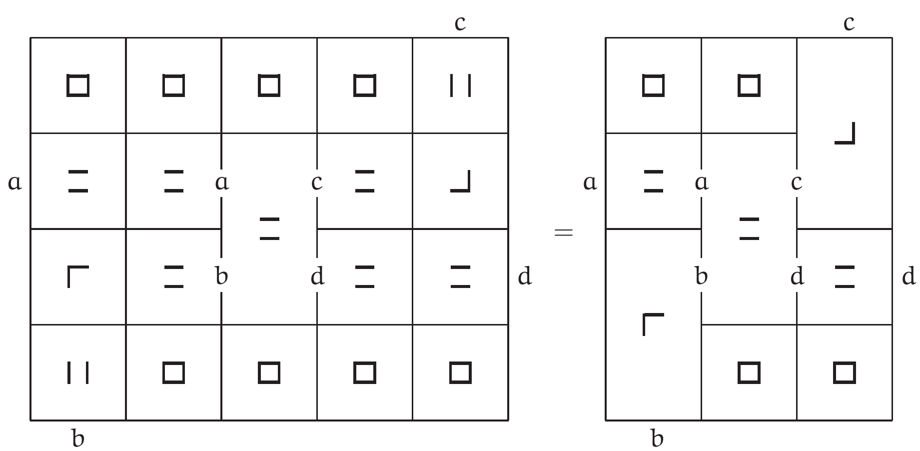


From this diagram, we may compose the second and third row using the transport law and then rearrange things, getting Θ_1 as indicated



$$= \Theta_1\left(a \begin{matrix} c \\ b \end{matrix} d \right).$$

Similarly, operating in the bottom left and the top right corner, we get



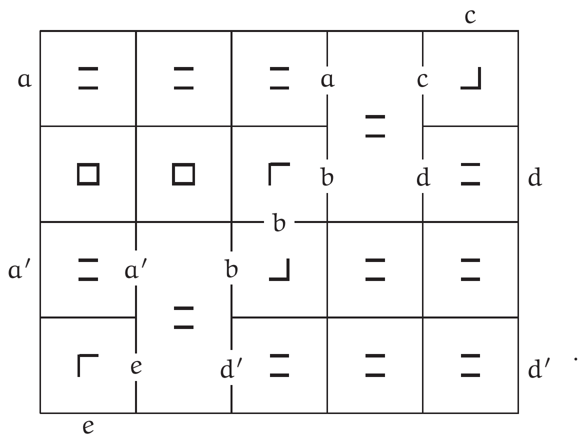
and this last diagram is, quite clearly Θ_2. We write Θ for the common value.

We would like to prove that Θ is a morphism. From any of its representations, it is clear that Θ commutes with faces and degeneracies. The only point we have to prove is that it commutes with both compositions. In this direction, it is good to have two definitions of Θ. First, we use $\Theta = \Theta_2$ to prove that Θ preserves the vertical composition. The use of $\Theta = \Theta_1$ to prove that it preserves the horizontal composition is similar.

So we want to prove

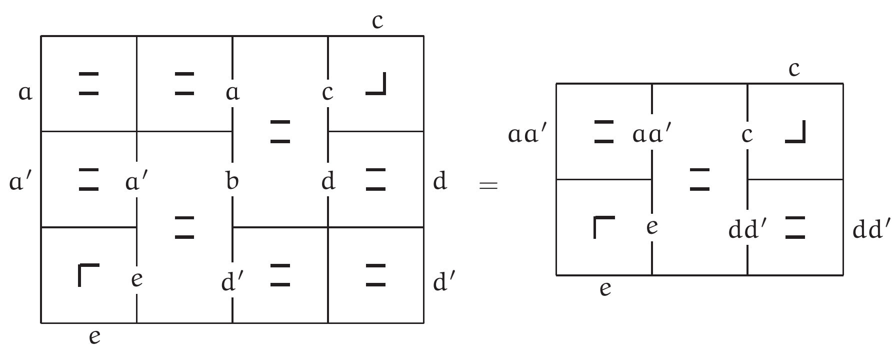
$$\Theta_2\left(a \begin{matrix} c \\ b \end{matrix} d \right) +_1 \Theta_2\left(a' \begin{matrix} b \\ e \end{matrix} d' \right) = \Theta_2\left(aa' \begin{matrix} c \\ e \end{matrix} dd' \right).$$

As before we compute a common subdivision in two ways. The common subdivision we choose is



If we compose the first two rows, they produce $\Theta_2\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix}$. Similarly, the two last rows give $\Theta_2\begin{pmatrix} & b & \\ a' & & d' \\ & e & \end{pmatrix}$.

On the other hand, making some easy adjusts on the three middle rows, we get



which clearly is $\Theta_2\begin{pmatrix} & c & \\ aa' & & dd' \\ & e & \end{pmatrix}$. □

6.6 Equivalence between XMod and DGpds: folding.

In this section, we prove the equivalence between the category DGpds of double groupoids of Definition 6.4.3 and that of crossed modules of groupoids XMod of Definition 6.2.1.

On the one hand, the crossed module associated to a double groupoid is given by the functor

$$\gamma : \mathsf{DGpds} \to \mathsf{XMod}.$$

restriction of the one defined in Section 6.2.

On the other hand, there is a double groupoid associated to each crossed module as was already hinted at the end of Section 6.1. We shall develop this idea in this Section. We recall that to generalise the category of shells in a category, we use 2-shells which commute up to some element in the image of the crossed module.

Let $\mathcal{M} = (\mu : M \to P)$ be a crossed module over a groupoid. There is an associated double groupoid $G = \lambda\mathcal{M}$ whose sets are

$$G_0 = P_0, \quad G_1 = P_1 \quad \text{and} \quad G_2 = \{(m, (b, a; c, d)) \mid \mu m = a^{-1}b^{-1}cd\}.$$

The elements of G_2 may be represented by

$$\begin{tikzcd} \cdot \arrow[r, dash, "c"] \arrow[d, dash, "b"'] \arrow[dr, phantom, "m"] & \cdot \arrow[d, dash, "d"] \\ \cdot \arrow[r, dash, "a"'] & \cdot \end{tikzcd}$$

where μm measures the lack of commutativity of the boundary, giving the composition of the sides of the boundary in clockwise direction starting from the bottom right corner, considered as base point of the square. This choice is conventional, and will influence many later formulae. You are invited to consider the effect of other conventions on formulae below.

The category structure in (G_1, G_0) is the same as that of (P_1, P_0), so it is a groupoid. The horizontal and vertical structures on (G_2, G_1) have source, target and identities defined as in $\square P$. The definitions of the compositions in dimension 2 is the key to the work.

For the 'horizontal' composition we require the boundaries to be given as follows

$$\begin{tikzcd} \cdot \arrow[r, dash, "g"] \arrow[d, dash, "h"'] \arrow[dr, phantom, "n"] & \cdot \arrow[d, dash, "b"] \\ \cdot \arrow[r, dash, "k"'] & \cdot \end{tikzcd} +_2 \begin{tikzcd} \cdot \arrow[r, dash, "c"] \arrow[d, dash, "b"'] \arrow[dr, phantom, "m"] & \cdot \arrow[d, dash, "d"] \\ \cdot \arrow[r, dash, "a"'] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, dash, "gc"] \arrow[d, dash, "h"'] \arrow[dr, phantom, "A"] & \cdot \arrow[d, dash, "d"] \\ \cdot \arrow[r, dash, "ka"'] & \cdot \end{tikzcd} \tag{6.6.1}$$

and for the 'vertical composition' we require

$$\begin{tikzcd} \cdot \arrow[r, dash, "g"] \arrow[d, dash, "f"'] \arrow[dr, phantom, "u"] & \cdot \arrow[d, dash, "e"] \\ \cdot \arrow[r, dash, "c"'] & \cdot \end{tikzcd} +_1 \begin{tikzcd} \cdot \arrow[r, dash, "c"] \arrow[d, dash, "b"'] \arrow[dr, phantom, "m"] & \cdot \arrow[d, dash, "d"] \\ \cdot \arrow[r, dash, "a"'] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, dash, "g"] \arrow[d, dash, "fb"'] \arrow[dr, phantom, "B"] & \cdot \arrow[d, dash, "ed"] \\ \cdot \arrow[r, dash, "a"'] & \cdot \end{tikzcd} \tag{6.6.2}$$

The problem is to find reasonable values in M for A, B. With our convention the boundary of the square A is:

$$(ka)^{-1}h^{-1}gcd = a^{-1}k^{-1}h^{-1}gcd = a^{-1}(k^{-1}h^{-1}gb)a(a^{-1}b^{-1}cd) = a^{-1}\mu(n)a\mu(m).$$

So a good choice is

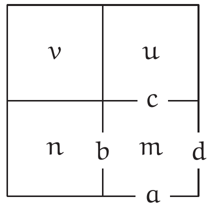
$$A = n^a\, m.$$

This agrees with intuition since n has to be 'moved to the right' by the edge a to have the same base point as m. Similarly, and the calculation is left to you, a good choice is

$$B = m\,u^d,$$

since u has to be 'moved down' by the edge d. Notice that we use the rule CM1) for a crossed module.

It is not difficult to check that with these compositions all three categories are groupoids. We now verify the interchange law, using the following diagram,



Evaluating the rows first gives the first component of the composition, in an abbreviated notation since the edges are omitted, as

$$\begin{bmatrix} v^c u \\ \\ n^a v \end{bmatrix} = (n^a m)(v^c u)^d$$

while evaluating the columns first gives the first component of the composition, in a similar notation, as

$$\begin{bmatrix} nv^b & mu^d \end{bmatrix} = (nv^b)^a m u^d.$$

So to prove the interchange law we have to verify that

$$mv^{cd} = v^{ba} m.$$

This follows from CM2) since $\mu m = a^{-1}b^{-1}cd$ and then

$$m^{-1}v^{ba}m = (v^{ba})^{\mu m} = v^{cd}.$$

Remark These 'childish calculations' were a key to the whole theory, and will be part of the higher dimensional theory in Chapter 13.

To finish, we define a thin structure on G by the obvious morphism

$$\Theta : \square P \to G_2$$

given by $\Theta(a, b, c, d) = (1, (a, b, c, d))$.

This gives a functor

$$\lambda : \mathsf{XMod} \to \mathsf{DGpds}$$

and that $\gamma\lambda\mathcal{M}$ is naturally isomorphic to $\mathcal{M}$ is trivial in dimensions 0,1 and in dimension 2 follows from

$$(\gamma\lambda\mathcal{M})_2 = \{(m, (1, 1, 1, \mu m)) \mid m \in M\} \cong M.$$

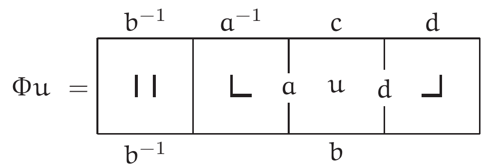
It is rather more involved to get a natural isomorphism from G to $\lambda\gamma G$ for any double groupoid G. In order to do this, we shall see first that a double groupoid is "generated" by the thin elements and those that have only one non-degenerate face, which we assume to be the top face. To this end we "fold" all faces to the chosen one.

Definition 6.6.1 Let G be a double groupoid. We define the *folding* map

$$\Phi : G_2 \to (\gamma G)_2 \subseteq G_2$$

by the formula $\Phi u = [-_2\varepsilon_1\partial_1^+ u,\ -_2\Gamma^-\partial_2^- u,\ u,\ \Gamma^-\partial_2^+ u]$. Notice that this can be defined only in the groupoid case because we are using $-_2$. □

In the usual description



Now let us see that the boundary of Φu is the one we expect. As a consequence Φ is well defined.

Proposition 6.6.2 *All faces of* Φu *are identities except the first in the vertical direction, and*

$$\partial_1^- \Phi u = \partial_1^+ u^{-1} \partial_2^- u^{-1} \partial_1^- u \partial_2^+ u.$$

Thus $\Phi u \in \gamma G_2$ *and* $\mathrm{Im}\ \Phi \subseteq \gamma G_2$.

Proof All are easy calculations which are left as exercises. □

Also from the definition, the following property is clear.

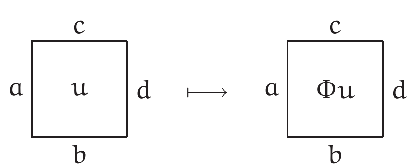
Proposition 6.6.3 *All* $u \in \gamma G_2$ *satisfy* $\Phi u = u$. *Thus* $\gamma G_2 = \mathrm{Im}\ \Phi$ *and* $\Phi\Phi = \Phi$.

Proof This is immediate since in this case all the elements making up Φu except u itself are identities. □

We are now able to define a map

$$\Psi : G_2 \to (\lambda\gamma)G_2$$

by mapping any element $u \in G_2$ to the element given by the folding map Φu and the shell of u:



We shall see that this map is an isomorphism between the two double groupoids.

It is clear that Ψ preserves faces. Also Ψ preserves thin elements since Φ of a thin element is a composition of thin elements and so is thin.

The most delicate part of the proof is the behaviour of the folding map Φ with respect to compositions. We obtain not a homomorphism but a kind of 'derivation', involving conjugacies, or, equivalently, the action in the crossed module γG.

Proposition 6.6.4 *Let* $u, v, w \in G_2$ *be such that* $u +_1 v, u +_2 w$ *exist, and let* $b = \partial_1^+ u$, $g = \partial_2^+ v$. *Then*

$$\Phi(u +_1 v) = [\Phi v, -_2\varepsilon_1 g, \Phi u, \varepsilon_1 g] = \Phi v +_2 (\Phi u)^g$$

$$\Phi(u +_2 w) = [-_2\varepsilon_1 b, \Phi u, \varepsilon_1 b, \Phi w] = (\Phi u)^b +_2 \Phi w.$$

Proof The proof of the second rule is simple, involving composition and cancelation in direction 2, so we prove in detail only the first rule. As before, this is done by constructing a common subdivision and computing it in two ways. Namely if both u, v are represented by

$$\begin{array}{ccc} & c & \\ a & \boxed{\;u\;} & d \\ & b & \end{array} \quad \text{and} \quad \begin{array}{ccc} & b & \\ e & \boxed{\;v\;} & g \\ & f & \end{array}$$

then

$$u +_1 v \quad = \quad \begin{array}{ccc} & c & \\ ae & \boxed{u +_1 v} & dg \\ & f & \end{array}$$

So we have

$$\Phi(u +_1 v) \quad = \quad \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{f^{-1}} & \multicolumn{1}{c}{(ae)^{-1}} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{dg} \\ \hline \| & \llcorner & u +_1 v & \lrcorner \\ \hline \end{array}$$

Applying both transport laws to the second and fourth square, we get a refinement

$$\begin{array}{|c|c|c|c|c|c|} \multicolumn{1}{c}{f^{-1}} & \multicolumn{1}{c}{e^{-1}} & \multicolumn{1}{c}{a^{-1}} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{d} & \multicolumn{1}{c}{g} \\ \hline \| & \| & \llcorner & u & \lrcorner & \| \\ \hline \| & \llcorner & = & v & = & \lrcorner \\ \hline \end{array}$$

having the same composition by Remark 6.1.5. Next we get another array

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|} \multicolumn{1}{c}{f^{-1}} & \multicolumn{1}{c}{e^{-1}} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{g} & \multicolumn{1}{c}{g^{-1}} & \multicolumn{1}{c}{b^{-1}} & \multicolumn{1}{c}{a^{-1}} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{d} & \multicolumn{1}{c}{g} \\ \hline \| & \| & \| & \| & \| & \| & \llcorner & u & \lrcorner & \| \\ \hline \| & \llcorner & v & \lrcorner & \| & \| & \square & \| & \square & \| \\ \hline \end{array}$$

having the same composite because each row has same composite in both cases (apply Remark 6.1.5). Now we can compose vertically in this last diagram to get

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|} \multicolumn{1}{c}{f^{-1}} & \multicolumn{1}{c}{e^{-1}} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{g} & \multicolumn{1}{c}{g^{-1}} & \multicolumn{1}{c}{b^{-1}} & \multicolumn{1}{c}{a^{-1}} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{d} & \multicolumn{1}{c}{g} \\ \hline \| & \llcorner & v & \lrcorner & \| & \| & \llcorner & u & \lrcorner & \| \\ \hline \end{array}$$

and this is clearly $\Phi v +_2 (\Phi u)^g$ as stated. □

The important consequence is that the map

$$\Psi : G_2 \to (\lambda\gamma G)_2$$

we are studying is a homomorphism with respect to both compositions since the equations proved in the preceding property are part of the definition of the compositions in $(\lambda\gamma G)_2$.

To end our proof of the equivalence between the categories of crossed modules over groupoids and double groupoids, it just remains to prove that the map Ψ is bijective, and preserves the thin structures. Let us start by characterising the thin elements of G_2 using the folding map.

Proposition 6.6.5 *An element $u \in G_2$ is thin if and only if $\Phi u = 1$.*

Proof As we pointed out in the Remark 6.5.4 an element $u \in G_2$ is thin if and only if it is a composition of identities and connections. By the preceding properties, it is clear that both identities and connections go to 1 under the folding map, so the same remains true for their compositions.

Conversely, if $u \in G_2$ is such that $\Phi u = 1$, by the definition of Φ, we have the following diagram

$$\begin{array}{|c|c|c|c|}\hline \mid\mid & \llcorner & u & \lrcorner \\ \hline \end{array} \quad = \quad 1$$

Solving this equation for u, we get that it is a product of identities and connections:

$$\begin{array}{ccccccccc} & & & b & b^{-1} & & & & \\ u & = & \begin{array}{|c|}\hline \lrcorner \\ \hline\end{array} & \begin{array}{|c|}\hline \mid\mid \\ \hline\end{array} & \begin{array}{|c|}\hline \mid\mid \\ \hline\end{array} & \begin{array}{|c|}\hline \llcorner \\ \hline\end{array} & \begin{array}{|c|}\hline u \\ \hline\end{array} & \begin{array}{|c|}\hline \lrcorner \\ \hline\end{array} & \begin{array}{|c|}\hline \llcorner \\ \hline\end{array} \end{array}$$

$$= \quad \begin{array}{|c|c|c|c|}\hline \lrcorner & \mid\mid & \Phi u & \llcorner \\ \hline \end{array}$$

□

Corollary 6.6.6 *The map Ψ preserves the thin structures.*

Thus we can conclude that an element $u \in G_2$ is uniquely determined by its boundary and its image under the folding map.

Proposition 6.6.7 *Given elements $(a, b, c, d) \in \square G_2$ and $m \in \gamma G_2$, there is an element $u \in G_2$ with boundary (a, b, c, d) and $\Phi u = m$ if and only if $\partial_1^- m = b^{-1}a^{-1}cd$. Moreover, in this case u is unique.*

Proof As before, we can solve the equation for u getting

$$u \quad = \quad \begin{array}{|c|c|c|c|}\hline \lrcorner & \mid\mid & \Phi u & \llcorner \\ \hline \end{array}$$

thus giving the construction of such element u. Uniqueness follows from the result before. □

Corollary 6.6.8 *The map $\Psi : G_2 \to (\lambda\gamma G)_2$ is bijective and determines a natural equivalence of functors $1 \simeq \lambda\gamma$.*

Thus we have completed the proof that the functors γ and λ give an equivalence of categories.

Corollary 6.6.9 *The functor γ preserves pushouts and, more generally, colimits.*

This allows us to prove first a 2-dimensional van Kampen Theorem for the fundamental double groupoid and then deduce a corresponding theorem for the fundamental crossed module.

Remark 6.6.10 This equivalence also gives another way of checking some equalities on double groupoids. To see that two elements are equal we just need to know that they have the same boundary and that they fold to the same element. Alternatively, we can just check the equations in a double groupoid of the form $\lambda(M \to P)$. □

6.7 Homotopy commutativity lemma.

As we saw in Chapter 1, the desire for the generalisation to higher dimensions of the concept of commutative square was one of the motivations behind the search for higher dimensional group theory.

Recall that when proving the classical van Kampen Theorem 1.6.1, the main idea in the second part was to divide a homotopy into smaller squares and change each one to give a commutative square in π_1. Then we applied the morphisms and got composable commutative 2-shells in K; the fact that in a groupoid any composition of commutative 2-shells is commutative gave the result.

To generalise this to a 2-dimensional van Kampen Theorem, we need several points:

- a concept of commutative 3-shell;

- to prove that the composition of 3-shells is commutative; and

- to relate commutative 3-shells with homotopy.

Those are the objectives of this section.

Before getting down to business, let us point out that there is a further generalisation to commutative n-shells for all n which will be explained in Part III (Chapter 13). Nevertheless, in the 3-dimensional case this can be done using connections with some careful handling.

The process generalises the construction of the double categories of 2-shells and commutative 2-shells seen in Example 6.1.8. In the 3-dimensional case we get what could be labeled a "triple category" but we are not formalising this concept at this stage because is not necessary now and can be done in a more natural way in a more general setting (see Chapter 13).

First we consider 3-shells, the definition of which does not use the thin structure. Let us start with the picture of a 3-cube (where we have drawn the directions to make things a bit easier to follow)

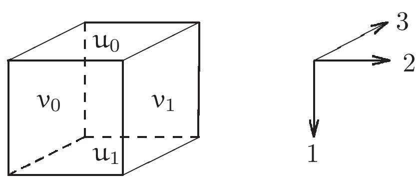


Figure 6.4: cube

Definition 6.7.1 Let D be a double category. A *cube* or (3-*shell)* in D,

$$\alpha = (u_0, u_1, v_0, v_1, w_0, w_1)$$

consists of squares in D_2 which fit together as do the faces of a 3-cube, i.e. such that

$$\partial_1^\sigma u_\tau = \partial_1^\tau v_\sigma;\ \partial_2^\sigma u_\tau = \partial_1^\tau w_\sigma;\ \partial_2^\sigma v_\tau = \partial_2^\tau w_\sigma$$

for $\sigma, \tau = \pm$. We also define the *faces* of the shell to be $\partial_1^\sigma \alpha = u_\sigma, \partial_2^\sigma \alpha = v_\sigma$, and $\partial_3^\sigma \alpha = w_\sigma$ for $\sigma = \pm$. Among these, the *even* faces are $\partial_1^+, \partial_2^-, \partial_3^+$ and the *odd* faces are $\partial_1^-, \partial_2^+, \partial_3^-$; thus the parity of a face ∂_i^σ is the parity of $i + l(\sigma)$ where $l(-) = 0, l(+) = 1$. □

Now we make these 3-shells into a triple category by defining three partial compositions of 3-shells as follows:

Definition 6.7.2 Let $\alpha = (u_0, u_1, v_0, v_1, w_0, w_1)$ and $\beta = (x_0, x_1, y_0, y_1, z_0, z_1)$ be cubes in D.

(i) If $u_1 = x_0$ we define

$$(u_0, u_1, v_0, v_1, w_0, w_1) +_1 (u_1, u_2, y_0, y_1, z_0, z_1) = (u_0, u_2, v_0 +_1 y_0, v_1 +_1 y_1, w_0 +_1 z_0, w_1 +_1 z_1).$$

(ii) If $v_1 = y_0$ we define

$$(u_0, u_1, v_0, v_1, w_0, w_1) +_2 (x_0, x_1, v_1, v_2, z_0, z_1) = (u_0 +_1 x_0, u_0 +_1 x_0, v_0, v_2, w_0 +_2 z_0, w_1 +_2 z_1).$$

(iii) If $w_1 = z_0$ we define

$$(u_0, u_1, v_0, v_1, w_0, w_1) +_3 (x_0, x_1, y_0, y_1, w_1, w_2) = (u_0 +_1 x_0, u_0 +_1 x_0, v_0 +_2 y_0, v_1 +_2 y_1, w_0, w_2).$$

□

It is easy to check that these compositions yield a triple category, in the obvious sense. (This construction will be extended to all dimensions in Chapter 13, Definition 13.5.5, using a notation more suitable for the general case.)

Now we have to formulate the notion of *commutative 3-shell*. From the square case it seems that the proper generalisation would be that the composition of the even faces of the shell equals the composition of the odd faces. We shall take a different route which works in the groupoid case, explaining the other route briefly later.

Let us try to give some meaning to one face of a cube being the composition of the remaining five. We can start by thinking of the picture we get by folding flat those five faces of the cube.

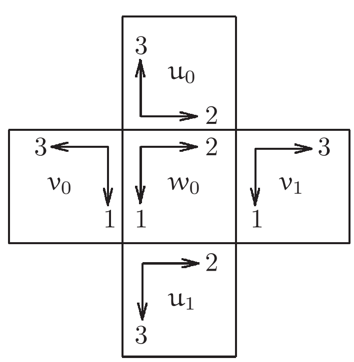


Figure 6.5: Five faces of a cube folded flat

First, notice that already in this figure we need that the double category we are using has all structures groupoids since we are using the inverse of some faces. Also, this Figure does not give a composable array in any obvious sense. However we can use the connections in a double groupoid with thin structure to fill the corners of the diagram to give a composable array:

$$\begin{array}{|c|c|c|}\hline \ulcorner & -_1 u_0 & \mathrm{tr} \\ \hline -_2 v_0 & w_0 & v_1 \\ \hline \llcorner & u_1 & \lrcorner \\ \hline \end{array}$$

We shall say that *the above* 3*-shell* α *in a double groupoid commutes* if the face $w_1 = \partial_3^+ \alpha$ is the composition of the previous array involving the other five faces.

Remark 6.7.3 For a corresponding theory in higher dimensions it seems easier to take the connections rather than thin structure as basic, since the properties of connections in all dimensions are easily expressed in terms of a finite number of axioms, each of which expresses simple geometric features of mappings of cubes. This is developed in Chapter 13. It is then a main feature of the algebra to develop the related notion of thin structure. The chief advantage of the latter is that complicated arguments involving multiple compositions of commuting shells of cubes are reduced to simple arguments on the composition of thin elements. □

Now we get two results on commuting cubes which are key to the proof of our 2-dimensional van Kampen Theorem 6.8.2, in particular in Lemma 6.8.4. The first one shows that two non-degenerate faces of a 'degenerate' commutative 3-shell are equal.

Theorem 6.7.4 *Let* α *be a commutative* 3*-shell in a double groupoid* G. *Suppose that all the faces of* α *not involving direction* 3 *are degenerate. Then* $\partial_3^- \alpha = \partial_3^+ \alpha$.

Proof In this case the array containing the five faces is

$$\begin{array}{|c|c|c|}\hline \square & \mid\,\mid & \square \\ \hline = & \partial_3^+ \alpha & = \\ \hline \square & \mid\,\mid & \square \\ \hline \end{array}$$

whose composition is clearly $\partial_3^+ \alpha$. Thus the commutativity of the 3-shell implies that $\partial_3^- \alpha = \partial_3^+ \alpha$. □

Our second result is about the composition of commutative 3-shells .

Theorem 6.7.5 *In a double groupoid with connections, any composition of commutative* 3*-shells is commutative.*

Proof It is enough to prove that any composition of two commutative 3-shells is commutative.

So, let us consider $\alpha = (u_0, u_1, v_0, v_1, w_0, w_1)$ and $\beta = (x_0, x_1, y_0, y_1, z_0, z_1)$ two commutative 3-shells in a double groupoid G. This means that w_1 and z_1 are respectively given by

$$
w_1 = \begin{array}{|c|c|c|}\hline \ulcorner & -_1 u_0 & \mathrm{tr} \\ \hline -_2 v_0 & w_0 & v_1 \\ \hline \llcorner & u_1 & \lrcorner \\ \hline \end{array}
\qquad
z_1 = \begin{array}{|c|c|c|}\hline \ulcorner & -_1 x_0 & \mathrm{tr} \\ \hline -_2 y_0 & z_0 & y_1 \\ \hline \llcorner & x_1 & \lrcorner \\ \hline \end{array}
$$

We are going to check that composing in any of the three possible directions gives a commutative 3-shell.

If $v_1 = y_0$, the face $\partial_3^+(\alpha +_2 \beta) = w_1 +_2 z_1$ of $\alpha +_2 \beta$ is given by

$$
w_1 +_2 z_1 = \begin{array}{|c|c|c|c|c|c|}\hline \ulcorner & -_1 u_0 & \mathrm{tr} & \ulcorner & -_1 x_0 & \mathrm{tr} \\ \hline -_2 v_0 & w_0 & v_1 & -_2 v_1 & z_0 & y_1 \\ \hline \llcorner & u_1 & \lrcorner & \llcorner & x_1 & \lrcorner \\ \hline \end{array}
$$

Adding first the central two columns of this array and then the central three columns of the resulting array, we get

$$
w_1 +_2 z_1 = \begin{array}{|c|c|c|c|c|}\hline \ulcorner & -_1 u_0 & \square & -_1 x_0 & \mathrm{tr} \\ \hline -_2 v_0 & w_0 & = & z_0 & y_1 \\ \hline \llcorner & u_1 & \square & x_1 & \lrcorner \\ \hline \end{array}
= \begin{array}{|c|c|c|}\hline \ulcorner & -_1(u_0 +_2 x_0) & \mathrm{tr} \\ \hline -_2 v_0 & w_0 +_2 z_0 & y_1 \\ \hline \llcorner & u_1 +_2 x_1 & \lrcorner \\ \hline \end{array}
$$

Thus $\alpha +_2 \beta$ is a commutative 3-shell.

Working vertically in the same way we can prove that $\alpha +_1 \beta$, when it is defined, is commutative if both α and β are commutative.

The case $\alpha +_3 \beta$ is a bit different. In this case $w_1 = z_0$, thus we have

$$w_1 \;=\; \begin{array}{|c|c|c|}\hline \ulcorner & -_1 u_0 & \mathrm{tr} \\ \hline -_2 v_0 & w_0 & v_1 \\ \hline \llcorner & u_1 & \lrcorner \\ \hline \end{array} \qquad w_2 \;=\; \begin{array}{|c|c|c|}\hline \ulcorner & -_1 x_0 & \mathrm{tr} \\ \hline -_2 y_0 & w_1 & y_1 \\ \hline \llcorner & x_1 & \lrcorner \\ \hline \end{array}$$

Substituting w_1 in the second array for the first array and subdividing the other blocks to get a composable array, we get that

$$w_2 \;=\; \begin{array}{|c|c|c|c|c|}\hline \ulcorner & = & -_1 u_0 & = & \mathrm{tr} \\ \hline \| & \ulcorner & -_1 x_0 & \mathrm{tr} & \| \\ \hline -_2 v_0 & -_2 y_0 & z_0 & y_1 & v_1 \\ \hline \| & \llcorner & x_1 & \lrcorner & \| \\ \hline \llcorner & = & u_1 & = & \lrcorner \\ \hline \end{array}$$

Now, we can compose by blocks and, using the transport law, we get

$$w_2 \;=\; \begin{array}{|c|c|c|}\hline \ulcorner & -_1(u_0 +_1 x_0) & \mathrm{tr} \\ \hline -_2(v_0 +_2 y_0) & w_0 & v_1 +_2 y_1 \\ \hline \llcorner & u_1 +_1 x_1 & \lrcorner \\ \hline \end{array}$$

Thus $\alpha +_3 \beta$ is also a commutative 3-shell. □

Let us go now to the case of the fundamental double groupoid of a triple (X, A, C). In particular, we will see that some 3-cubes $h : I^3 \to X$ produce a commutative 3-shell in $\rho(X, A, C)$. This we call a 'homotopy commutativity lemma' reserving the term homotopy addition lemma which we give later for a result expressing the boundary of a cube or simplex in terms of a 'sum' of the faces.

For the statement of the lemma we introduce some notation that represents the changes of coordinates suggested by Figure 6.5. So, if $h : I^3 \to X$ is a cube in X, then the faces of h are given by restriction to the corresponding faces of the cube, i.e.

$$\partial_i^\alpha h = h \circ \eta_i^\alpha,$$

where $\eta_i^\alpha(x_1, x_2) = (y_1, y_2, y_3)$, the y_j being defined by $y_j = x_j$ for $j < i$, $y_i = \alpha$, and $y_j = x_{j-1}$ for $j > i$.

Also in some of the cases we are going to need some switching of coordinates, so let us consider $\tilde{\eta}_1^\alpha(x_1, x_2) = (\alpha, x_2, x_1)$.

Proposition 6.7.6 (the Homotopy Commutativity Lemma) *Let (X, A, C), ρ be as in section 6.3. Let h be a cube in X with edges in A and vertices in C. Then the 3-shell in $\rho(X, A, C)$ given by the boundary of h is commutative.*

Proof What the proposition says is that if the elements $u_\alpha, v_\alpha, w_\alpha$ of ρ_2 represented by its faces are respectively the classes of $h \circ \tilde{\eta}_1^\alpha, h \circ \eta_2^\alpha, h \circ \eta_3^\alpha$ $(\alpha = 0, 1)$. Then

$$w_1 = \begin{bmatrix} \ulcorner & -_1 u_0 & \mathrm{tr} \\ -_2 v_0 & w_0 & v_1 \\ \llcorner & u_1 & \lrcorner \end{bmatrix}$$

in ρ_2 where the corner elements are thin elements as above.

Consider the maps $\varphi_0, \varphi_1 : I^2 \to I^3$ defined by

$$\varphi_0 = \begin{bmatrix} -_2 -_1 \Gamma & -_1(\tilde{\eta}_1^0) & -_1\Gamma \\ -_2\eta_2^0 & \eta_3^- & \eta_2^1 \\ -_2\Gamma & \tilde{\eta}_1^1 & \Gamma \end{bmatrix}, \qquad \varphi_1 = \begin{bmatrix} -_2 -_1 \Gamma & 1 & -_1\Gamma \\ 0 & \eta_3^1 & 0 \\ -_2\Gamma & 1 & \Gamma \end{bmatrix}.$$

where Γ is the map induced by $\gamma : I^2 \to I$ given by $\gamma(x_1, x_2) = \max(x_1, x_2)$. Notice that φ_0, φ_1 agree on ∂I^2 and so, since I^3 is convex, the linear homotopy

$$\begin{array}{rcl} F : I^2 \times I & \to & I^3 \\ (x_1, x_2), t & \mapsto & t\varphi_0(x_1, x_2) + (1 - t)\varphi_1(x_1, x_2) \end{array}$$

gives an homotopy rel ∂I^2 between φ_0 and φ_1.

Now $h\phi_0, h\phi_1$ are the two compositions given in the next Figure:

$$\begin{array}{|c|c|c|} \hline \ulcorner & -_1 u_0 & \mathrm{tr} \\ \hline -_2 v_0 & w_0 & v_1 \\ \hline \llcorner & u_1 & \lrcorner \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \hline \square & \| & \square \\ \hline = & w_1 & = \\ \hline \square & \| & \square \\ \hline \end{array}$$

Figure 6.6: Two arrays with the same boundary

Hence $\langle\!\langle h\varphi_0 \rangle\!\rangle = \langle\!\langle h\varphi_1 \rangle\!\rangle$ in ρ_2. So the 3-shell defined by h is commutative. $\square$

Remark 6.7.7 In the case where D is a double category with thin structure, we cannot get a formula of the above type, because of the lack of inverses. What we can expect as commuting boundary is a formula saying that some composition of the even faces is the same as a composition of the odd faces. Let us investigate this case.

If we fold flat the faces of the 3-cube, the six faces look like:

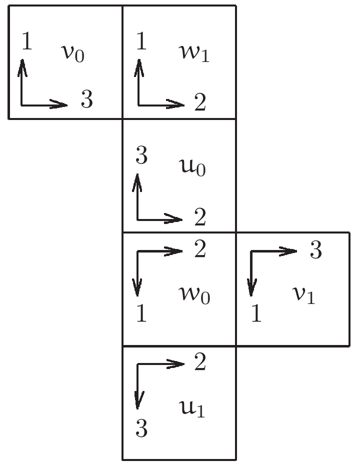


Figure 6.7: Cube boundary folded in a plane

This diagram can be nicely cut in two pieces such that each one can be transformed into a composable array using connections as follows:

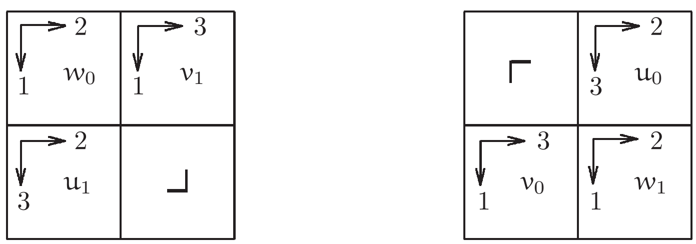
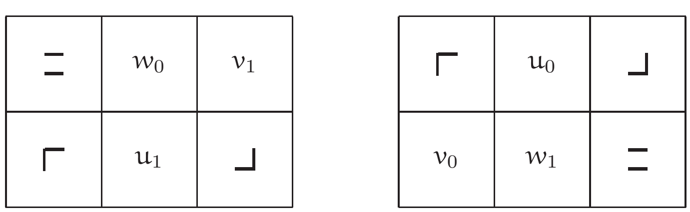


Figure 6.8: Cube boundary decomposed in two

It seems that we could say that a 3-shell is commutative if both compositions are the same, but this does not work because the two squares have different boundary. We expand both squares to get the same boundaries:



Therefore, we can say that *a 3-shell α in a double category with thin structure commutes* if the above compositions are equal. This definition works (see the Notes). □

6.8 Proof of the 2-dimensional van Kampen Theorem.

In this last section of Part I we shall prove a 2-dimensional van Kampen Theorem (6.8.2) which includes as a particular case Theorem 2.3.1 some of whose algebraic consequences have been studied in Chapters 4 and 5.

Theorem 6.8.2 is true for triples of spaces (X, A, C) satisfying some connectivity conditions which can be expressed as algebraic conditions on the π_0 and π_1 functors.

Definition 6.8.1 We say that the triple (X, A, C) is *connected* if the following conditions hold:

$(\ddagger)_0$. The maps $\pi_0(C) \to \pi_0(A)$ and $\pi_0(C) \to \pi_0(X)$ are surjective.

$(\ddagger)_1$. The morphism of groupoids $\pi_1(A, C) \to \pi_1(X, C)$ is piecewise surjective.

Notice that condition $(\ddagger)_0$ is equivalent to saying that C intersects all path components of X and all of A. Also condition $(\ddagger)_1$ just says that the function $\pi_1(A)(x, y) \to \pi_1(X)(x, y)$ induced by inclusion is surjective for all $x, y \in C$. It may be shown that given $(\ddagger)_0$, condition $(\ddagger)_1$ may be replaced by

$(\ddagger')_1$. For each $x \in C$, the homotopy fibre over x of the inclusion $A \to X$ is path connected.

That both conditions can be stated in terms of connectivity, explains the origin of the term 'connected'. $\square$

Let us introduce some notation which will be helpful in both the statement and the proof of Theorem 6.8.2. Suppose we are given a cover $\mathcal{U} = \{U^\lambda\}_{\lambda \in \Lambda}$ of X such that the interiors of the sets of $\mathcal{U}$ cover X. For each $\nu = (\lambda_1, \cdots, \lambda_n) \in \Lambda^n$ we write

$$U^\nu = U^{\lambda_1} \cap \ldots \cap U^{\lambda_n}.$$

An important property of this situation is that a continuous function f on X is entirely determined by a family of continuous functions $f^\lambda : U^\lambda \to X$ which agree on all pairwise intersections $U^{\lambda_1} \cap U^{\lambda_2}$. This is expressed by saying that the following diagram

$$\bigsqcup_{\lambda_1, \lambda_2 \in \Lambda} U^{\lambda_1} \cap U^{\lambda_2} \underset{i_2}{\overset{i_1}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} U^\lambda \xrightarrow{\ i\ } X$$

is a coequaliser in the category of topological spaces. The functions i_1, i_2 are determined by the inclusions $U^\nu = U^{\lambda_1} \cap U^{\lambda_2} \to U^{\lambda_1}$, and $U^\nu \to U^{\lambda_2}$ for each $\nu = (\lambda_1, \lambda_2) \in \Lambda^2$, and i is determined by the inclusions $U^\lambda \to X$ for each $\lambda \in \Lambda$.

It is not difficult to extend this to the case of a triple (X, A, C). If we define $A^\nu = U^\nu \cap A$, and $C^\nu = U^\nu \cap C$, we get a similar coequaliser diagram in the category of triples of spaces:

$$\bigsqcup_{\nu \in \Lambda^2} (U^\nu, A^\nu, C^\nu) \underset{i_2}{\overset{i_1}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} (U^\lambda, A^\lambda, C^\lambda) \xrightarrow{\ i\ } (X, A, C).$$

Now we move from this to the homotopical situation, by applying ρ to the coequaliser diagram of triples. So the homotopy double groupoids in the following ρ-*sequence of the cover* are well-defined:

$$\bigsqcup_{\nu \in \Lambda^2} \rho(U^\nu, A^\nu, C^\nu) \underset{i_2}{\overset{i_1}{\rightrightarrows}} \bigsqcup_{\lambda \in \Lambda} \rho(U^\lambda, A^\lambda, C^\lambda) \xrightarrow{\ i\ } \rho(X, A, C). \qquad (6.8.1)$$

Here $\bigsqcup$ denotes disjoint union, which is the coproduct in the category of double groupoids. It is an advantage of the approach using a set of base points that the coproduct in this category is so simple to describe. The morphisms i_1, i_2 are determined by the inclusions $U^\nu = U^{\lambda_1} \cap U^{\lambda_2} \to U^{\lambda_1}$, and $U^\nu \to U^{\lambda_2}$ for each $\nu = (\lambda_1, \lambda_2) \in \Lambda^2$, and i is determined by the inclusions $U^\lambda \to X$ for each $\lambda \in \Lambda$.

Theorem 6.8.2 (2-dimensional van Kampen Theorem for the fundamental double groupoid) *Assume that for every finite intersection U^ν of elements of $\mathcal{U}$ the triple (U^ν, A^ν, C^ν) is connected. Then*

(Con) *the triple (X, A, C) is connected, and*

(Iso) *in the above ρ-sequence of the cover, i is the coequaliser of i_1, i_2 in the category of double groupoids.*

Proof The proof follows the pattern of the 1-dimensional case (Theorem 1.6.1) and it will take several stages.

We shall be aiming for the coequaliser result (Iso) because the connectivity part (Con) is obtained along the way. So we start with a double groupoid G and a morphism of double groupoids

$$f' : \bigsqcup_{\lambda \in \Lambda} \rho(U^\lambda, A^\lambda, C^\lambda) \to G$$

such that $f' i_1 = f' i_2$. We have to show that there is a unique morphism of double groupoids

$$f : \rho(X, A, C) \to G$$

such that $fi = f'$.

Recall that by the structure of coproduct in the category of double groupoids, the map f' is just the disjoint union of maps $f^\lambda : \rho(U^\lambda, A^\lambda, C^\lambda) \to G$ and the condition $f' i_1 = f' i_2$ translates to f^{λ_1} and f^{λ_2} being the same when restricted to $\rho(U^\nu, A^\nu, C^\nu)$ for $\nu = (\lambda_1, \lambda_2)$.

To define f on $\rho(X, A, C)$ we shall describe how to construct an $F(\alpha) \in G_2$ for all $\alpha \in R_2(X, A, C)$. Then we define $f(\langle\!\langle \alpha \rangle\!\rangle) = F(\alpha)$ and prove independence of all choices.

Stage 1.- Define $F(\alpha) \in G_2$ when $\alpha = [\alpha_{ij}]$ such that each α_{ij} lies in some $R_2(U^\lambda, A^\lambda, C^\lambda)$.

The easiest case is when the image of α lies in some U^λ of $\mathcal{U}$. Then α determines uniquely an element $\alpha^\lambda \in R_2(U^\lambda, A^\lambda, C^\lambda)$. The only way to have $fi = f'$ is by defining

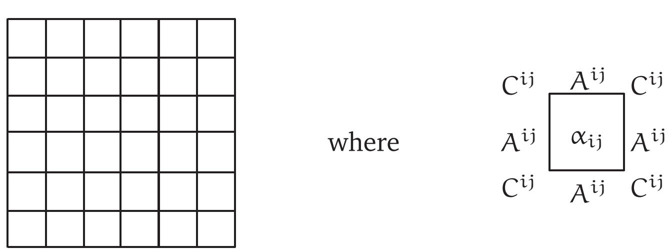
$$F(\alpha) = f^\lambda(\langle\!\langle \alpha^\lambda \rangle\!\rangle).$$

This definition does not depend on the choice of λ, because of the condition $f' i_1 = f' i_2$.

Next, suppose that the element $\alpha \in R_2(X, A, C)$ may be expressed as the composition of an array

$$\alpha = [\alpha_{ij}]$$

such that each α_{ij} belongs to $R_2(X, A, C)$, and also the image of α_{ij} lies in some U^λ of $\mathcal{U}$ which we shall denote by U^{ij}.



$\alpha = [\alpha_{ij}]$

Figure 6.9: Case $\alpha = [\alpha_{ij}]$ with $\alpha_{ij} \in R_2(U^{ij}, A^{ij}, C^{ij})$

We can define $F(\alpha_{ij})$ for each ij as before. Since the composite $[\alpha_{ij}]$ is defined, it is easy to check using $f'i_1 = f'i_2$, that the elements $F(\alpha_{ij})$ compose in G_2. We define $F(\alpha)$ to be the composite of these elements of G_2, i.e.

$$F(\alpha) = F([\alpha_{ij}]) = [F(\alpha_{ij})],$$

although *a priori* this definition could depend on the subdivision chosen.

Stage 2.- Define $F(\alpha) \in G_2$ by changing α by an f-homotopy to a map of the type used in Stage 1.

This is done analogously to the 1-dimensional case (Theorem 1.6.1). So, first we apply the Lebesgue covering lemma to get a subdivision $\alpha = [\alpha_{ij}]$ such that for each i, j, α_{ij} lies in some element U^{ij} of the covering. In general, we will not have $\alpha_{ij} \in R_2(U^{ij}, A^{ij}, C^{ij})$, so we have to deform α to another β satisfying this condition. The homotopy for this is given by the next lemma. In this we use the cell-structure on I^2 determined by a subdivision of α as in Remark 6.3.2, and also refer to the 'domain' of α_{ij} as defined there.

Lemma 6.8.3 *Let $\alpha \in R_2(X, A, C)$ and let $\alpha = [\alpha_{ij}]$ be a subdivision of α such that each α_{ij} lies in some U^{ij} of $\mathcal{U}$. Then there is an f-homotopy $h : \alpha \equiv \alpha'$, with $\alpha' \in R_2(X, A, C)$, such that, in the subdivision $h = [h_{ij}]$ determined by that of α, each homotopy $h_{ij} : \alpha_{ij} \simeq \alpha'_{ij}$ satisfies:*

(i) *h_{ij} lies in U^{ij};*

(ii) *α'_{ij} belongs to $R_2(X, A, C)$, and so can be considered an element of $R_2(U^{ij}, A^{ij}, C^{ij})$;*

(iii) *if a vertex v of the domain of α_{ij} is mapped into C, then h is constant on v;*

(iv) *if a cell e of the domain of α_{ij} is mapped by α into A(resp. C), then $e \times I$ is mapped by h into A(resp. C), and hence $\alpha'(e)$ is contained in A(resp. C).*

Proof Let K be the cell-structure on I^2 determined by the subdivision $\alpha = [\alpha_{ij}]$, as in Remark 6.3.2. We define h inductively on $K^n \times I \cup K \times \{0\} \subseteq K \times I$ using the connectivity conditions of the statement, where K^n is the n-skeleton of K for $n = 0, 1, 2$.

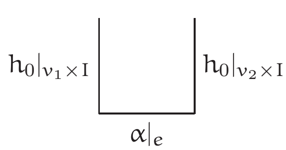
Step 1.- Extend $\alpha|_{K^- \times \{0\}}$ to $h_0 : K^- \times I \to C$.

Since the triples (U^ν, A^ν, C^ν) are connected for all finite sets $\nu \subseteq \Lambda$, the map $\pi_0(C^\nu) \to \pi_0(U^\nu)$ is surjective. For each vertex $v \in K$ we can choose a path lying in the intersection of all the U^λ corresponding to all the 2-cells of K containing v (one to four according to the situation of v) and going from $\alpha(v)$ to a point of C.

In particular, when $\alpha(v) \in C$ we choose the constant path and if $\alpha(v) \in A$, using that $\pi_0(C^\nu) \to \pi_0(A^\nu)$ is also surjective, we choose the path lying in A . These paths give a map $h_0 : K^- \times I \to C$.

Step 2.- Extend $\alpha|_{K^+ \times \{0\}} \cup h_0$ to $h_1 : K^+ \times I \to A$.

For each 1-cell $e \in K$ with vertices v_1 and v_2, we have the following diagram



where on the three sides of $e \times I$ the definition of h_1 is given as indicated. We proceed to extend to $e \times I$ with some care.

If $\alpha(e) \subseteq A$ we consider two cases. When v_1, v_2 are mapped into C, we extend to $e \times I$ using α at each level $e \times \{t\}$. If $\alpha(e) \subseteq A$, and v_1, v_2 are not both mapped into C, since all edges go to A, then we can use a retraction to extend the homotopy.

Otherwise, the product of these three paths defines an element of $\pi_1(U^\nu, C^\nu)$ where U^ν is the intersection of the U^λ corresponding to all the 2-cells containing e (1 or 2 according to the situation of e). Using the condition on the surjectivity of the π_1, we have a homotopy rel $\{0,1\}$ to a path in (A^ν, C^ν). This homotopy gives $h_1|_{e \times \{1\}}$.

Step 3.- Extend $\alpha|_{K\times\{0\}} \cup h_1$ to $h : K \times I \to X$.

This is done using for each 2-cell e the retraction of $e \times I$ to $\partial e \times I \cup e \times 0$

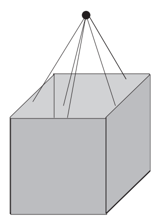


Figure 6.10: Projecting from above in a 3-cube

given by projecting from a point above the centre of the top face. □

The three steps in the construction of h in this Lemma are indicated in Figure 6.11 where the third and fourth diagrams look the same from this direction but from the back the third one looks like a hive with square cells while the fourth diagram is solid.

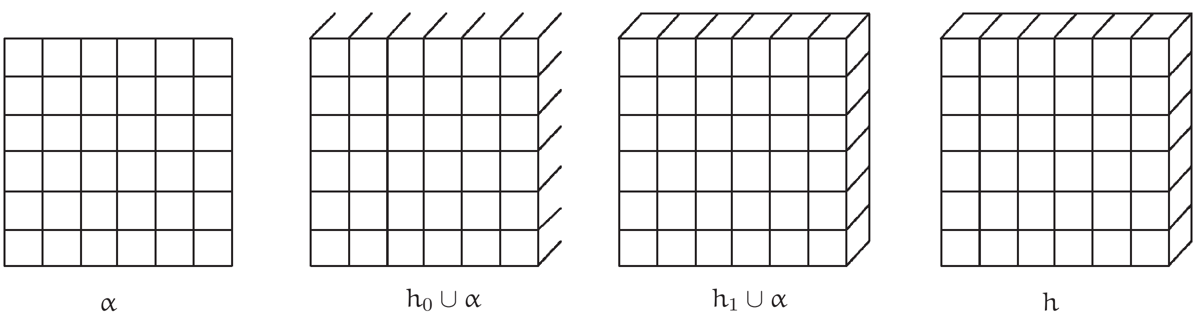


Figure 6.11: Steps in constructing h in Lemma 6.8.3

Notice that the connectivity result (Con) follows immediately from this lemma, particularly (iv), applied to doubly degenerate or to degenerate squares representing elements of an appropriate π_0 or π_1.

We can now define F for an arbitrary element $\alpha \in R_2(X, A, C)$ as follows. First we choose a subdivision $[\alpha_{ij}]$ of α such that for each i, j, α_{ij} lies in some U^{ij}. Then we apply Lemma 6.8.3 to get an element $\alpha' = [\alpha'_{ij}]$ and an f-homotopy $h : \alpha \equiv \alpha'$ decomposing as $h = [h_{ij}]$, the image of each h_{ij} lying in some U^{ij}.

We define

$$F(\alpha) = F(\alpha') = [F(\alpha'_{ij})],$$

i.e the composition of the array in G got by applying the appropriate f^{λ} to the decomposition resulting on the back face of the last diagram in Figure 6.11. Since this in principle depends on the subdivision and the homotopy h we will sometimes write this element as $F(\alpha, (h_{ij}))$.

Stage 3.- Key lemmas

The tools for our independence of choices are going to be proved at this stage. They are two lemmas considering a homotopy H of maps in $\alpha, \beta \in R_2(X, A, C)$ with a given subdivision $H = [H_{ijk}]$. They are represented in the Figure 6.12.

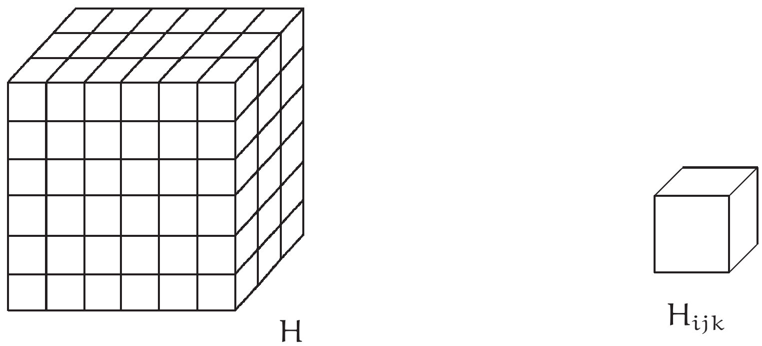


Figure 6.12: Decomposition of a homotopy $H = [H_{ijk}]$

The first lemma is a rather short application of previous results on commutative cubes and states that $F(\alpha) = F(\beta)$ gives particular conditions on α, β and on an f-homotopy $H : \alpha \equiv \beta$.

Lemma 6.8.4 *Let $H : I^3 \to X$ be an f-homotopy of maps $\alpha, \beta : (I^2, \partial I^2, \partial^2 I^2) \to (X, A, C)$. Suppose given a subdivision $H = [H_{ijk}]$ of H such that each H_{ijk} maps its domain D^{ijk} of I^3 into a set U^{ijk} of the cover and maps the vertices and edges of D^{ijk} into C and A respectively, i.e. all its faces lie in $R_2(U^{ijk}, A^{ijk}, C^{ijk})$. Then for the induced subdivisions $\alpha = [\alpha_{ij}]$, $\beta = [\beta_{ij}]$ we have in G that*

$$F(\alpha) = F(\beta). \tag{*}$$

Proof The assumptions imply that each H_{ijk} satisfy the conditions of the homotopy commutativity lemma (6.7.6) and thus defines a commutative 3-shell in $\rho(U^{ijk}, A^{ijk}, C^{ijk})$. This is mapped by f^{ijk} to give a commutative 3-shell in G. The condition $f'i_1 = f'i_2$ implies that these 3-shells are composable in G, and so, by Theorem 6.7.5, their composition is a commutative cube in G. The assumption that H is an f-homotopy allows us to apply Theorem 6.7.4, and to deduce (*), as required. □

Now we have to prove that we can always obtain from a general f-homotopy between two maps an f-homotopy between associated maps that satisfies the conditions of the previous Lemma. This is where our connectivity assumptions are used again.

Lemma 6.8.5 *Let $H : I^3 \to X$ be an f-homotopy of maps $\alpha, \beta : (I^2, \partial I^2, \partial^2 I^2) \to (X, A, C)$. Suppose given a subdivision $H = [H_{ijk}]$ of H such that each H_{ijk} maps its domain D^{ijk} of I^3 into a set U^{ijk} of the cover. Then there is a homotopy Φ of H to a homotopy H' such that such that in the cell structure K determined by the subdivision of H,*

(i) *H' maps the 0-cells of K into C and the 1-cells into A;*

(ii) *if a 0-cell v of K is mapped by H into C, then Φ is constant on v, and if v is mapped into A by H, then so also is $v \times I$ by Φ;*

(iii) *if a 1-cell e of K is mapped by H into C, then Φ is constant on e, and if e is mapped into A by H, then so also is $e \times I$ by Φ.*

Proof As in Remark 6.3.2, but now in dimension 3, there is a cell structure K on I^3 appropriate to the subdivision of H. We define a homotopy $\Phi : K \times I \to X$ of H by induction on $K^n \times I \cup k \times \{0\} \subseteq K$. The first two steps are as in Lemma 6.8.3. This takes us up to $K^1 \times I \cup K \times \{0\}$. Finally, we extend Φ over the 2- and 3-skeleta of K by using retractions, i.e. by a careful use of the Homotopy Extension Property. □

Remark 6.8.6 The map H' constructed in the Lemma gives an f-homotopy from $\alpha' = H'_0$ to $\beta' = H'_1$. Also there is a decomposition of $\alpha' = [\alpha'_{ij}]$ and $\beta' = [\beta'_{ij}]$ which has each element lying in some $R_2(U^\lambda, A^\lambda, C^\lambda)$. Moreover, the homotopy Φ induces homotopies $h : \alpha \equiv \alpha'$ and $h' : \beta \equiv \beta'$ of the type described in Lemma 6.8.3 and later used to define $F(\langle\!\langle \alpha \rangle\!\rangle)$.

In particular, if all the maps in the induced subdivisions $\alpha = [\alpha_{ij}]$ and $\beta = [\beta_{ij}]$ lie in some $R_2(U^\lambda, A^\lambda, C^\lambda)$, the map H' constructed in the lemma gives an f-homotopy $H' : \alpha \equiv \beta$. □

Stage 4.- Independence of choices inside the same f-homotopy class.

Now we can prove that f is well defined, proving independence of two choices.

1.- Independence of the subdivision and the homotopy h of Lemma 6.8.3.

Let us consider two subdivisions of the same map $\alpha \in R_2(X, A, C)$. As there is a common refinement we can assume that one is a refinement of the other. We shall write them $\alpha = [\alpha_{ij}]$ and $\alpha = [\alpha^{ij}_{kl}]$ where for a fixed ij we have $\alpha_{ij} = [\alpha^{ij}_{kl}]$.

Using Lemma 6.8.3, we get f-homotopies $h : \alpha \equiv \alpha'$, with $\alpha' \in R_2(X, A, C)$, such that, in the subdivision $h = [h_{ij}]$ determined by that of α, each homotopy $h_{ij} : \alpha_{ij} \simeq \alpha'_{ij}$ and $h' : \alpha \equiv \alpha''$, with $\alpha'' \in R_2(X, A, C)$, such that, in the subdivision $h' = [h'^{ij}_{kl}]$ determined by that of α, each homotopy $h'^{ij}_{kl} : \alpha^{ij}_{kl} \simeq \alpha''^{ij}_{kl}$. We want to prove that

$$[F(\alpha'_{ij})] = [F(\alpha''^{ij}_{kl})].$$

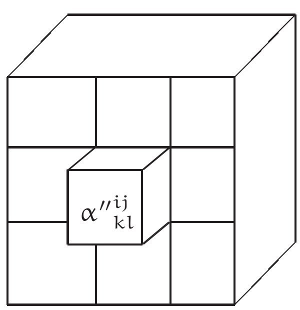


Figure 6.13: Independence of subdivision

The situation for a fixed ij is described in Figure 6.13 where the smaller cube at the front represents h'^{ij}_{kl} and the larger cube at the back is h_{ij}.

If we denote by h'_{ij} the composition of the array $h'_{ij} = [h'^{ij}_{kl}]$ and by α'_{ij} the composition of the array $\alpha'_{ij} = [\alpha'^{ij}_{kl}]$, we have $h'_{ij} : \alpha_{ij} \simeq \alpha''_{ij}$.

Now $\overline{h}h'$ gives an f-homotopy satisfying the conditions of Lemma 6.8.5 if we denote by $\overline{h}$ the homotopy given by $\overline{h}(x, y, t) = h(x, y, 1 - t)$. First, we change this homotopy using Lemma 6.8.5

and we then apply Lemma 6.8.4, to get

$$[F(\alpha'_{ij})]) = [F(\alpha''_{ij})].$$

On the other hand since the second is a refinement of the first, we have

$$[F(\alpha''_{ij})]) = [F(\alpha''^{ij}_{kl})].$$

As a consequence to define the element $F(\alpha)$ we can choose whatever subdivision and homotopy we want insofar as the conditions of Lemma 6.8.3 are met.

2.- Independence of the choice inside the same f-homotopy class.

Let $H : \alpha \equiv \beta$ be an f-homotopy of elements of $R_2(X, A, C)$. We choose a subdivision $H = [H_{ijk}]$ of H so that each H_{ijk} maps into a set of $\mathcal{U}$. On both extremes there are induced subdivisions $\alpha = [\alpha_{ij}]$, $\beta = [\beta_{ij}]$. We apply Lemma 6.8.3 to H, getting $H' : \alpha' \equiv \beta'$.

As indicated in the Remark 6.8.6, these α', β' satisfy the conditions to be used when defining $F(\alpha)$ and $F(\beta)$. Also H' satisfies the conditions of Lemma 6.8.4. Thus

$$F(\alpha) = [F(\alpha'_{ij})] = [F(\beta'_{ij})] = F(\beta).$$

<u>Stage 5</u>.- End of proof

Now we have proved that there is a well-defined map $f : \rho(X, A, C)_2 \to G_2$, given by $f(\langle\!\langle\alpha\rangle\!\rangle) = F(\alpha, (h_{ij}))$, which satisfies $fi = f'$ at least on the 2-dimensional elements of ρ.

The remainder of the proof of (Iso), that is the verification that f is a morphism, and is the only such morphism, is straightforward. It is easy to check that f preserves addition and composition of squares, and it follows from (iii) of Lemma 6.8.3 that f preserves thin elements.

It is now easy to extend f to a morphism $f : \rho(X, A, C) \to G$ of double groupoids, since the 1- and 0-dimensional parts of a double groupoid determine degenerate 2-dimensional parts. Clearly this f satisfies $fi = f'$ and is the only such morphism.

This completes the proof of Theorem 6.8.2. □

Of especial interest (but not essentially easier to prove) is the case of the Theorem in which the cover $\mathcal{U}$ has only two elements; in this case Theorem 6.8.2 gives a push-out of double groupoids. In the applications in previous chapters we have considered only path-connected spaces and assumed that $C = \{x\}$ is a singleton. Taking x as base point, the double groupoids can then be interpreted as crossed modules of groups to give the 2-dimensional analogue of the Seifert-van Kampen Theorem given as Theorem 2.3.1 earlier. We do not know how to prove that theorem without using groupoids in some form. A higher dimensional form of this proof and theorem is given in the second part of this book.

Proof of Theorem 2.3.1 In the case where (X, A) is a based pair with base point x, $\rho(X, A, x)$ is abbreviated to $\rho(X, A)$. That we obtain a pushout of crossed modules under the hypothesis of Theorem 2.3.1 is simply a special case of Theorem 6.8.2, together with Proposition 6.3.7, which gives the equivalence between double groupoids and crossed modules.

The corresponding result of Theorem 2.3.3 follows from Theorem 2.3.1 by standard techniques using mapping cylinders. For analogues of these techniques for the fundamental groupoid, see Chapter 8 of [Bro06].

Remark 6.8.7 An examination of the proof of Theorem 6.8.2 shows that conditions $(\ddagger)_0$ and $(\ddagger)_1$ are required only for 8-fold intersections of elements of $\mathcal{U}$. However, it has been shown by Razak-Salleh [RS76] that in fact one need only assume $(\ddagger)_0$ for 4-fold intersections and $(\ddagger)_1$ for 3-fold intersections. Further, these conditions are best possible. The reader may like to try to recover these results using the tool of Lebesgue covering dimension as in the paper [BRS84].

Remark 6.8.8 Theorem 6.8.2 contains 1-dimensional information which includes most known results expressing the fundamental group of a space in terms of an open cover, but it does not assume that the spaces of the cover or their intersections are path-connected. That is, it contains the classical van Kampen Theorem on $\pi_1(X, A)$ given in Chapter 1.

Thus we have completed the aims of Part I, to give a reasonably full and we hope comprehensible account of what we understand as 2-dimensional nonabelian algebraic topology, which is essentially the theory and application to algebraic topology of crossed modules, double groupoids and related structures.

Now in Part II we move on to the higher dimensional theory. The situation is more complicated because there are several generalisations of crossed modules and double groupoids, with applications to algebraic topology, basically in terms of crossed complexes, or in terms of crossed n-cubes of groups. The theory of crossed complexes is limited in its applications, because it starts as being a purely 'linear' theory. However, even this theory has advantages, in the range of applications, its relation to well known theories, such as chain complexes with a group of operators, its use of groupoids, and its intuitive basis as a development of the methods of Part I. So this is the account we give, in the space we have here.

Notes

(In this we diverge from the definition given in [BH78].)

(they were called special double groupoids with special connections in [BS76a], since more general connections were discussed there)

The name 'transport laws' was given because they were initially borrowed from a transport law for path connections in differential geometry, as explained in [BS76b].

, and it was already done by Brown and Spencer ([BS76a]) in the case that all structures are groupoids,

We follow the proof given by Brown and Mosa for the case of double categories ([BM99]). It is easier for double groupoids and in this case the proof may be traced back to Brown-Higgins ([BH78]). Nevertheless it is interesting to give the proof in the more general case for the possible applications in other situations.

This is how properties of rotations were verified in [BS76b]. The direct proofs are due to Philip Higgins.

Another aspect of the equivalence of categories is that it gives us a large source of double groupoids. Indeed one motivation of the equivalence in the work of [BS76b, BS76a] was simply to find new examples of double groupoids and these were found since there is a large source of crossed modules.

Work on the double category case, proving the equivalence with 2-categories, was done by Spencer in [Spe77] and additional work by Brown and Mosa [BM99]. This work has been extended to all dimensions in [AABS02].

In fact, as has already been pointed out, one of the reasons for introducing connections in the paper [BS76b] was to be able to discuss the notion of commutative 3-shell in a double groupoid.

Part II

Crossed complexes

Introduction to Part II

The utility of crossed modules for certain nonabelian homotopical calculations in dimension 2 has been shown in Part I, mainly as applications of a 2-dimensional van Kampen Theorem. In Part II, we obtain homotopical calculations using what we call *crossed complexes*. Again, a Higher Homotopy van Kampen Theorem (HHvKT) plays a key role, but we have to cover also a range of new techniques.

One new tool is the classifying space functor

$$B : \mathsf{Crs} \to \mathsf{Top}.$$

We shall also set up a notion of homotopy in the category Crs of crossed complexes. A major application of crossed complexes is a *homotopy classification theorem*, that there is a bijection of homotopy classes

$$[X, BC] \cong [\Pi X_*, C]$$

for a crossed complex C and skeletal filtration X_* of a CW-complex X. This we explain in Chapter 10. Its proof requires almost all of the major properties of crossed complexes, of which the key ones are fully justified in Part III.

Chapter 7.5 sets up the basic theory of crossed complexes, on which the rest of Parts II and III are built. Crossed complexes are made up of groupoids, crossed modules over groupoids, and modules over groupoids. In order to handle various constructions on these, particularly the calculation of colimits, we use results on fibrations and cofibrations of categories give in Appendix A.7. This enables one theory for a number of examples. Another important part of this chapter is section 7.5 on the relation between crossed complexes and chain complexes with operators, In accordance with the spirit of this book, and necessitated by the mathematics, we use a groupoid rather than the group of operators traditional in algebraic topology. This is also important mathematically, in relating the monoidal closed structures of crossed complexes and chain complexes with operators, in Chapter 9.

Chapter 8 is devoted to the statement and immediate applications of the Higher Homotopy van Kampen Theorem for crossed complexes.

Chapter 9 introduces a crucial monoidal closed structure on the category of crossed complexes.

Chapter 10 deals with the classifying space of a crossed complex. We find it convenient to define this cubically, and so set up the needed cubical theory. Some of the results on collapsings in this chapter are used in Part III, Chapter 14.

There is one contrast between the 2-dimensional case and that in higher dimensions. Homotopy 2-types are well modeled by crossed modules, but, for $n \geqslant 3$, crossed complexes give only a partial model of homotopy n-types. Thus crossed complexes give only a limited amount of information on many problems we would like to study. Further, the HHvKT itself has strong connectivity assumptions which again limit its utility. However the theorem does yield applications which are not obtainable by other means.

A major point is that crossed complexes can be used for some explicit calculations involving nonabelian information coming from the fundamental group. The fact that a van Kampen Theorem holds shows that this algebra gives a tool for some nonabelian local to global problems in dimensions > 1. Thus our aim is not just solve some specific problems in homotopy theory or algebraic topology, but also to suggest the kind of tools that might be required for a wider investigation of local to global problems, in view of the important part such problems play in mathematics and its applications.

Recall that Part I has given a transition from 1-dimensional to 2-dimensional homotopy theory. Dimension 1 involves in a natural way the fundamental groupoid and the theory of groupoids. Dimension 2 homotopy has been expressed principally using the fundamental crossed module

$$\Pi(X, A, x) = (\pi_2(X, A, x) \xrightarrow{\partial} \pi_1(A, x))$$

of a pointed pair of spaces $x \in A \subseteq X$, and the theory of crossed modules. A Higher Homotopy van Kampen Theorem (HHvKT) in dimension 2 allowed for many nonabelian computations of this crossed module, and so of these second relative homotopy groups. However for the proof of the HHvKT we had to move to the category of double groupoids, and in this respect it was convenient to use also the fundamental crossed module of groupoids

$$\Pi(X, A, C) = (\pi_2(X, A, C) \xrightarrow{\partial} \pi_1(A, C))$$

for a triple of spaces $C \subseteq A \subseteq X$, so that C is thought of as a *set of base points*. This groupoid viewpoint will be essential in this Part II.

It is also an important feature of crossed modules over groupoids that they model all weak homotopy 2-types. This modeling is done via the classifying space functor on crossed modules of groupoids

$$B : \mathsf{XMod} \to \mathsf{Top}.$$

However the definition of this functor and the proof of its properties requires the work of this Part II, since it involves the generalisation from crossed modules to crossed complexes (see Chapter 10).

Some aspects of the situation in dimensions $\geqslant 3$ are analogous to that in lower dimensions. In particular, a convenient generalisation of a triple of spaces is a *filtered space*

$$X_* : X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X,$$

so that the case of a triple is where $X_n = X_{n+1} = X$ for $n \geqslant 2$. The above fundamental crossed module of a triple generalises to the *fundamental crossed complex* of a filtered space

$$\Pi(X_*) : \cdots \xrightarrow{\partial} \pi_n(X_n, X_{n-1}, X_0) \xrightarrow{\partial} \pi_{n-1}(X_{n-1}, X_{n-2}, X_0) \xrightarrow{\partial} \cdots \pi_2(X_2, X_1, X_0) \xrightarrow{\partial} \pi_1(X_1, X_0).$$

Our first task is to explain the algebraic notion of crossed complex and the structure and properties of this $\Pi(X_*)$.

Chapter 7 sets up the basic structures we need. In order to relate these, and in particular to discuss induced constructions in general, it is convenient to use the language of fibrations and cofibrations of categories. This area is covered in a Appendix A.7.

The first major result we state and apply in Chapter 8, is that this functor

$$\Pi : \mathsf{FTop} \to \mathsf{Crs}$$

satisfies a Higher Homotopy van Kampen Theorem (HHvKT): Π preserves certain colimits. Part of Chapter 7 thus discusses colimits of crossed complexes and in particular the notion of *free crossed complex*.

Consequence of the HHvKT are some results which are commonly regarded as basic in algebraic topology and homotopy theory, for example:

1. the Brouwer degree theorem: homotopy classes of maps $S^n \to S^n$ are classified by their degree while any map $S^i \to S^n$ is inessential if $i < n$ (see Corollary 8.3.11);
2. the relative Hurewicz theorem: if $\pi_i(X, A) = 0$ for $1 < i < n$ and X, A are connected, then the Hurewicz map
$$\pi_n(X, A, x) \to H_n(X, A)$$
determines $H_n(X, A)$ as $\pi_n(X, A, x)$ factored by the action of $\pi_1(A, x)$ (see Theorem 8.3.18);
3. if $n \geqslant 2$, then $\pi_n(A \cup \{e^n_\lambda\}, A, x)$ is a free $\pi_1(A, x)$-module (crossed $\pi_1(A, x)$-module if $n = 2$) on the characteristic maps of the n-cells (see Theorem 8.3.13);
4. if X_* is the skeletal filtration of a CW-complex, then $\Pi(X_*)$ is a free crossed complex.

Put in another way, these results lend some support to the idea that crossed complexes give a convenient algebraic formalism for the theory and applications of relative homotopy theory.

This was first recognised in a 1946 paper by Blakers ([Bla48]) using the name "group system" for what is now called crossed complex (of groups). The applications were next developed by J.H.C.Whitehead in his 1949 paper ([Whi49b]) This paper has been considerably neglected in sharp contrast with the first part ([Whi49a]) where the definition of CW-complexes has contributed a basic tool for algebraic topology. (For more history see [BH82].)

A second major set of results, stated in Chapter 9 and applied there and in later chapters, deals with *homotopies* for filtered spaces and for crossed complexes. We introduce the latter homotopies briefly in subsection 7.1.5, and discuss them later in the context of monoidal closed categories and the cylinder construction.

The category of filtered spaces has a *tensor product* $X_* \otimes Y_*$ defined by
$$(X_* \otimes Y_*)_n = \bigcup_{p+q=n} X_p \times Y_q$$
for filtered spaces X_*, Y_*. This models the skeletal filtration of a product $X \times Y$ of CW-complexes, where the n-cells of the product are of the form $e^p \times e^q$ for all $p + q = n$ and p-cells e^p of X, q-cells e^q of Y.

This tensor product allows one to define homotopies for filtered spaces as maps $I_* \otimes X_* \to Y_*$ where I_* is the unit interval with its usual cell structure. There is also an *internal hom* filtered space $\mathsf{FTOP}(Y_*, Z_*)$, with total space $\mathsf{Top}(X, Y)$, and an exponential law
$$\mathsf{FTop}(X_* \otimes Y_*, Z_*) \cong \mathsf{FTop}(X_*, \mathsf{FTOP}(Y_*, Z_*)).$$
We require analogous structures for crossed complexes, and indeed it is these which give crossed complexes extra power.

In Chapter 9 we define for crossed complexes D, E an *internal hom* crossed complex $\mathsf{CRS}(D, E)$, whose elements in dimension 0 are morphisms $D \to E$ of crossed complexes, in dimension 1 are homotopies of morphisms, and in higher dimensions are forms of 'higher homotopies'. The definition is completely explicit, except that the verification of the axioms for a crossed complex is left till Part III. Then a *tensor product* $C \otimes D$ of crossed complexes is defined precisely so that the exponential law holds, i.e. there is a natural bijection
$$\mathsf{Crs}(C \otimes D, E) \cong \mathsf{Crs}(C, \mathsf{CRS}(D, E))$$
for all crossed complexes C, D, E. This adjoint relation enables us to prove that a tensor product of free crossed complexes is free.

Here $C \otimes D$ is generated in dimension n by all elements $c \otimes d$ for $c \in C_p, d \in D_q$ and with $p + q = n$. The complication of the rules required to hold reflects the complications of the geometry which is being modeled, namely that of the product $E^p \times E^q$ where the cell E^p has a cell subdivision

$$E^0 = e^0, \quad E^1 = e^0_{\pm} \cup e^1, \quad E^p = e^0 \cup e^{p-1} \cup e^p,\ p \geqslant 2.$$

Thus in general the product has 9 cells, and a 'cylinder' $E^1 \times E^p$ has cells in dimensions 0, 1, $p-1$, p, $p+1$. The capacity to contain information on what happens in a range of dimensions is what gives crossed complexes the extra power over chain complexes, even with a group of operators. A precise comparison with the latter and crossed complexes is given in Chapter 9, Section 7.5.

We have to assume some properties of the tensor product and of the functor Π whose proofs are deferred to Part III. There we give a precise algebraic relation between crossed complexes and a cubical algebraic theory; this relation is the engine behind the power of the crossed complexes theory and applications.

One of the results we are able to prove on this basis is that the functor Π preserves homotopies. This uses an important natural transformation

$$\eta : \Pi X_* \otimes \Pi Y_* \to \Pi(X_* \otimes Y_*)$$

which enables us to translate homotopies from filtered spaces to crossed complexes. Further, η is an isomorphism if $\Pi X_*, \Pi Y_*$ are free crossed complexes, for example if X_*, Y_* are CW-filtered spaces.

One consequence of these results is *the Homotopy Addition Lemma for a simplex*: intuitively, this says that the boundary $\partial\sigma$ of an n-simplex σ is the 'sum' of all its faces. How to express this intuition in precise terms was one of the problems that led to the foundation of algebraic topology. One longstanding description of this in chain complexes and homology is that for all $n \geqslant 0$

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \partial_i \sigma$$

where for any n-simplex σ, $\partial_i\sigma$ is the face opposite to the i-th vertex, so that the vertices are given the structure of a totally ordered set. However, for homotopical purposes we need to involve not the abelian homology groups, but instead the fundamental groupoid and its action on relative homotopy groups. This leads to differing formulae for $n = 1, 2, 3$ and $n \geqslant 4$. We shall show that we can model in crossed complexes the inductive construction of an n-simplex as a cone on the $(n-1)$-simplex, and hence give an algebraic deduction of these formulae, based on the algebraic formulae in the tensor product of crossed complexes (see Theorem 9.9.4).

In Chapter 10, we start the development of cubical theory to define the classifying space functor

$$B : \mathsf{Crs} \to \mathsf{Top}.$$

The main result is a homotopy classification theorem stating that if X_* is a CW-filtered space, and C is a crossed complex, then there is natural bijection of homotopy classes

$$[X, BC] \cong [\Pi X_*, C].$$

We show how this leads to some specific calculations of homotopy classes of maps. We also develop the notion of fibration of crossed complexes, and its relation with fibration of spaces.

It is time to remember from Part I that crossed modules had two origins: from Topology the fundamental crossed module of a triple of spaces, and, from Algebra, combinatorial group theory, i.e. presentations of groups and identities among relations. Until now, we have presented the generalisation to crossed complexes of the first source. It is time to turn to Algebra.

In Chapter 11 we move in this direction by considering the notion of *free crossed resolution* of a group or groupoid. This develops the work on Identities among Relations in Chapter 3 of Part I. We prove that any two of these resolutions are homotopy equivalent, thus making the homotopy computations independent of the resolution chosen. In particular, we give some examples of resolutions that have few generators and so make computations feasible.

Also related to the concept of resolution is the theory of *acyclic models* for crossed complexes; we introduce this theory and give several important applications.

Chapter 11 gives a procedure for generating free crossed resolutions associated to a given presentation of a finite group G. It generalises the classical procedure of constructing trees in Cayley graphs. This procedure works by constructing for the *universal covering groupoid* $\widetilde{G}$ of G, obtained from the adjoint action of G on itself, a free crossed resolution $F_*(\widetilde{G})$ together with a contracting homotopy of this resolution. This can be interpreted as using 'Cayley graphs with relations', and with still higher syzygies. For these higher syzygies , *covering morphisms* of crossed complexes give a useful algebraic model and one appropriate for calculations.

We end this Chapter by applying this procedure to get some free resolutions. In particular, for each group G, we get the free standard crossed resolution $F_*^{st}(G)$, a crossed complex version of the bar construction much used in homological algebra. Also, we get free crossed resolutions with few generators that have been used in Chapter 11.

As a consequence of the general theory of this Chapter, we get a relation between the free standard crossed resolution and the bar resolution of a group. In particular, for a crossed complex C and chain complex L there is a natural isomorphism

$$\mathsf{CRS}(C, \Theta L) \cong \Theta\mathsf{CHN}(\nabla C, L),$$

giving a bijection

$$[C, \Theta L] \cong [\nabla C, L]$$

between the corresponding homotopy classes.

Since the chain complex associated by ∇ to the free standard crossed resolution of a group G is the bar resolution, we can redefine the cohomology of a group G with coefficients in a G-module using free crossed resolutions. This is carried out at the beginning of Chapter 12.

This natural theory generalises in a number of ways, for example giving cohomology of a group or space with coefficients in a crossed complex. This somewhat nonabelian theory allows for more calculations than are easily possible in the abelian theory, for example of the k-invariant of a crossed module.

This ends our account of the theory and application of crossed complexes, apart from the proofs in Part III for crossed complexes and associated functors of a number of major properties which have been used extensively in Part II.

Chapter 7

The basics of crossed complexes

Introduction.

This first Chapter of Part II gives the background on crossed complexes which is required for the statement and applications of the Higher Homotopy van Kampen Theorem (HHvKT) for the functor

$$\Pi : (\text{filtered spaces}) \to (\text{crossed complexes})$$

given in the next chapter.

This is a substantial chapter, so you are encouraged to read the definition of crossed complex and of the functor Π, and then skip to the next chapters, returning to this one for further information as required.

Recall that, as said in the preface:

"Some of the main aims of the book can be summarised by stating that we construct a diagram, which we call the *Main Equivalence*(ME):

$$\begin{tikzcd} & (\text{filtered spaces}) \arrow[dl, "\Pi"'] \arrow[dr, "\rho"] & \\ (\text{crossed complexes}) \arrow[rr, shift left, "\lambda"] & & (\omega\text{-groupoids}) \arrow[ll, shift left, "\gamma"] \end{tikzcd} \tag{ME}$$

such that

(A) γ, λ give an equivalence of categories;

(B) $\gamma\rho$ is naturally equivalent to Π;

(C) ρ, and hence also Π, preserves certain colimits.

The final statement we call a Higher Homotopy van Kampen Theorem (HHvKT)."

We deal in this Chapter with the part of the diagram (ME) involving Π, but here giving only its definition. Even the proof that Π gives a crossed complex is left to Part III, since that proof fits with the Main Equivalence, which is the engine driving this Part II, purring quietly in the background.

This Chapter contains generalisations to all dimensions of concepts introduced in Chapter 2 of Part I, that is of the fundamental crossed module of a pair of spaces, and of some results of Chapters 4 and 5.

A basic construction for all of Parts II and III is the following: associated to a filtration

$$X_* := \quad X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X$$

of a space X, and to a base point $x \in X_0$, there is a sequence of morphisms of relative homotopy groups

$$\cdots \xrightarrow{\delta_{n+1}} \pi_n(X_n, X_{n-1}, x) \xrightarrow{\delta_n} \pi_{n-1}(X_{n-1}, X_{n-2}, x) \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_3} \pi_2(X_2, X_1, x) \xrightarrow{\delta_2} \pi_1(X_1, x) \ .$$

All these sequences, for varying x, and with an additional structure of operations from the fundamental groupoid $\pi_1(X_1, X_0)$, form what we call the *fundamental crossed complex* $\Pi(X_*)$ of the filtered space X_*.

Accordingly, the first Section of this Chapter contains:

- a quick introduction of the category FTop of filtered topological spaces paying attention to the standard example, the skeletal filtration of a CW-complex, and using the tensor product as a source of examples;
- an introduction to Crs the category of crossed complexes;
- the definition in terms of relative homotopy groups of the fundamental crossed complex functor $\Pi : \mathsf{FTop} \to \mathsf{Crs}$.

The statement and applications of the Higher Homotopy van Kampen Theorem (HHvKT), which says in general terms that the functor Π preserves some colimits (coequalisers of some special maps) is left to the next Chapter. This Theorem allows calculation with this homotopically defined functor; it is thus central to the main theme of this book. Analogously to Part I, the proof of the HHvKT requires cubical techniques of what are called ω-groupoids, and is delayed to Part III.

We define these categories and this functor in the first section of this chapter, and proceed to explain in following sections how to compute the colimits which arise naturally in the applications. The HHvKT and immediate applications are given in the next chapter. It is therefore quite reasonable for the reader to make sure of the basic definitions, and then skip to the next chapter, returning to this chapter as necessary.

The next two Sections detail the algebraic facts necessary to apply the HHvKT to deduce the homotopical consequences given in Section 8.2. In Section 7.3 we define some categories related to Crs which we need to analyse colimits of crossed complexes, namely: Crs_n the category of n-*truncated crossed complexes*; and Mod the category of *modules over groupoids*.

The n-truncated crossed complexes have the structure of the first n dimensions of a crossed complex. In particular, Crs_2, the 2-truncated crossed complexes, are simply the crossed modules over groupoids as seen in Chapter 6. This category thus includes the crossed modules over groups studied in Chapters 2–5.

We are also interested for $n \geqslant 3$ in the phenomena arising in a crossed complex just in dimensions 1 and n. In these dimensions, a crossed complex gives a pair (M, G) where G is a groupoid and M is a G-module, and so we need to study Mod, the category of these objects. This category is not so well known even for the case of groups, since we are varying G.

The generalisation to groupoids is necessary for many of the applications of the fundamental crossed complex since it allows the use of several base points and takes into account the full action of the fundamental groupoid.

The second algebraic Section (7.3.2) is devoted to the study of colimits in the category Crs; the results are necessary to do any computations with the HHvKT.

It is an easy consequence of the appropriate functors having right adjoints (and hence preserving colimits) that the colimits in Crs can be computed by taking colimits in three categories. First a colimit in groupoids, then a colimit of crossed modules and, last, colimits of modules in all dimensions $n \geqslant 3$.

Moreover, the computation of colimits in Mod (and XMod) can be done in two steps. First a change of base groupoid (via the *induced module construction*), and then, a colimit in the category of modules over a fixed groupoid.

We proceed to explain a bit further how to compute induced modules and how to define free modules as a special kind of induced modules (indicating the same results for crossed modules) and end the Section with some examples of colimits.

The last Section 7.4.3, gives the notion of *free crossed complex*. As is usual for such a concept, morphisms from a free crossed complex can be constructed in terms of their values on a free basis, so this is a basic concept for many homotopy classification results. A consequence of our results is that the skeletal filtration of a CW-complex X is a connected filtration, and that its fundamental crossed complex is a free crossed complex, on a basis determined by the characteristic maps of the cells of X.

For these results it is essential to use groupoids rather than groups, and so we set up enough of the general theory of fibred categories to handle the notions of pullback and induced constructions which arise in a variety of situations.

In a final section, we relate the notion of crossed complex to the more widely familiar notion of chain complex with operators. The usual notion is that of a group of operators, but in order to model the geometry, and to have better properties, it is again *essential* to generalise this to a *groupoid of operators*.

7.1 Our basic categories and functors.

7.1.1 The category FTop of filtered topological spaces.

By a *space* is meant a compactly generated topological space X, i.e. one which has the final topology with respect to all continuous functions $K \to X$ for all compact Hausdorff spaces K. The category of spaces and continuous maps will be written Top. We will assume the basic properties of these spaces and this category given in, for example, [Bro06, Section 5.9].

Definition 7.1.1 A *filtered space* X_* consists of a space X and an increasing sequence of subspaces of X:

$$X_* := \quad X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X$$

which we call a *filtration* of X.

A *filtration preserving map*

$$f : X_* \to Y_*$$

is a continuous map $f : X \to Y$ such that $f(X_n) \subseteq Y_n$ for all $n \geqslant 0$.

These objects and morphisms form the *category* FTop *of filtered spaces and filtered maps*. □

Definition 7.1.2 A standard way of constructing a new filtered space from given ones X_*, Y_* is the *tensor product* with total space $X \times Y$ and filtration given by

$$(X_* \otimes Y_*)_n = \bigcup_{p+q=n} X_p \times Y_q.$$

We write I for the unit interval $[0,1]$ with its skeletal filtration with 0-cells 0,1, and one 1-cell. We write I^n for the standard n-cube,

Remark 7.1.3 The category FTop is, like the category Top, both complete and complete. Colimits are calculated filtration wise: that is, if $T : C \to \mathsf{FTop}$ is a small diagram in FTop, then $T_n : C \to \mathsf{Top}$ is well defined, and $L = \operatorname{colim} T$ is the filtered space with L_n as the colimit in Top $\operatorname{colim} T_n$, provided L_n is a subspace of L_{n+1}. This will happen in the cases we use. (That this is not so in general is shown by subspaces of adjunction spaces.)

Example 7.1.4 Here are some standard filtered spaces.

1) We denote the standard n-*simplex* by Δ^n. We take this to be the subset of $\mathbb{R}^{n+1}$ of points $(x_0, x_1, \dots, x_n)$ for which $x_i \geqslant 0$ and $x_0 + \cdots + x_n = 1$. We set $\Delta^n_r = \Delta^n$ for $r \geqslant n$, and for $0 \leqslant r < n$ we let it be the set of points $(x_0, \dots, x_n)$ for which $n - r$ of the x_i are 0. This defines the filtered space $\boldsymbol{\Delta}^n_*$.

2) The filtered space $I = [0,1]$ has $I_0 = \{0,1\}$ and $I_1 = I$, and we write I^n for the n-fold product of I with itself and $\mathbf{I}^n_*$ for the corresponding tensor product filtered space, which we call the skeletal filtration of the standard n-*cube*.

3) To define the filtered n-ball, we fix some notation. The standard n-*ball and* $(n-1)$-*sphere* are the usual subsets of the Euclidean space of dimension n:

$$\mathbf{E}^n = \{x \in \mathbb{R}^n \mid \ ||x|| \leqslant 1\}, \ \mathbf{S}^{n-1} = \{x \in \mathbb{R}^n \mid \ ||x|| = 1\}$$

where $||x||$ is the standard Euclidean norm.

We write $\mathbf{E}^n_*$ for the filtered space of the filtration of the n-ball given by the base point up to dimension $n-2$, $\mathbf{S}^{n-1}$ in dimension $n-1$ and $\mathbf{E}^n$ in dimensions $\geqslant n$. Thus

$$\mathbf{E}^1_* = \mathbf{I}_* = \boldsymbol{\Delta}^1_*,$$

and for $n \geqslant 2$, $\mathbf{E}^n_*$ is the filtration

$$\underset{0}{\{1\}} = \cdots = \underset{n-2}{\{1\}} \subseteq \underset{n-1}{\mathbf{S}^{n-1}} \subseteq \underset{n}{\mathbf{E}^n}$$ □

Further standard examples of filtered spaces which include all the previous ones are the skeletal filtrations of *CW-complexes*. These are spaces built up in inductive fashion by attaching cells. We recall their construction, which also gives a preparation for Section 7.4.3 where we work analogously with crossed complexes.

We begin by explaining the process of attaching cells to a space. For background to this idea, we refer to [Bro06, Section 4.7]. Let A be a space, Λ a set of indexes, $\{f_\lambda\}_{\lambda\in\Lambda}$ a family of continuous maps $f_\lambda : \mathbf{S}^{m_\lambda-1} \to A$. We form the adjunction space

$$X = A \cup_{\{f_\lambda\}} \{e^{m_\lambda}\}_{\lambda\in\Lambda},$$

given by the pushout diagram,

$$\begin{tikzcd} \bigsqcup_{\lambda\in\Lambda} \mathbf{S}^{m_\lambda-1} \arrow[r, "(f_\lambda)"] \arrow[d] & A \arrow[d, "j"] \\ \bigsqcup_{\lambda\in\Lambda} \mathbf{E}^{m_\lambda} \arrow[r, "(h_\lambda)"] & X. \end{tikzcd}$$

Then we say that the space X is *obtained from* A *by attaching cells*. By standard properties of adjunction spaces (see [Bro06, Chapter 4]), the map j is a closed injection, and so we usually assume it is an inclusion. As examples, we have

$$\mathbf{E}^1 = \{0,1\} \cup e^1 \quad \text{and} \quad \mathbf{E}^n = e^0 \cup e^{n-1} \cup e^n \quad \text{for } n \geqslant 2.$$

The maps $h_\lambda : \mathbf{E}^{m_\lambda} \to X$ are called the *characteristic maps* of the cells. It is a standard fact that they are homeomorphisms on the interior of each $\mathbf{E}^{m_\lambda}$. The images $e^{m_\lambda} = h_\lambda(\mathbf{E}^{m_\lambda})$ in X are called the closed *cells of* X *relative to* A. We say that X is obtained from A by attaching the cells $\{e^{m_\lambda}\}_{\lambda\in\Lambda}$. It is important to notice that a map $f : X \to Y$ is continuous if and only if $f|_A$ is continuous and each composite fh_λ is continuous.

We construct a *relative CW-complex* (X^*, A) by attaching cells in the following inductive process. We start with a space A and form a sequence of spaces X^n by setting X^0 to be the disjoint union of A and a discrete space Λ_0. Then, inductively, we form X^n by 'attaching' to X^{n-1} a family of n-cells indexed by a set Λ_n. That is for each $n \geqslant 0$ we choose a family of maps $f_\lambda : \mathbf{S}^{n-1} \to X^{n-1}$, $\lambda \in \Lambda_n$, and define

$$X^n = X^{n-1} \cup_{\{f_\lambda\}} \{e^n_\lambda\}_{\lambda\in\Lambda_n} \quad \text{and} \quad X = \operatorname{colim} X^n.$$

The canonical map $j : A \to X$ is also called a *relative CW-complex*. Clearly, the X^n (called the *relative* n*-skeleton*) form a filtration of X which we write X_*. If $A = \emptyset$, we say that X is a *CW-complex*.

The cells, characteristic maps, etc., are regarded as part of the structure of a relative CW-complex. The advantage of this structure is that it allows proofs by induction on n. For example, a map $f : X \to Y$ is continuous if and only each restriction $f_n : X^n \to Y$ is continuous and this holds if and only if $f|A$ is continuous and each composite fh_λ is continuous, for all $\lambda \in \Lambda_n$ and all $n \geqslant 0$. Thus we may construct a map $f : X \to Y$ by induction on skeleta starting with X_0, which is just the disjoint union of A and Λ_0.

We can conveniently write

$$X = A \cup \{e^n_\lambda\}_{\lambda\in\Lambda_n, n\geqslant 0},$$

and may abbreviate this in some cases, for example to $X = A \cup e^n \cup e^m$.

All filtered spaces given in Example 7.1.4 are CW-complexes. More detail of this, including the characteristic maps, is given, for example, in [Bro06].

7.1.2 The category Crs of crossed complexes.

The structure of *crossed complex* is suggested by the canonical example, the *fundamental crossed complex* $\Pi(X_*)$ of the filtered space X_*, in particular that of a *CW*-complex with its skeletal filtration. We discuss this further in the next subsection.

Definition 7.1.5 Let C_1 be a groupoid. We write C_0 for its objects, $C_1(x,y)$ for the set of morphisms from x to y $(x, y \in C_0)$ and $C_1(x)$ for the group $C_1(x,x)$. The source and target maps are $s, t : C_1 \to C_0$ and the composition of $a : x \to y$ and $b : y \to z$ is written $ab : x \to z$.

A *crossed complex* C over C_1 is written as a sequence

$$\cdots \longrightarrow C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots\cdots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1$$

and it is given by the following three sets of data:

1.- For $n \geqslant 2$, C_n is a totally disconnected groupoid (abelian if $n \geqslant 3$) with the same set of objects as C_1, namely C_0. This is equivalent to say that C_n is a family of groups $\{C_n(x)\}_{x \in C_0}$ and for $n \geqslant 3$, the groups $C_n(x)$ are abelian. The *target* or *base point* map $t : C_n \to C_0$ sends $C_n(x)$ to x.

We shall generally use additive notation for all groups $C_n(x), n \geqslant 3$, and multiplicative notation for $n = 1, 2$, and we shall use the symbol 0 or 1 for their respective identity elements. However, in dealing with the tensor product in the next chapter, it is often convenient to use additive notation in all dimensions $\geqslant 1$.

2.- For $n \geqslant 2$, an action of the groupoid C_1 on the right of each C_n,

$$C_n \times C_1 \to C_n$$

written $(c, c_1) \mapsto c^{c_1}$, such that if $c \in C_n(x)$ and $c_1 \in C_1(x,y)$ then $c^{c_1} \in C_n(y)$.

Later on, we shall see that this property is equivalent to say that C_n is a C_1-module (see Definition **??**).

We shall always consider C_1 as acting on C_1 by conjugation, i.e. $c^{c_1} = c_1^{-1} c c_1$ for all $c \in C_1(x)$ and $c_1 \in C_1(x,y)$.

As an example of our use of notation, two of the conditions for an action are written $c^{c_1 c_1'} = (c^{c_1})^{c_1'}$ and $c^1 = c$ in all dimensions, but the third condition is expressed as $(cc')^{c_1} = c^{c_1} c'^{c_1}$ for $n = 1, 2$, and $(c + c')^{c_1} = c^{c_1} + c'^{c_1}$ for $n \geqslant 3$.

A consequence of the existence of this action is that $C_n(x) \cong C_n(y)$ if there is a morphism in $C_1(x,y)$, i.e. when x and y lie in the same component of the groupoid C_1.

3.- For $n \geqslant 2$, $\delta_n : C_n \to C_{n-1}$ is a morphism of groupoids over C_0 and preserves the action of C_1.

These three sets of data have to satisfy two conditions:

CX1) $\delta_{n-1}\delta_n = 0 : C_n \to C_{n-2}$ for $n \geqslant 3$ (thus C has analogies with chain complexes);

CX2) $\operatorname{Im} \delta_2$ acts by conjugation on C_2, and trivially on C_n for $n \geqslant 3$.

Notice that CX2) actually has two parts. The first part, together with the condition that δ_2 preserves the action of C_1, says that C_2 is a crossed module over the groupoid C_1, since for $c, c' \in C_2$, $c^{\delta_2 c'} = c'^{-1} c c'$. The second part implies that for $n \geqslant 3$, C_1 acts on C_n through $\pi_1(C) = \operatorname{Cok} \delta_2$.

These two axioms give a good reason for the name 'crossed complex': it has a 'root' which is a crossed module (over C_1) and a 'trunk' that is a (kind of) chain complex (over $\pi_1(C)$). The interplay of these two actions is important in what follows.

A *morphism of crossed complexes* $f : C \to D$ is a family of morphisms of groupoids $f_n : C_n \to D_n$ $(n \geqslant 1)$ all inducing the same map of vertices $f_0 : C_0 \to D_0$, and compatible with the boundary maps and the actions of C_1 and D_1. This means that $\delta_n f_n(c) = f_{n-1}\delta_n(c)$ and $f_n(c^{c_1}) = f_n(c)^{f_1(c_1)}$ for all $c \in C_n$ and $c_1 \in C_1$. We represent a morphism of crossed complexes by the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & C_n \arrow[r, "\delta_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \arrow[r, "\delta_3"] & C_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"] & C_1 \arrow[d, "f_1"] \\ \cdots \arrow[r] & D_n \arrow[r, "\delta_n"'] & D_{n-1} \arrow[r, "\delta_{n-1}"'] & \cdots \arrow[r, "\delta_3"'] & D_2 \arrow[r, "\delta_2"'] & D_1 \end{tikzcd}$$

We denote by Crs the resulting *category of crossed complexes*.

In the case when C_0 is a single point we call C a *reduced crossed complex*, or a *crossed complex over a group*. These crossed complexes give a full subcategory of Crs, which we write $\mathsf{Crs}_{\mathrm{red}}$.

We can also fix the groupoid C_1 to be a groupoid G and restrict the morphisms to those inducing the identity on G, getting then the category Crs_G of *crossed complexes over* G. □

7.1.3 The fundamental crossed complex functor.

As explained in the Introduction to this Part, for any filtered space X_* and any $x \in X_0$, there is a sequence of groups and homomorphisms (abelian for $n \geqslant 3$):

$$\cdots \xrightarrow{\delta_{n+1}} \pi_n(X_n, X_{n-1}, x) \xrightarrow{\delta_n} \pi_{n-1}(X_{n-1}, X_{n-2}, x) \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_3} \pi_2(X_2, X_1, x) \xrightarrow{\delta_2} \pi_1(X_1, x) .$$

In this sequence, the $\pi_n(X_n, X_{n-1}, x)$ are the relative homotopy groups, δ_n are the boundary maps defined

$$\pi_n(X_n, X_{n-1}, x) \xrightarrow{\partial_n} \pi_{n-1}(X_{n-1}, x) \xrightarrow{i_{n-1}} \pi_{n-1}(X_{n-1}, X_{n-2}, x),$$

for $n > 2$ and $\delta_2 : \pi_2(X_2, X_1, x) \to \pi_1(X_1, x)$; the composition in these groups and the action of the groupoid $\pi_1(X_1, X_0)$ on these relative groups for varying $x \in X_0$ were studied in Section 2.1 of Part I.

It is convenient to combine these structures over all base points $x \in X_0$ and so to use crossed complexes over groupoids. So we get groupoids $\pi_n(X_n, X_{n-1}, X_0)$ for $n \geqslant 2$, and the groupoid $\pi_1(X_1, X_0)$, all having the same set X_0 of objects.

Definition 7.1.6 This structure of groupoids, morphisms, and actions define the *fundamental crossed complex of the filtered space* X_*:

$$\Pi(X_*) : \cdots \xrightarrow{\delta_{n+1}} \pi_n(X_n, X_{n-1}, X_0) \xrightarrow{\delta_n} \cdots\cdots \xrightarrow{\delta_3} \pi_2(X_2, X_1, X_0) \xrightarrow{\delta_2} \pi_1(X_1, X_0) .$$

That $\Pi(X_*)$ has the properties of a crossed complex can be proved directly, in a manner similar to that of Chapter 2. Instead, we shall deduce these properties from the full construction and properties of the *homotopy ω-groupoid* $\rho(X_*)$, since the relation between these constructions, given in Part III, is a kind of engine which drives this book. It turns out that $\Pi(X_*)$ can be considered as a substructure $\gamma\rho(X_*)$ of $\rho(X_*)$, and in this way we obtain in Chapter 14 a verification that ΠX_* is a crossed complex. In fact the intuition for the required structure on $\rho(X_*)$ is clear, but the proof that this intuition works is not simple. The relations between these two structures form a basis for this whole work, even though this may be disguised in Part II, in which our main object is the study and use of Π.

The homotopical definition of this crossed complex implies immediately that it gives a functor

$$\Pi : \mathsf{FTop} \to \mathsf{Crs}.$$

Note also that in each of the categories $\mathsf{FTop}, \mathsf{Crs}$, disjoint unions are the coproducts; this is one of the advantages of a groupoid approach. The homotopical definition of Π implies easily that it preserves disjoint unions.

An obvious property of Π is that it is preserved by isomorphisms of filtered space.

Proposition 7.1.7 *An isomorphism* $f : X_* \to Y_*$ *in the category* FTop *induces an isomorphism of crossed complexes* $\Pi f : \Pi X_* \to \Pi Y_*$.

A more subtle property is the following:

Proposition 7.1.8 *Let* $f : X_* \to Y_*$ *be a map of filtered space such that* $f_0 : X_0 \to Y_0$ *is a bijection, and for* $n \geqslant 1$, $f_n : X_n \to Y_n$ *is a homotopy equivalence. Then* $\Pi f : \Pi X_* \to \Pi Y_*$ *is an isomorphism.*

Proof This follows from basic properties of relative homotopy groups. □

Thus the advantage of the functor Π is that its topological and indeed homotopical invariance in the above sense is clear. The fact that we can calculate to some extent with Π comes from the Higher Homotopy van Kampen Theorem in the next Chapter.

We will discuss in a later chapter how the functor Π behaves with respect to homotopies of filtered maps (such a homotopy is a map in FTop, $I_* \otimes X_* \to Y_*$). This discussion requires the development of more algebraic machinery, and in particular the tensor product of crossed complexes.

We emphasise that the use of crossed complexes over groupoids is central to this theory, both for the development of the algebra and for the modeling of the topology.

7.1.4 Homotopy and homology groups of crossed complexes.

Let us recall some definitions and define some new functors giving direct algebraic and set theoretic invariants of crossed complexes. The first one expresses the connectivity of the basic groupoid C_1.

Definition 7.1.9 The *set of components* of the crossed complex C, written $\pi_0(C)$, is just the set of components of the groupoid C_1. This definition gives a functor

$$\pi_0 : \mathsf{Crs} \to \mathsf{Sets}. \qquad \square$$

Example 7.1.10 It is easy to see that for the skeletal filtration of a CW-complex, $\pi_0\Pi(X_*)$ is bijective with $\pi_0(X)$. □

The next invariant we have used is $\pi_1(C)$, the cokernel of the crossed module part of the crossed complex.

Definition 7.1.11 The *fundamental groupoid* of a crossed complex C is the groupoid $\pi_1(C)$ given by the cokernel of δ_2,

$$\pi_1(C) = \mathrm{Cok}\,\delta_2 = \frac{C_1}{\mathrm{Im}\,\delta_2}.$$

A morphism $f : C \to D$ of crossed complexes induces morphisms $f_* : \pi_1(C) \to \pi_1(D)$, giving a functor

$$\pi_1 : \mathsf{Crs} \to \mathsf{Gpds}. \qquad \Box$$

Example 7.1.12 By the homotopy long exact sequence of a pair, it is clear that if X_* is a filtered space such that the morphism induced by inclusion $\pi_1(X_1, x) \to \pi_1(X_2, x)$ is surjective for all $x \in X_0$, then $\pi_1 \Pi(X_*) \cong \pi_1(X_2, X_0)$. The 2-dimensional van Kampen theorem for the fundamental groupoid implies, in particular, that for the skeletal filtration of a CW-complex, we have $\pi_1 \Pi(X_*) \cong \pi_1(X, X_0)$. $\Box$

Now we consider the homology of the Abelian part of a crossed complex, getting $\pi_1(C)$-modules associated to a crossed complex C.

Definition 7.1.13 For any crossed complex C and for $n \geqslant 2$ there is a totally disconnected groupoid $H_n(C)$ given by the family of abelian groups

$$H_n(C, x) = \frac{\operatorname{Ker} \delta_n(x)}{\operatorname{Im} \delta_{n+1}(x)}$$

for all $x \in C_0$. This is called the family of n-*homology groups* of the crossed complex C.

For $n \geqslant 2$, a morphism $f : C \to D$ of crossed complexes induces morphisms $f_* : H_n(C) \to H_n(D)$. $\Box$

Exercise 7.1.14 Prove that for a crossed complex C and $n \geqslant 2$, the homology groups are a family of Abelian groups, and that there is an induced action of $\pi_1(C)$ on the family $H_n(C)$ of homology groups making $H_n(C)$ a $\pi_1(C)$-module. Thus each such homology group gives a functor

$$H_n : \mathsf{Crs} \to \mathsf{Mod}. \qquad \Box$$

Definition 7.1.15 A morphism $f : C \to D$ is a *weak equivalence* if it induces a bijection $\pi_0(C) \to \pi_0(D)$ and isomorphisms $\pi_1(C, x) \to \pi_1(D, fx)$, $H_n(C, x) \to H_n(D, fx)$ for all $x \in C_0$ and $n \geqslant 2$. $\Box$

Example 7.1.16 We shall see in the next chapter (Section 8.4) that if X_* is the skeletal filtration of a CW-complex, then $H_n(\Pi X_*, x)$ is isomorphic to $H_n(\widetilde{X}_x)$, the nth homology group of the universal cover of X based at x. $\Box$

Remark 7.1.17 In the next subsection, we shall introduce the notion of homotopy of morphisms of crossed complexes. It is then an easy exercise to define homotopy equivalences of crossed complexes and check that they are weak equivalences. The converse is true for free crossed complexes (which we define later (subsection 7.4.3), but this is a non trivial result. $\Box$

7.1.5 Homotopies of morphisms of crossed complexes

It is useful at this stage to complete some basic properties of the category Crs by defining the notion of homotopy of morphisms. However the justification of this definition will have to wait till Chapter 9 where it is put in the context of a tensor product structure on Crs and a homotopy can then be seen as a morphism $\mathcal{I} \otimes C \to D$ from a 'cylinder object' $\mathcal{I} \otimes C$ to D.

Definition 7.1.18 A *homotopy* $f^- \simeq f$ of morphisms $f^-, f : C \to D$ of crossed complexes is a pair (h, f) where h is a family of functions $h_n : C_n \to D_{n+1}$ with the following properties, in which tc for $c \in C$ is c, if $c \in C_0$, is tc, if $c \in C_1$, and is x if $c \in C_n(x),\ n \geqslant 2$. So we require:

$$
\begin{array}{rlr}
th_n(c) = tf(c) & \text{for all } c \in C; & \text{(i)} \\
h_1(cc') = h_1(c)^{fc'}\, h_1(c') & \text{if } c, c' \in C_1 \text{ and } cc' \text{ is defined;} & \text{(ii)} \\
th_n(c) = tf(c) & \text{for all } c \in C; & \text{(iii)} \\
h_2(cc') = h_2(c) + h_2(c') & \text{if } c, c' \in C_2 \text{ and } cc' \text{ is defined;} & \text{(iv)} \\
th_n(c) = tf(c) & \text{for all } c \in C; & \text{(v)} \\
h_n(c + c') = h_n(c) + h_n(c') & \text{if } c, c' \in C_n,\ n \geqslant 3 \text{ and } c + c' \text{ is defined;} & \text{(vi)} \\
th_n(c) = tf(c) & \text{for all } c \in C; & \text{(vii)} \\
h_n(c^{c_1}) = (h_n c)^{fc_1} & \text{if } c \in C_n, n \geqslant 2,\ c_1 \in C_1,\ \text{and } c^{c_1} \text{ is defined.} & \text{(viii)} \\
th_n(c) = tf(c) & \text{for all } c \in C; & \text{(ix)}
\end{array}
$$

Then f^-, f are related by

$$
f^-(c) = \begin{cases}
sh_0 c & \text{if } c \in C_0, \\
(h_0 sc)(fc)(\delta_2 h_1 c)(h_0 tc)^{-1} & \text{if } c \in C_1, \\
\{(fc)(h_1 \delta_2 c)(\delta_3 h_2 c)\}^{(h_0 tc)^{-1}} & \text{if } c \in C_2, \\
\{fc + h_{n-1}\delta_n c + \delta_{n+1} h_n c\}^{(h_0 tc)^{-1}} & \text{if } c \in C_n,\ n \geqslant 3.
\end{cases} \qquad \text{(x)}
$$

Exercise 7.1.19 Prove directly from this definition that homotopy of morphisms is an equivalence relation.

Example 7.1.20 *Retracting homotopies*
From the above we can deduce formulae for a retraction. Suppose then in the above formulae we take $C = D,\ f^0 = 1_C,\ f = 0$ where 0 denotes the constant morphism on C mapping everything to a base point 0. Then the homotopy $h : 1 \simeq 0$ must satisfy:

$$
\begin{array}{rlr}
 & & \text{(ri)} \\
sh_0 c = c & \text{if } c \in C_0, & \text{(rii)} \\
\delta_2 h_1 c = (h_0 sc)^{-1} c\,(h_0 tc) & \text{if } c \in C_1, & \text{(riii)} \\
\delta_3 h_2 c = (h_1 \delta_2 c)^{-1} c^{h_0 tc} & \text{if } c \in C_2, & \text{(riv)} \\
\delta_{n+1} h_n c = -h_{n-1}\delta_n c + c^{h_0 tc} & \text{if } c \in C_n, n \geqslant 3, & \text{(rv)} \\
h_n(c^{c_1}) = (h_n c) & \text{if } c \in C_n, n \geqslant 2,\ c_1 \in C_1,\ \text{and } c^{c_1} \text{ is defined.} & \text{(rvi)}
\end{array}
$$

Further, in this case h_1 is a morphism by (ii) and for $n \geqslant 2$, h_n is by (vi) a morphism which by (viii) trivialises the operations of C_1. All these conditions (ri)-(rvi) are necessary and sufficient for h to be a contracting homotopy. □

A connected groupoid G is known to be isomorphic to $G(x_0) * T$ where $x_0 \in \mathrm{Ob}\, G$ and T is a wide tree subgroupoid of G, [Bro06, 8.1.5]. See also equation (1.7.1). Further, T determines a strong deformation retraction $G \to G(x_0)$. We now show the same applies to crossed complexes. We extend the term 'tree subgroupoid' to a subcrossed complex T of C such that T_1 is a tree groupoid, and $T_n(x)$ is trivial for all $x \in T_0$ and $n \geqslant 2$. The final part of the following Proposition generalises [Bro06, 6.7.3]. It is related to Proposition 1.7.1.

Proposition 7.1.21 *Let C be a connected crossed complex, let $x_0 \in C_0$ and let T be a wide tree sub-crossed complex of C. Let $C(x_0)$ be the subcrossed complex of C at the base point x_0. Then the natural morphism*

$$\phi : C(x_0) * T \to C$$

determined by the inclusions is an isomorphism, and T determines a strong deformation retraction

$$r : C \to C(x_0).$$

Further, if $f : C \to D$ is a morphism of crossed complexes which is the identity on $C_0 \to D_0$ then we can find a retraction $r' : D \to D(x_0)$ giving rise to a pushout square

$$\begin{tikzcd} C \arrow[r, "r"] \arrow[d, "f"'] & C(x_0) \arrow[d, "f'"] \\ D \arrow[r, "r'"'] & D(x_0) \end{tikzcd} \tag{7.1.1}$$

in which f' is the restriction of f.

Proof Let $i : C(x_0) \to C$, $j : T \to C$ be the inclusions. We verify the universal property of the free product. Let $\alpha : C(x_0) \to E, \beta : T \to E$ be morphisms of crossed complexes agreeing on x_0. Suppose $g : C \to E$ satisfies $gi = a, gj = b$. Then g is determined on C_0. Let $c \in C_1(x, y)$. Then

$$c = (\tau x)((\tau x)^{-1} c(\tau y))(\tau y)^{-1} \tag{*}$$

and so

$$\begin{aligned} gc &= g(\tau x) g((\tau x)^{-1} c(\tau y)) g(\tau y)^{-1} \\ &= \beta(\tau x)\alpha((\tau x)^{-1} c(\tau y))\beta(\tau y)^{-1}. \end{aligned}$$

If $c \in C_n(x), n \geqslant 2$, then

$$c = (c^{\tau x})^{(\tau x)^{-1}} \tag{**}$$

and so

$$g(c) = \alpha(c^{\tau x})^{\beta(\tau x)^{-1}}.$$

This proves uniqueness of g, and conversely one checks that this formula defines a morphism g as required.

In effect, equations (*) and (**) give for the elements of C normal forms in terms of elements of $C(x_0)$ and of T.

This isomorphism and the constant map $T \to \{x_0\}$ determine the strong deformation retraction $r : C \to C(x_0)$.

The retraction r' is defined by the elements $f\tau(x), x \in C_0$, and then the diagram (7.1.1) is a pushout since it is a retract of the pushout square

$$\begin{tikzcd} C \arrow[r, "1"] \arrow[d, "f"'] & C \arrow[d, "f"] \\ D \arrow[r, "1"'] & D \end{tikzcd}$$

□

7.2 Some fibrations of categories

In order to describe further basic properties of the relations between crossed complexes and various parts of the structure, such as groupoids, crossed modules, modules, and so forth, it is convenient to use the language of fibred categories, or fibrations of categories. These notions are developed in the Appendix A.7, and these sections assume that language.

7.2.1 Groupoids bifibred over sets

We see in Example A.7.3 that the functor $\mathrm{Ob} : \mathsf{Gpds} \to \mathsf{Sets}$ is a fibration. It also has a left adjoint D assigning to a set I the discrete groupoid on I, and a right adjoint assigning to a set I the codiscrete groupoid on I.

It follows from general theorems on algebraic theories that the category Gpds is cocomplete, and in particular admits pushouts, and so it follows from previous results that $\mathrm{Ob} : \mathsf{Gpds} \to \mathsf{Sets}$ is also a cofibration. A construction of the cocartesian liftings of $u : I \to J$ for G a groupoid over I is given in terms of words, generalising the construction of free groups and free products of groups, in [Hig71, Bro06]. In these references the cocartesian lifting of u to G is called a *universal morphism*, and is written $u_* : G \to U_u(G)$. This construction is of interest as it yields a normal form for the elements of $U_u(G)$, and hence u_* is injective on the set of non-identity elements of G.

A homotopical application of this cocartesian lifting is the following theorem on the fundamental groupoid. It shows how identification of points of a discrete subset of a space can lead to 'identifications of the objects' of the fundamental groupoid:

Theorem 7.2.1 *Let (X, A) be a pair of spaces such that A is discrete and the inclusion $A \to X$ is a closed cofibration. Let $f : A \to B$ be a function to a discrete space B. Then the induced morphism*

$$\pi_1(X, A) \to \pi_1(B \cup_f X, B)$$

is the cocartesian lifting of f.

This theorem immediately gives the fundamental group of the circle S^1 as the infinite cyclic group C, since S^1 is obtained from the unit interval $[0, 1]$ by identifying 0 and 1. The theorem is a translation of [Bro06, 9.2.1], where the words 'universal morphism' are used instead of 'cocartesian lifting'. Section 8.2 of [Bro06] shows how free groupoids on directed graphs are obtained by a generalisation of this example.

The calculation of colimits in a fibre Gpds_I is similar to that in the category of groups, since both categories are protomodular, [BB04]. Thus a colimit is calculated as a quotient of a coproduct, where quotients are themselves obtained by factoring by a normal subgroupoid. Quotients are discussed in [Hig71, Bro06].

Theorem A.9.4 now shows how to compute general colimits of groupoids.

We refer again to [Hig71, Bro06] for further developments and applications of the algebra of groupoids; we generalise some aspects to modules, crossed modules and crossed complexes in later subsections.

7.2.2 Abelianisation of groupoids

We will need in sections 7.5.3 and 14.7 the notion of abelianisation of a groupoid.

Let Ab, Groups, Gpds denote respectively the categories of abelian groups, groups, and groupoids. Each of the inclusions

$$\mathsf{Ab} \to \mathsf{Groups} \to \mathsf{Gpds} \tag{7.2.1}$$

has a left adjoint. That from groupoids to groups is called the *universal group* UG of a groupoid G and is described in detail in [Bro06, Chapter 8] and [Hig71]. In particular, the universal group of a groupoid G is the free product of the universal groups of the transitive components of G.

It follows that we have what we call the *universal abelianisation* G^{totab} of a groupoid, namely the usual abelianisation of the group UG. It is isomorphic to the direct sum of the G_i^{totab} over all components G_i of G. Any transitive groupoid G may be written in a non canonical way as the free product $G(a_0) * T$ of a vertex group $G(a_0)$ and an indiscrete or tree groupoid T (This result has been used to suggest that 'groupoids reduce to groups'; but this is analogous to suggesting that vector spaces reduce to numbers!). Then

$$UG \cong G(a_0) * UT$$

and UT is the free group on the elements $x : a_0 \to a$ in T for all $a \in \mathrm{Ob}(T)$, $a \neq a_0$. So for a transitive groupoid G with $a_0 \in \mathrm{Ob}\, G$

$$G^{\mathrm{totab}} \cong G(a_0)^{\mathrm{ab}} \oplus F$$

where F is the free abelian group on the elements $x : a_0 \to a$ in T for all $a \in \mathrm{Ob}(T), a \neq a_0$, for T a wide tree subgroupoid of G.

However we shall also need a more restrictive abelianisation of a groupoid G with object set I, which we write G^{ab}. Here the abelianisation takes place in the category of groupoids with object set I, and an abelian groupoid over I is one in which all vertex groups are abelian. It is this construction which we shall apply to C_2 as part of the abelianisation ∇C of a crossed complex C, giving a chain complex with $\pi_1 C$ as groupoid of operators, in section 7.5.3.

Exercise 7.2.2 A groupoid G is *abelian* if all its vertex groups are abelian. Show that the abelian groupoids for a reflexive subcategory of the category of all groupoids. □

7.2.3 Groupoid modules bifibred over groupoids

Modules over groupoids are a useful generalisation of modules over groups, and also form part of the basic structure of crossed complexes. Homotopy groups $\pi_n(X; X_0), n \geqslant 2$, of a space X with a set X_0 of base points form a module over the fundamental groupoid $\pi_1(X, X_0)$, as do the homotopy groups $\pi_n(Y, X : X_0), n \geqslant 3$, of a pair (Y, X).

Definition 7.2.3 A *module over a groupoid* is a pair (M, G), where G is a groupoid with set of objects G_0, M is a totally disconnected abelian groupoid with the same set of objects as G, and with a given action of G on M. Thus M comes with a *target function* $t : M \to G_0$, and each $M(x) = t^{-1}(x), x \in G_0$, has the structure of Abelian group. The G-action is given by a family of maps

$$M(x) \times G(x, y) \to M(y)$$

for all $x, y \in G_0$. These maps are denoted by $(m, p) \mapsto m^p$ and satisfy the usual properties, i.e. $m^1 = m$, $(m^p)^{p'} = m^{(pp')}$ and $(m + m')^p = m^p + m'^p$, whenever these are defined. In particular, any $p \in G(x, y)$ induces an isomorphism $m \mapsto m^p$ from $M(x)$ to $M(y)$.

A *morphism of modules* is a pair $(\theta, f) : (M, G) \to (N, H)$, where $f : G \to H$ and $\theta : M \to N$ are morphisms of groupoids and preserve the action. That is, θ is given by a family of group morphisms

$$\theta(x) : M(x) \to N(f(x))$$

for all $x \in G_0$ satisfying $\theta(y)(m^p) = (\theta(x)(m))^{f(p)}$, for all $p \in G(x, y), m \in M(x)$.

This defines the category Mod having modules over groupoids as objects and the morphisms of modules as morphisms. If (M, G) is a module, then $(M, G)_0$ is defined to be G_0. □

We have a forgetful functor $\Phi_M : \mathsf{Mod} \to \mathsf{Gpds}$ in which $(M, G) \mapsto G$.

Proposition 7.2.4 *The forgetful functor* $\Phi_M : \mathsf{Mod} \to \mathsf{Gpds}$ *has a left adjoint and is fibred and cofibred.*

Proof The left adjoint of Φ_M assigns to a groupoid G the module written $0 \to G$ which has only the trivial group over each $x \in G_0$.

Next, we give the pullback construction to prove that Φ_M is fibred. This is entirely analogous to the group case, but taking account of the geometry of the groupoid.

So let $\nu : G \to H$ be a morphism of groupoids, and let (N, H) be a module. We define $(M, G) = \nu^*(N, H)$ as follows. For $x \in G_0$ we set $M(x) = \{x\} \times N(\nu x)$ with addition given by that in $N(\nu x)$. The operation is given by $(x, n)^p = (y, n^{\nu p})$ for $p \in G(x, y)$.

The construction of $N = \nu_*(M, G)$ for (M, G) a G-module is as follows.

For $y \in H_0$ we let $N(y)$ be the abelian group generated by pairs (m, q) with $m \in M, q \in H$, and $t(q) = y, s(q) = \nu(t(m))$, so that $N(y) = 0$ if no such pairs exist. The operation of H on N is given by $(m, q)^{q'} = (m, qq')$, addition is $(m, q) + (m', q) = (m + m', q)$ and the relations imposed are $(m^p, q) = (m, \nu(p)q)$ when these make sense. The cocartesian morphism over ν is given by $\psi : m \mapsto (m, 1_{\nu t(m)})$. □

Remark 7.2.5 The relation between a module over a groupoid and the restriction to the vertex groups is discussed in Theorem 7.1.21 in the general context of crossed complexes. However it is useful to give the general situation of many base points to describe the relative homotopy group $\pi_n(X, A, a_0)$ when X is obtained from A by adding n-cells at various base points. The natural invariant to consider is then $\pi_n(X, A, A_0)$ where A_0 is an appropriate set of base points.

We now describe free modules over groupoids in terms of the inducing construction. The interest of this is two fold. Firstly, induced modules arise in homotopy theory from a HHvKT, and we get new proofs of results on free modules in homotopy theory. Secondly, this indicates the power of the HHvKT since it gives new results.

Definition 7.2.6 Let Q be a groupoid. A *free basis* for a module (N, Q) consists of a pair of functions $t_B : B \to Q_0, i : B \to N$ such that $t_N i = t_B$ and with the universal property that if (L, Q) is a module and $f : B \to L$ is a function such that $t_L f = t_N$ then there is a unique Q-module morphism $\phi : N \to L$ such that $\phi i = f$. □

Proposition 7.2.7 *Let* B *be an indexing set, and* Q *a groupoid. The free* Q*-module* $(FM(t), Q)$ *on* $t : B \to Q_0$ *may be described as the* Q*-module induced by* $t : B \to Q$ *from the discrete free module* $\mathbb{Z}B = (\mathbb{Z} \times B, B)$ *on* B*, where* B *denotes also the discrete groupoid on* B.

Proof This is a direct verification of the universal property. □

Remark 7.2.8 Proposition A.8.7 shows that the universal property for a free module can also be expressed in terms of morphisms of modules $(FM(t), Q) \to (L, R)$. □

7.2.4 Crossed modules bifibred over groupoids

Out homotopical example here is the family of second relative homotopy groups of a pair of spaces with many base points.

A *crossed module over a groupoid*, [BH81d], consists first of a morphism of groupoids $\mu : M \to P$ of groupoids with the same set P_0 of objects such that μ is the identity on objects, and M is a family of groups $M(x), x \in P_0$; second, there is an action of P on the family of groups M so that if $m \in M(x)$ and $p \in P(x, y)$ then $m^p \in M(y)$. This action must satisfy the usual axioms for an action with the additional properties:
CM1) $\mu(m^p) = p^{-1}\mu(m)p$, and
CM2) $m^{-1}nm = n^{\mu m}$
for all $p \in P$, $m, n \in M$ such that the equations make sense. These form the objects of the category XMod in which a morphism is a commutative square of morphisms of groups

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\mu"'] & N \arrow[d, "\nu"] \\ P \arrow[r, "f"'] & Q \end{tikzcd}$$

which preserve the action in the sense that $g(m^p) = (gm)^{fp}$ whenever this makes sense.

The category XMod is equivalent to the well known category $2 - \mathsf{Gpd}$ of 2-groupoids, [BH81b]. However the advantages of XMod over 2-groupoids are:

- crossed modules are closer to the classical invariants of relative homotopy groups;
- the notion of freeness is clearer in XMod and models a standard topological situation, that of attaching 1- and 2-cells;
- the category XMod has a monoidal closed structure which helps to define a notion of homotopy; these constructions are simpler to describe in detail than those for 2-groupoids, and they extend to all dimensions.

We have a forgetful functor $\Phi_1 : \mathsf{XMod} \to \mathsf{Gpds}$ which sends $(M \to P) \mapsto P$. Our first main result is:

Proposition 7.2.9 *The forgetful functor* $\Phi_1 : \mathsf{XMod} \to \mathsf{Gpds}$ *is fibred and has a left adjoint.*

Proof The left adjoint of Φ_1 assigns to a groupoid P the crossed module $0 \to P$ which has only the trivial group over each $x \in P_0$.

Next, we give the pullback construction to prove that Φ_1 is fibred. So let $f : P \to Q$ be a morphism of groupoids, and let $\nu : N \to Q$ be a crossed module. We define $M = \nu^*(N)$ as follows.

For $x \in P_0$ we set $M(x)$ to be the subgroup of $P(x) \times N(fx)$ of elements (p, n) such that $fp = \nu n$. If $p_1 \in P(x, x')$, $n \in N(fx)$ we set $(p, n)^{p_1} = (p_1^{-1}pp_1, n^{f(p_1)})$, and let $\mu : (p, n) \mapsto p$. We leave the reader to verify that this gives a crossed module, and that the morphism $(p, n) \mapsto n$ is cartesian. □

The following result in the case of crossed modules of groups appeared in our earlier chapter 5, described in terms of the crossed module $\partial : u_*(M) \to Q$ *induced* from the crossed module $\mu : M \to P$ by a morphism $u : P \to Q$.

Proposition 7.2.10 *The forgetful functor* $\Phi_1 : \mathsf{XMod} \to \mathsf{Gpds}$ *is cofibred.*

Proof We prove this by a direct construction, generalising that given earlier.

Let $\mu : M \to P$ be a crossed module, and let $f : P \to Q$ be a morphism of groupoids. The construction of $N = f_*(M)$ and of $\partial : N \to Q$ requires just care to the geometry of the partial action in addition to the construction for the group case.

Let $y \in Q_0$. If there is no $q \in Q$ from a point of $f(P_0)$ to y, then we set $N(y)$ to be the trivial group.

Otherwise, we define $F(y)$ to be the free group on the set of pairs (m, q) such that $m \in M(x)$ for some $x \in P_0$ and $q \in Q(fx, y)$. If $q' \in Q(y, y')$ we set $(m, q)^{q'} = (m, qq')$. We define $\partial' : F(y) \to Q(y)$ to be $(m, q) \mapsto q^{-1}(fm)q$. This gives a precrossed module over $\partial : F \to Q$, with function $i : M \to F$ given by $m \mapsto (m, 1)$ where if $m \in M(x)$ then 1 here is the identity in $Q(fx)$.

We now wish to change the function $i : M \to F$ to make it an operator morphism. For this, factor F out by the relations

$$\begin{aligned}(m, q)(m', q) &= (mm', q),\\ (m^p, q) &= (m, (fp)q),\end{aligned}$$

whenever these are defined, to give a projection $F \to F'$ and $i' : M \to F'$. As in the group case, we have to check that $\partial' : F \to Q$ induces $\partial'' : F' \to H$ making this a precrossed module. To make this a crossed module involves factoring out Peiffer elements, whose theory is as for the group case in [BH82]. This gives a crossed module morphism $(\phi, f) : (M, P) \to (N, Q)$ which is cocartesian. □

We recall the algebraic origin of free crossed modules, but in the groupoid context.

Let P be a groupoid, with source and target functions written $s, t : P \to P_0$. A subgroupoid N of P is said to be *normal* in P, written $N \lhd P$, if N is wide in P, i.e. $N_0 = P_0$, and for all $x, y \in P_0$ and $a \in P(x, y)$, $a^{-1}N(x)a = N(y)$. If N is also totally intransitive, i.e. $N(x, y) = \emptyset$ when $x \neq y$, as we now assume, then the quotient groupoid P/N is easy to define. (It may also be defined in general but we will need only this case.)

Now suppose given a family $R(x), x \in P_0$ of subsets of $P(x)$. Then the normaliser $N_P(R)$ of R is well defined as the smallest normal subgroupoid of P containing all the sets $R(x)$. Note that the elements of $N_P(R)$ are all *consequences* of R in P, i.e. all well defined products of the form

$$c = (r_1^{\epsilon_1})^{a_1} \dots (r_n^{\epsilon_n})^{a_n}, \quad \epsilon_i = \pm 1, a_i \in P, n \geqslant 0 \tag{7.2.2}$$

and where b^a denotes $a^{-1}ba$. The quotient $P/N_P(R)$ is also written P/R, and called the *quotient* of P by the relations $r = 1, r \in R$.

As in group theory, we need also to allow for repeated relations. So we suppose given a set R and a function $\omega : R \to P$ such that $s\omega = t\omega = \beta$, say. This 'base point function', saying where the relations are placed, is a useful part of the information.

We now wish to obtain 'syzygies' by replacing the normal subgroupoid by a 'free object' on the relations $\omega : R \to P$. As in the group case, this is done using *free crossed modules*.

Remark 7.2.11 There is a subtle reason for this use of crossed modules. A normal subgroupoid N of P (as defined above) gives a quotient object P/N in the category Gpds_X of groupoids with object set $X = P_0$. Alternatively, N defines a congruence on P, which is a particular kind of equivalence relation. Now an equivalence relation is in general a particular kind of subobject of a product, but in this case, we must take the product in the category Gpds_X. As a generalisation of this, one should take a groupoid object in the category Gpds_X. Since these totally disconnected normal subgroupoids determine equivalence relations on each $P(x, y)$ which are congruences, it seems clear that a groupoid object internal to Gpds_X is equivalent to a 2-groupoid with object set X. □

Definition 7.2.12 A *free basis* for a crossed module $\partial : C \to P$ over a groupoid P is a set R, function $\beta : R \to P_0$ and function $i : R \to C$ such that $i(r) \in C(\beta r), r \in R$, with the universal property that if $\mu : M \to P$ is a crossed module and $\theta : R \to M$ a function over the identity on P_0 such that $\mu\theta = \partial i$, then there is a unique morphism of crossed P-modules $\phi : C \to M$ such that $\phi i = \theta$. □

Example 7.2.13 Let R be a set and $\beta : R \to P_0$ a function. Let $\mathrm{id} : F_1(R) \to F_2(R)$ be the identity crossed module on two copies of $F(R)$, the disjoint union of copies $C(r)$ of the infinite cyclic group C with generator $c_r \in C(r)$. Thus $F_2(R)$ is a totally intransitive groupoid with object set R. Let $i : R \to F_1(R)$ be the function $r \mapsto c_r$. Let $\beta : R \to R$ be the identity function. Then $\mathrm{id} : F_1(R) \to F_2(R)$ is the free crossed module on i. The verification of this is simple from the diagram

$$\begin{tikzcd} R \arrow[r, "i"'] \arrow[rr, bend left, "\theta"] & F_1(R) \arrow[r, dashed, "f"'] \arrow[d, "\mathrm{id}"'] & M \arrow[d, "\mu"] \\ & F_2(R) \arrow[r, "\mathrm{id}"'] & F_2(R) \end{tikzcd}$$

The morphism f simply maps the generator c_r to θr. □

Proposition 7.2.14 *Let* R *be a set, and* $\mu : M \to P$ *a crossed module over the groupoid* P. *Let* $\beta : R \to P_0$ *be a function. Then the functions* $i : R \to M$ *such that* $s\mu = t\mu = \beta$ *are bijective with the crossed module morphisms* (f, g)

$$\begin{tikzcd} F_1(R) \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & M \arrow[d, "\mu"] \\ F_2(R) \arrow[r, "g"'] & P \end{tikzcd}$$

such that $sg = \beta$.

Further, the free crossed module $\partial : C(\omega) \to P$ *on a function* $\omega : R \to P$ *such that* $s\omega = t\omega = \beta$ *is determined as the crossed module induced from* $\mathrm{id} : F_1(R) \to F_2(R)$ *by the extension of* ω *to the groupoid morphism* $F_2(R) \to P$.

Proof The first part is clear since $g = \mu f$ and f and i are related by $f(c_r) = i(r), r \in R$.

The second part follows from the first part and the universal property of induced crossed modules as shown in the following diagram:

$$\begin{tikzcd} F_1(R) \arrow[r, "f"'] \arrow[rr, bend left, "\theta"] \arrow[d, "\mathrm{id}"'] & C(\omega) \arrow[r, dashed, "\phi"'] \arrow[d, "\partial"] & M \arrow[d, "\mu"] \\ F_2(R) \arrow[r, "g"'] & P \arrow[r, "="'] & P \end{tikzcd}$$

□

7.3 Some substructures of crossed complexes and their interrelations.

Previous sections have shown groupoids as having related structures in dimensions 0 and 1, and this was used to study calculations with groupoids; in particular we studied subcategories Gpds_X of Gpds

for various sets X, and their relationships through functions $X \to Y$. This was seen in the general context of fibred categories, studied in Appendix A.7.

In this section, we carry out a similar study on crossed complexes, using the idea that crossed complexes have algebraic structure in a range of dimensions. The particular functors of truncation, skeleton, coskeleton, cotruncation, have importance not only for particular calculations of colimits of crossed complexes, but also for the theoretical studies of Part III on the equivalence of crossed complexes and ω-groupoids, whose utility is at the heart of this book, even if at this stage in a way which may be mysterious.

First, we study the category Crs_n of n-truncated crossed complexes and the functors relating it to Crs. The category Crs_2 of 2-truncated crossed complexes is the same as the category XMod of crossed modules over groupoids studied in Chapter 6 and in earlier sections of this chapter.

7.3.1 n-truncated crossed complexes.

We will use finite-dimensional versions of crossed complexes.

Definition 7.3.1 An n-*truncated crossed complex* is a finite sequence

$$C_n \xrightarrow{\delta_n} C_{n-1} \longrightarrow \cdots\cdots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1$$

satisfying all the axioms for a crossed complex in so far as they make sense. In a similar way, we define *morphism between* n*-truncated crossed complexes*. They define Crs_n the category of n-truncated crossed complexes. □

Notice that a 1-truncated crossed complex is simply a groupoid, and a 2-truncated crossed complex is a crossed module over a groupoid. Thus we can write $\mathsf{Crs}_1 = \mathsf{Gpds}$ and $\mathsf{Crs}_2 = \mathsf{XMod}$.

Definition 7.3.2 We define the n-*truncation functor*

$$\mathrm{tr}_n : \mathsf{Crs} \to \mathsf{Crs}_n$$

which applied to a crossed complex C gives only its n-dimensional part. □

There is also a functor in the other direction:

Definition 7.3.3 The n-*skeleton functor*

$$\mathrm{sk}^n : \mathsf{Crs}_n \to \mathsf{Crs}$$

maps an n-truncated crossed complex $C: \ C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots \longrightarrow C_2 \xrightarrow{\delta_2} C_1$ to

$$\mathrm{sk}^n(C): \ \cdots \longrightarrow 0 \longrightarrow C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots \longrightarrow C_2 \xrightarrow{\delta_2} C_1 \ ,$$

which agrees with C up to dimension n and is trivial thereafter.

It is also convenient to write $\mathrm{Sk}^n = \mathrm{sk}^n \, \mathrm{tr}_n$, so that an n-truncated crossed complex is also thought of as a crossed complex C with $C_k = 0$ for $k > n$. This n-Skeleton functor allows us to consider Crs_n as a full subcategory of Crs.

Proposition 7.3.4 *The* n*-skeleton functor* sk^n *is left adjoint to the* n*-truncation functor* tr_n.

Proof For any crossed complex D and n-truncated crossed complex C there is an obvious bijection

$$\mathsf{Crs}(\mathrm{sk}^n(C), D) \to \mathsf{Crs}_n(C, \mathrm{tr}_n(D))$$

because a morphism of crossed complexes

$$f : \mathrm{sk}^n(C) \to D$$

is given just by the first n maps f_i, since all the others are the 0 maps as in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & C_n \arrow[r, "\delta_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots\cdots \arrow[r, "\delta_3"] & C_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"] & C_1 \arrow[d, "f_1"] \\
\cdots \arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r, "\delta'_n"'] & D_{n-1} \arrow[r, "\delta'_{n-1}"'] & \cdots\cdots \arrow[r, "\delta'_3"'] & D_2 \arrow[r, "\delta'_2"'] & D_1
\end{tikzcd}$$

□

The n-truncation functor has also a right adjoint which is a modification of the n-skeleton functor.

Definition 7.3.5 We define the n*-coskeleton functor*

$$\mathrm{cosk}^n : \mathsf{Crs}_n \to \mathsf{Crs},$$

on an n-truncated crossed complex C: $C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots\cdots \longrightarrow C_2 \xrightarrow{\delta_2} C_1$ by

$$\mathrm{cosk}^n(C) := \ \cdots \longrightarrow 0 \longrightarrow \mathrm{Ker}\,\delta_n \longrightarrow C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots\cdots \longrightarrow C_1$$

for $n \geqslant 2$ and by

$$\mathrm{cosk}^1(C) := \ \cdots \longrightarrow 0 \longrightarrow \mathrm{Inn}(C_1) \longrightarrow C_1\ ,$$

where $\mathrm{Inn}(C_1)$ is the totally disconnected groupoid formed by the object groups of C_1. □

We also write $\mathrm{Cosk}^n = \mathrm{cosk}^n\,\mathrm{tr}_n$ as a functor $\mathsf{Crs} \to \mathsf{Crs}$. Notice that the only difference of the coskeleton from the skeleton functor is in the existence of elements of dimension $n+1$. The importance of this is realised when proving adjointness.

Proposition 7.3.6 *The* n*-coskeleton functor* cosk^n *is right adjoint to the* n*-truncation functor* tr_n.

Proof Let $n \geqslant 2$. For any crossed complex C and n-truncated crossed complex C' there is an obvious bijection

$$\mathsf{Crs}(C, \mathrm{cosk}^n(D)) \to \mathsf{Crs}_n(\mathrm{tr}_n(C), D)$$

because a morphism f from C to $\mathrm{cosk}^n(D)$ is just given by the first n maps since the $(n+1)$st has to be the restriction of $f_n\delta_{n+1}$ to its image and all others have to be the 0 maps as in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+2} \arrow[r] \arrow[d, "0"] & C_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "f_n\delta_{n+1}"] & C_n \arrow[r, "\delta_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots\cdots \arrow[r, "\delta_3"] & C_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"] & C_1 \arrow[d, "f_1"] \\
\cdots \arrow[r] & 0 \arrow[r] & \mathrm{Ker}\,\delta'_n \arrow[r] & D_n \arrow[r, "\delta'_n"'] & D_{n-1} \arrow[r, "\delta'_{n-1}"'] & \cdots\cdots \arrow[r, "\delta'_3"'] & D_2 \arrow[r, "\delta'_2"'] & D_1
\end{tikzcd}$$

Notice that we need the $(n+1)$st dimensional part of $\mathrm{cosk}^n(D)$ to be $\mathrm{Ker}\,\delta'_n$, in order to be able to define f_{n+1} as above because $\delta'_n f_n \delta_{n+1} = f_{n-1}\delta_n\delta_{n+1} = 0$.

We leave the case $n = 1$ to the reader. □

We will use later a left adjoint to sk^n.

Proposition 7.3.7 *For $n \geqslant 1$, the cotruncation functor $\mathrm{cotr}_n : \mathsf{Crs} \to \mathsf{Crs}_n$ which assigns to a crossed complex C the n-truncated crossed complex over C_0*

$$\mathrm{cotr}_n\, C := \quad (C_n/\operatorname{Im}\delta_{n+1}) \to C_{n-1} \to \cdots \to C_2 \to C_1$$

is left adjoint to sk^n.

Proof We leave details to the reader, including the proof that $\mathrm{cotr}_n\, C$ inherits the structure of crossed complex. □

In summary, we have functors

$$\mathrm{tr}_n, \mathrm{cotr}_n : \mathsf{Crs} \to \mathsf{Crs}_n, \qquad \mathrm{sk}^n, \mathrm{cosk}^n : \mathsf{Crs}_n \to \mathsf{Crs}$$

such that tr_n has left adjoint sk^n, and right adjoint cosk^n, while sk^n has right adjoint tr_n and left adjoint cotr_n.

Corollary 7.3.8 *The functors $\mathrm{sk}^n, \mathrm{tr}_n$ preserve limits and colimits; cosk^n preserves limits; cotr_n preserves colimits. In particular, the fundamental groupoid functor $\pi : \mathsf{Crs} \to \mathsf{Gpds}$, which coincides with $\mathtt{cotr}_1$, preserves colimits.*

Definition 7.3.9 For $n \geqslant 2$, we define the *restriction to dimension n functor*

$$\mathrm{res}_n : \mathsf{Crs} \to \mathsf{Mod}$$

to be given on objects by

$$\mathrm{res}_n(C) = \begin{cases} (C_n, \pi_1 C) & \text{if } n \geqslant 3, \\ (C_2^{\mathrm{ab}}, \pi_1 C) & \text{if } n = 2. \end{cases}$$

and with the obvious extension to morphisms. □

Definition 7.3.10 For each $n \geqslant 3$, we define the functor

$$\mathbb{F}_n : \mathsf{Mod} \to \mathsf{Crs}$$

to have value on a module (M, G) the crossed complex

$$\cdots \longrightarrow 0 \longrightarrow \underset{n}{M} \xrightarrow{1_M} \underset{n-1}{M} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{1}{G}\ ,$$

where the two Ms are in dimensions n and $n-1$, the map between them is the identity, and all other boundary maps are 0. The value of $\mathbb{F}_n$ on morphisms is defined similarly. Notice that for $n \geqslant 3$, $\mathbb{F}_n(\mathbb{Z}, 0) = \mathbb{F}(n)$ as defined earlier, and this explains our notation. □

Proposition 7.3.11 *For $n \geqslant 2$, the functor res_n is left adjoint to $\mathbb{F}_{n+1}$.*

Proof Suppose $n \geqslant 3$. We need to study $\mathsf{Crs}(C, \mathbb{E}_{n+1}(M, G))$, i. e. morphisms of crossed complexes

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+2} \arrow[r, "\delta_{n+2}"] \arrow[d, "0"] & C_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\delta_n"] \arrow[d, "f_n"] & \cdots \arrow[r, "\delta_3"] & C_2 \arrow[r, "\delta_2"] \arrow[d, "0"] & C_1 \arrow[d, "f_1"] \\
\cdots \arrow[r] & 0 \arrow[r, "0"'] & M \arrow[r, "1_M"'] & M \arrow[r, "0"'] & \cdots \arrow[r, "0"'] & 0 \arrow[r, "0"'] & G
\end{tikzcd}$$

Since $f_1\delta_2 = 0$) and $\theta = f_n$, this diagram produces a morphism of modules

$$(\theta, f) : (C_n, \pi_1(C)) \to (M, G)$$

where $f : \pi_1(C) \to G$ is induced by f_1.

On the other hand, given a morphism of modules

$$(\theta, f) : (C_n, \operatorname{Cok} \delta_2) \to (M, G)$$

we get a morphism of crossed modules on putting $f_1 = f\phi$ (ϕ being the projection $\phi : C_1 \to \operatorname{Cok} \delta_2$), $f_n = \theta$ and $f_{n+1} = f_n\delta_{n+1}$.

These correspondences give the adjointness for this case. We leave the case $n = 2$ to the reader. □

Corollary 7.3.12 *For* $n \geqslant 2$*, the functor* $\mathrm{res}_n : \mathsf{Crs} \to \mathsf{Mod}$ *preserves colimits.*

Exercise 7.3.13 Give the proof of the case $n = 2$ of the last proposition. □

Exercise 7.3.14 There is another restriction functor for $n \geqslant 3$

$$\mathrm{res}'_n : \mathsf{Crs} \to \mathsf{Mod}$$

given by $\mathrm{res}'_n\, C = ((C_n)/(\delta_{n+1}C_{n+1}), C_1)$. This is useful in two ways: Firstly, the inclusion

$$\mathbb{K}_n : \mathsf{Mod} \to \mathsf{Crs}$$

defined on objects by

$$\mathbb{K}_n(M, G) := \quad \cdots \longrightarrow 0 \longrightarrow \underset{n}{M} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{1}{G}$$

gives an embedding of Mod as a full subcategory of Crs for any dimension $n \geqslant 3$, since it is the right inverse of res'_n. Secondly, the functor res'_n is right adjoint to $\mathbb{K}_n$. This functor will be used (with a slightly different notation) in Section 12.5 in connection with the homotopy classification of maps. □

7.3.2 Colimits of crossed complexes.

The HHvKT Theorem 8.1.5 in the next Chapter states that the functor $\Pi : \mathsf{FTop} \to \mathsf{Crs}$ preserves certain colimits. The proof, which we give in Part III, does not require knowledge of the existence of colimits in the category Crs. It is true that these colimits exist: this follows from general facts on algebraic theories which do not need to go into here. However, in order to apply the HHvKT we need to know, not that they exist in general, but how to compute colimits of crossed complexes in more familiar terms and in specific situations.

As an illustration, recall that we have regarded as topologically and algebraically significant the description of the infinite cyclic group C_∞ as given by a pushout of groupoids:

$$\begin{tikzcd} \{0,1\} \arrow[r] \arrow[d] & \{0\} \arrow[d] \\ \mathcal{I} \arrow[r] & C_\infty \end{tikzcd}$$

Here we give a similar example involving modules, the details of which we leave to the reader.

Example 7.3.15 Consider for $n \geqslant 2$ the following pair of maps of spaces:

$$S^n \to S^n \vee [0,1] \sqcup \{2\} \to S^n \vee S^1$$

the first being an inclusion and the second an identification of the three vertices. A first basic test of our theory is to be able to model directly simple examples like this. Here is how the modelling goes. Consider the diagram of modules over groupoids consisting of two pushouts:

$$\begin{tikzcd}
(0,0) \arrow[r, "{(0,f)}"] \arrow[d] & (0, \mathfrak{I} \sqcup \{2\}) \arrow[r, "{(0,g)}"] \arrow[d] & (0, C_\infty) \arrow[d] \\
(\mathbb{Z}, 0) \arrow[r] & (f_*(\mathbb{Z}), \mathfrak{I} \sqcup \{2\}) \arrow[r] & (g_* f_*(\mathbb{Z}), C_\infty)
\end{tikzcd}$$

each defining an induced module. Here $(\mathbb{Z}, 0)$ is the module consisting of the integers, with generator a, together with the action of the trivial group $G = 0$. Let H be the groupoid with three objects $0, 1, 2$ and exactly one arrow $\iota : 0 \to 1$, so that $H = \mathfrak{I} \sqcup \{2\}$. Let $G \to H$ be the inclusion. Let C_∞, regarded as a groupoid with object 0, have generator c. Let $G \xrightarrow{f} H \xrightarrow{g} C_\infty$ be the morphisms in which $f(0) = 0, g(\iota) = c$. Then $f_*(M)$ is the H-module consisting of the integers over each of $0, 1$ with generators a, a^ι, and the zero group over 2. Finally, $(gf)_*(M) = g_* f_*(M)$ is the free C_∞-module on one generator a, say, and so as an abelian group is the direct sum of copies of $\mathbb{Z}$, one for each element of C_∞, and the action of this group is given by translation among these copies. $\Box$

We can now easily show that the determination of colimits in Crs can be reduced to the determination of colimits in:

(i) the category Gpds of groupoids;

(ii) the category XMod of crossed modules over groupoids, and

(iii) the category Mod of modules over groupoids.

We have already proved using the notion of fibred and cofibred category prove that constructing colimits of connected diagrams in either of Mod and XMod may be done in two steps. First, we change the base groupoids of the modules of a diagram so that they become the same for all modules or crossed modules and then we take the colimit in Mod_G or XMod_G.

Proposition 7.3.16 *Let $C = \operatorname{colim} C^\lambda$ be a colimit in the category Crs of crossed complexes. Then*

(i) *for $n = 1$, the groupoid $\mathrm{tr}_1 C = C_1$ is the colimit in Gpds of the groupoids $\mathrm{tr}_1 C^\lambda = C_1^\lambda$, i.e.*

$$C_1 = \operatorname{colim}_{\mathsf{Gpds}} C_1^\lambda;$$

(ii) *for $n = 2$, the crossed complex $\mathrm{tr}_2 C$ is the colimit in XMod of the crossed modules $\mathrm{tr}_2 C^\lambda$, i.e.*

$$(C_2 \to C_1) = \operatorname{colim}_{\mathsf{XMod}} (C_2^\lambda \to C_1^\lambda);$$

(iii) *for each $n \geqslant 3$, the groupoid C_n as a module over $\pi_1(C)$ is the colimit in the category Mod of the groupoids C_n^λ as modules over $\pi_1(C_n^\lambda)$, i.e.*

$$(C_n, \pi_1(C)) = \operatorname{colim}_{\mathsf{Mod}} (C_n^\lambda, \pi_1(C^\lambda)).$$

Proof All these facts hold because the functors appropriate to each case have right adjoints and, consequently, they preserve colimits:

(i) and (ii) follow because we have proved in Proposition 7.3.6 that the coskeleton functors $\mathrm{cosk}_1 : \mathsf{Gpds} \to \mathsf{Crs}$ and $\mathrm{cosk}_2 : \mathsf{XMod} \to \mathsf{Crs}$ are the right adjoints of the truncation functors $\mathrm{Tr}_1 : \mathsf{Crs} \to \mathsf{Gpds}$ and $\mathrm{Tr}_2 : \mathsf{Crs} \to \mathsf{XMod}$.

(iii) follows because we have proved in Proposition 7.3.11 that the functor $\mathbb{E}_{n+1} : \mathsf{Mod} \to \mathsf{Crs}$ is right adjoint to the restriction functor $(-)_n : \mathsf{Crs} \to \mathsf{Mod}$.

Note that, this description of $\mathrm{Tr}_2\, C$ and C_n for $n \geqslant 3$ as colimits gives not only the modules, but also the boundary maps $\delta : C_n \to C_{n-1}$; these can be recovered as induced by the maps $\delta^\lambda : C_n^\lambda \to C_{n-1}^\lambda$, for all λ. □

7.4 Free constructions.

In Part I we have used free groups, and studied free crossed modules over groups; free modules over a group are common knowledge. Now we generalise all this to the groupoid case, in order to arrive at the notion of free crossed complexes. These are important in their own right in algebra, and also in topology because they gives a useful algebraic model *CW*-complexes.

In particular, free constructions given here model the process of attaching cells to a space.

Attaching 1-cells to a discrete space gives graphs, with the well known free groupoids as algebraic models. In higher dimensions, for a space A we may form $X = A \cup_{f_i} e_i^2$ where the cells e_i^2 are attached by maps $f_i : S^1 \to A$. We may take the base point of S^1 to be say 1, and set $a_i = f_i(1), A_0 = \{a_i\}$. We then want to express $\pi_2(X, A, A_0)$ as a free crossed module over the fundamental groupoid $\pi_1(A, A_0)$. We also want to see, if A has itself a base point say a_0, how to calculate the crossed module of groups $\pi_2(X, A, a_0) \to \pi_1(A, a_0)$.

So we must extend the notions of free groupoid to the higher dimensions of free crossed modules and free crossed complexes. Because of the given geometric structure of crossed complexes, this extension is quite simple.

We first need to say something on free groupoids.

7.4.1 Free groupoids.

We explain the notion of free groupoid on a graph – this is used implicitly in combinatorial group theory, for example in paths in a Cayley graph, and is required for combinatorial groupoid theory. We will exploit this in a later chapter, when calculating crossed resolutions.

Definition 7.4.1 By a *graph* $\Gamma = (E(\Gamma), V(\Gamma), s, t)$ we mean a set $E(\Gamma)$ of *edges*, a set $V(\Gamma)$ of *vertices* and two functions $s, t : E(\Gamma) \to V(\Gamma)$ called the *source and target maps*.

A *morphism* $f : \Gamma \to \Gamma'$ *of graphs* is a pair of functions $E(f) : E(\Gamma) \to E(\Gamma'), V(f) : V(\Gamma) \to V(\Gamma')$ which commute with the source and target maps. This gives the category Grphs of graphs. □

Remark 7.4.2 This is commonly called a *directed graph*, but we shall use only these. Also we shall, in keeping with the terminology for categories and groupoids, use also the term objects of the graph instead of vertices. As for groupoids, we write $a : x \to y$ if a is an edge and $sa = x, ta = y$, say a is from x to y, and we write $\Gamma(x, y)$ for the set of edges from x to y in Γ.

Proposition 7.4.3 *The forgetful functor* $\mathsf{U} : \mathsf{Gpds} \to \mathsf{Grphs}$ *has a left adjoint, whose arrows can be seen as paths in the graph.*

Proof We outline a proof and leave the details as an exercise.

Let 2 denote the graph with two vertices $0, 1$ and one edge $\iota : 0 \to 1$. A given graph Γ can be regarded as obtained from a disjoint union of copies of the graph 2 by an appropriate identification of the vertices. The same identification for a similar disjoint union of copies of the groupoid $\mathcal{I}$ gives the free groupoid $F(\Gamma)$ on Γ.

Then $F(\Gamma)(x, y)$ can be seen as made up of classes of 'paths' from x to y in Γ, where such a path is a sequence of edges or a formal inverse of an edge, which are composable, and which starts at x and ends at y. (See also [Bro06, Section 8.2].) □

Exercise 7.4.4 Use the functor f_* to construct and verify the properties of a free groupoid on a directed graph. In particular give the universal property, and show that the graph morphism $\Gamma \to F(\Gamma)$ is injective on vertices and on edges. □

Exercise 7.4.5 Formulate and discuss the notion of generating graph for a groupoid. □

7.4.2 Free crossed modules, and free modules, over groupoids.

We first note that a set of generators for a crossed module $\mu : M \to P$ must be a family $R(x)$ of subsets of $M(x)$ for all $x \in P_0$ such that M itself is the smallest subcrossed P-module of M containing R. This means that M consists of all consequences of R, where a consequence, analogously to the group case, is an element of the form

$$c = \prod_{i=1}^{n} ((r_i)^{\varepsilon_i})^{p_i}$$

where $r_i \in R, \varepsilon_i = \pm 1, p_i \in P_1$. The difference from the group case is that here if $c \in M(x)$ then $r_i \in M(x_i)$ if and only if $p_i \in P(x_i, x)$. Thus a set of generators comes with a function $\omega : R \to P$ such that $s\omega = t\omega$.

Definition 7.4.6 A function $\omega : R \to P$ together with a function $i : R \to M$ such that $\mu i = \omega$ is said to give a *set of free generators* of the crossed module $\mu : M \to P$ if it has the universal property that for any crossed P-module $\nu : N \to P$ and function $j : R \to N$ such that $\nu j = i$, there is a unique morphism $\phi : M \to N$ of crossed P-modules such that $\phi i = j$.

Proposition 7.4.7 *Given a groupoid* P*, and a function* $\omega : R \to P$ *such that* $s\omega = t\omega$*, then the free crossed* P*-module on* ω *exists.*

Proof We present this free crossed module as an induced module, analogously to the group case.

Let $F = F(R, t')$ be the (totally disconnected) free groupoid over P_0 on the family of sets given by $t' = t\omega : R \to P_0$. Thus for each $x \in P_0$, $F(R, t')(x)$ is the free group on $t'^{-1}(x)$. Let $\omega' : F \to P$ be the groupoid morphism over P_0 determined by ω.

Since F is totally disconnected, we can form the 'conjugacy crossed module' $1 : F \to F$, in which F acts on itself by conjugation. Then the free crossed P-module on ω is given by the induced crossed module, as in the diagram:

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, "1"'] & C(\omega) \arrow[d, "\partial"] \\ F \arrow[r, "\omega'"'] & P \end{tikzcd}$$

We leave the reader to check the defining universal property. □

We now have to give the analogous, but simpler, theory for free modules. Let P be a groupoid, and $t : R \to P_0$ be a function. The development is missing.

7.4.3 Free crossed complexes.

Free crossed complexes model algebraically the topological notion of inductively attaching cells, as in relative CW-complexes.

Definition 7.4.8 We write $\mathbb{F}(n)$ for the *crossed complex freely generated by one generator* c_n *in dimension* n. So

- $\mathbb{F}(0)$ is $\{1\}$ in dimension $n = 0$ and trivial elsewhere;

- $\mathbb{F}(1)$ is the crossed complex $\mathrm{sk}^1 \mathcal{I}$, where $\mathcal{I}$ is the groupoid which has only two objects 0, 1 and non-identity elements $c_1 : 0 \to 1$ and its inverse $c_1^{-1} : 1 \to 0$. Thus $\mathbb{F}(1)$ has $\{0, 1\}$ in dimension $n = 0$, $\mathcal{I}$ in dimension $n = 1$, the trivial crossed module $0_{\mathcal{I}}$ in dimension $n = 2$ and trivial elsewhere;

- $\mathbb{F}(n)$ for $n \geqslant 2$ is in dimensions n and $n - 1$ an infinite cyclic group with generators c_n and $c_{n-1} = \delta c_n$ respectively, and is otherwise trivial. The only non trivial δ is defined by $\delta_n(c_n) = c_{n-1}$. Notice that $\mathbb{F}(n)$ is just another name for $\mathbb{E}_n(\mathbb{Z})$.

Also, we shall write $\mathbb{S}(n-1)$ for the subcomplex of $\mathbb{F}(n)$ which agrees with $\mathbb{F}(n)$ up to dimension $n - 1$ and is trivial otherwise. Thus the only difference between $\mathbb{F}(n)$ and $\mathbb{S}(n-1)$ is at dimension n, where $\mathbb{F}(n)_n$ is isomorphic to $\mathbb{Z}$ and $\mathbb{S}(n-1)_n$ is trivial. □

Remark 7.4.9 1.- Notice that this definition satisfies what we would assume is a characterisation of free in this context, i.e. for any crossed complex C and any element $c \in C_n$ there is a unique morphism of crossed modules

$$\hat{c} : \mathbb{F}(n) \to C$$

such that $\hat{c}(c_n) = c$. That is, there is a natural bijection of sets $C_n \cong \mathsf{Crs}(\mathbb{F}(n), C)$.

2.- It is a straightforward consequence of convexity of the interval $\mathbf{E}^1$ that $\Pi(\mathbf{E}^1_*) \cong \mathbb{F}(1)$. It will be proved in Corollary 8.3.12 in the next Chapter that $\Pi(\mathbf{E}^n_*) \cong \mathbb{F}(n)$ for all n.

3.- That $\Pi(\mathbf{S}^1_*) \cong \mathbb{S}(1)$, is in essence the fact that the fundamental group of $\mathbf{S}^1$ is isomorphic to $\mathbb{Z}$, as has been proved in Section 1.7. It will be proved in Corollary 8.3.11 in the next Chapter that $\Pi(\mathbf{S}^n_*) \cong \mathbb{S}(n)$ for $n \geqslant 1$ and it follows from this that $\Pi(\mathbf{E}^n_*) \cong \mathbb{F}(n)$. □

Now, using $\mathbb{F}(n)$, we may define the notion of 'adding free generators in dimension n'.

Definition 7.4.10 Let A be a crossed complex. We say that a morphism of crossed complexes $f : A \to C$ is of *pure relative free type of dimension* $n \geqslant 0$ if there is a set of indexes Λ and a family of morphisms $f^\lambda : \mathbb{S}(n-1) \to A$ for $\lambda \in \Lambda$, such that the following square is a pushout in Crs:

$$\begin{tikzcd} \bigsqcup_{\lambda\in\Lambda} \mathbb{S}(n-1) \arrow[r, "(f^\lambda)"] \arrow[d] & A \arrow[d, "j"] \\ \bigsqcup_{\lambda\in\Lambda} \mathbb{F}(n) \arrow[r] & C. \end{tikzcd}$$

We write

$$C = A \cup \{x^n_\lambda\}_{\lambda\in\Lambda},$$

and may abbreviate this in some cases, for example to $C = A \cup x^n$. □

Example 7.4.11 To have a clearer view of the last definition it is good to take some pains and describe the change on adding just one free generator, i.e. $C = A \cup x^n$. Let us see the cases according to dimension

1. When $n = 0$, adding one generator is just adding an extra point x to the set of objects. Thus $C_0 = A_0 \cup \{x\}$ and $C_n = A_n$ for all $n \geqslant 0$.
2. The case $n = 1$ is most delicate, because when we are adding a free generator x to the groupoid A_1, we are also adding all elements a^x for $a \in A_n$, $n \in \mathbb{N}$.
3. In the case $n > 1$ we add only a free generator in dimension n. This gives a coproduct with $f_*(\mathbb{F})$. [This is a bit unclear. There is an identification of base points, which is where the coproduct comes in, but one has to explain the category in which the coproduct takes place.]

$\square$

We consider now the situation when the free generators are added in increasing order of dimension. In the limit we get a special kind of 'subcomplexes' $j : A \to C$ which we shall call a *crossed complex morphism of relative free type*.

Definition 7.4.12 Let A be any crossed complex. We define a sequence of complexes C_n and morphisms $j_n : C^{n-1} \to C^n$ starting with $C^0 = A$, and choosing a family of morphisms $f_n^\lambda : \mathbb{S}(n-1) \to C^{n-1}$ for $\lambda \in \Lambda_n$ such that C_n is got by forming the pushout

$$\begin{tikzcd}
\bigsqcup_{\lambda\in\Lambda_n} \mathbb{S}(n-1) \arrow[r, "(f_n^\lambda)"] \arrow[d] & C^{n-1} \arrow[d, "j_n"] \\
\bigsqcup_{\lambda\in\Lambda_n} \mathbb{F}(n) \arrow[r] & C^n .
\end{tikzcd}$$

Let $C = \operatorname{colim} \ C^n$, and let $j : A \to C$ be the canonical morphism. We call $j : A \to C$ a *crossed complex morphism of relative free type*. The images x^n of the elements c_n in C are called *basis elements of* C *relative to* A. We can conveniently write

$$C = A \cup \{x^n\}_{\lambda\in\Lambda_n, n\geqslant 0},$$

and may abbreviate this in some cases, for example to $C = A \cup x^n \cup x^m$, analogously to standard notation for *CW*-complexes. $\square$

Remark 7.4.13 Let us describe the structure we get on C in each dimension.

1. C_0 is the disjoint union of A_0 and Λ_0;
$$C_0 = A_0 \sqcup \Lambda_0 \quad \text{in Sets}$$
2. C_1 is the coproduct of two groupoids with set of objects C_0,
$$C_1 = A_1^* \sqcup_{C_0} F(\Lambda_1) \quad \text{in } \mathsf{Gpds}_{C_0},$$
where A_1^* is the groupoid obtained from A_1 by adjoining the objects of C not already in A, and $F(\Lambda_1)$ is the free groupoid on Λ_1 considered as a graph over C_0 via the maps f_1^λ;

3. C_2 is the coproduct of two crossed C_1-modules

$$C_2 = A_2^* * F(\Lambda_2) \quad \text{in } \mathsf{XMod}_{C_1},$$

where A_2^* is the C_1-crossed module induced from the A_1-crossed module A_2 by the morphism of groupoids $A_1 \to C_1$, and $F(\Lambda_2)$ is the free crossed C_1-module on Λ_2 via the maps f_2^λ;

4. C_n, for $n \geqslant 3$, is the direct sum of two $\pi_1(C_1)$-modules

$$C_n = A_n^* \oplus F(\Lambda_n) \quad \text{in } \mathsf{Mod}_{\pi_1(C)_1},$$

where A_n^* is the module induced from the $\pi_1(A_1)$-module A_n by the morphism of groups $\pi_1(A_1) \to \pi_1(C_1)$, and $F(\Lambda_n)$ is the free $\pi_1(C_1)$-module on Λ_n.

The boundary maps are in all cases induced by the boundary maps in A and by the maps f_n^λ.

Thus at each dimension C_n is the coproduct in the suitable category of the n-dimensional part of A (appropriately modified) and a free structure with as many generators as the n-cells we are attaching. □

Example 7.4.14 It will be a corollary of the HHvKT in the next chapter that for the skeletal filtration X_* of a CW-complex X, the crossed complex ΠX_* is free; and that if Y_* is a subcomplex of X_* then the induced morphism $\Pi Y_* \to \Pi X_*$ is relatively free. □

Of course, the advantage of a having a free basis X_* for a crossed complex C is that a morphism $f : C \to D$ is defined completely by the values of f on X_* provided the following conditions are satisfied:

(i) They have the appropriate source and target,

i.e. $sf_1x = f_0sx$ and $tf_1x = f_0tx$, for all $x \in X_1$, and $tf_n(x) = f_0(tx)$ for all $x \in X_n, n \geqslant 2$.

(ii) They produce a morphism of crossed complex,

i.e. $\delta_n f_n(x) = f_{n-1}\delta_n(x), x \in X_n, n \geqslant 2$.

Notice that in (ii), f_{n-1} has to be constructed on all of C_{n-1} from its values on the basis for C_{n-1}, before this condition can be verified.

If further D is free, then to specify $f_n(x)$ we simply have to give the expression of $f_n(x)$ in terms of the basis in dimension n for D_n.

Later we will see that homotopies can be specified similarly (see Corollary 9.6.5).

We end this Section by stating some results about the good behaviour of relatively free morphisms with respect to some colimits. The way to prove all of them is to check for the case when all generators have the same dimension and go to the general case by a colimit argument.

First such morphisms are preserved by composition:

Proposition 7.4.15 *Given two morphism of relative free type, so is their composite.*

Then they are preserved by pushout:

Proposition 7.4.16 *If in a pushout square*

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & D' \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

the morphism $C' \to C$ *is of relative free type, so is the morphism* $D' \to D$.

Last, they are preserved by sequential colimits:

Proposition 7.4.17 *If in a commutative diagram*

$$\begin{tikzcd} C^0 \arrow[r] \arrow[d] & C^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & C^n \arrow[r] \arrow[d] & \cdots \\ D^0 \arrow[r] & D^1 \arrow[r] & \cdots \arrow[r] & D^n \arrow[r] & \cdots \end{tikzcd}$$

each vertical morphism is of relative free type, so is the induced morphism $\mathrm{colim}_n C^n \to \mathrm{colim}_n D^n$.

In particular:

Corollary 7.4.18 *If in a sequence of morphisms of crossed complexes*

$$C^0 \to C^1 \to \cdots \to C^n \to \cdots$$

each morphism is of relative free type, so are the composites $C^0 \to C^n$ *and the induced morphism* $C^0 \to \mathrm{colim}_n C^n$.

7.5 Crossed complexes and chain complexes

As we have seen in Section 7.3, a crossed complex is a kind of nonabelian chain complex with operators, the nonabelian features being confined to dimensions $\leqslant 2$. In this section, we begin to make the relation between the two complexes more precise. The adjoint constructions will be used later in section 9.5 to help understand tensor products of crossed complexes.

Definition 7.5.1 Let G be a groupoid. A *chain complex* $A = (A_n, \partial_n)_{n \geqslant 0}$ over G is a sequence

$$\cdots \xrightarrow{\partial} A_n \xrightarrow{\partial} A_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} A_1 \xrightarrow{\partial} A_0$$

of G-modules and G-morphisms satisfying $\partial\partial = 0$. A *morphism of chain complexes* $f : (A, G) \to (B, H)$ is a family of morphisms $f_n : (A_n, G) \to (B_n, H)$ (over some $f_0 : G \to H$, independent of n) satisfying $\partial f_n = f_{n-1}\partial$. These form a category Chn and, for a fixed groupoid G, we have a subcategory Chn_G of chain complexes over G. Also reducing the basis to groups we get back the category Chn_* of chain complexes over groups. □

Our aim now is to construct a functor

$$\nabla : \mathsf{Crs} \to \mathsf{Chn}$$

which gives a form of 'semiabelianisation' of the crossed module part of a crossed complex, which keeps information on the fundamental groupoid. It will be important later that this functor ∇ has a right adjoint (called Θ). We will use this adjoint pair in later chapters to investigate the tensor product of crossed complexes, and the homotopy classification of maps from a free crossed complex.

The definition of ∇ is easy in dimensions $\geqslant 3$, when we set $(\nabla C)_n = C_n$, with boundary ∂ just δ: we shall leave everything as it is. We are left with

$$C_3 \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1$$

where δ_2 is a crossed module with cokernel $\phi : C_1 \to G$ and we want to change it to get

$$C_3 \xrightarrow{\partial_3} L_2 \xrightarrow{\partial_2} L_1 \xrightarrow{\partial_1} L_0$$

where L_2, L_1 and L_0 are G-modules. We can use ϕ to associate to C_2 the G-module $L_2 = C_2^{ab}$. It is more difficult to get the correct candidates for L_1 and L_0, but again they crucially involve ϕ.

In the first subsection, we study the candidates for L_0, the 'adjoint module' and the 'augmentation module' and prove that they give functors which have right adjoints. In subsection 7.5.2 we study the candidate for L_1, the 'derived module'. A big advantage of working with the category Mod (which includes modules over all groupoids) is that we can exploit the formal properties of the functorial constructions used.

In the penultimate subsection we give the right adjoint Θ of $\nabla : \mathsf{Crs} \to \mathsf{Chn}$, and the last subsection illustrates with a specific calculation the fact that ∇ preserves colimits.

7.5.1 Adjoint module and augmentation module.

Basic constructions used to linearise the theory of groups in homological algebra are, for a group G, the group ring $\mathbb{Z}G$ and the augmentation ideal IG. We extend these constructions to the case of groupoids: however for a groupoid G we obtain not a 'groupoid ring' but what we call the 'adjoint module' $\overrightarrow{\mathbb{Z}}G$, and from this we get the 'augmentation module' $\overrightarrow{I}G$. We use the distinctive notation for the groupoid case, even though if G is a group then the constructions of $\mathbb{Z}G$ and $\overrightarrow{\mathbb{Z}}G$, IG and $\overrightarrow{I}G$, coincide: one reason is that they denote different structures, and another is that there is a second generalisation to a groupoid G of the usual group ring of a group, in which we obtain a 'ring with several objects' $\mathbb{Z}G$ which is an additive category with objects the same as those of G and in which $\mathbb{Z}G(x, y)$ is the free abelian group on $G(x, y)$.

We will often write G_0 for $\mathrm{Ob}\, G$ for a groupoid G.

Definition 7.5.2 Let G be a groupoid. For $q \in G_0$, we define $\overrightarrow{\mathbb{Z}}G(q)$ to be the free Abelian group on the elements of G with target q. Thus an element has uniquely the form of a finite sum $\Sigma_i n_i g_i$ with $n_i \in \mathbb{Z}$ and $g_i \in G$ with $t(g_i) = q$.

Clearly, $\overrightarrow{\mathbb{Z}}G$ becomes a (right) G-module under the action

$$(a, g) \mapsto ag$$

of G on basis elements. Thus

$$\overrightarrow{\mathbb{Z}}G = \{\overrightarrow{\mathbb{Z}}G(q)\}_{q \in G_0}$$

is a G-module, which we call the *adjoint module* of G (over $\mathbb{Z}$), since it involves the adjoint action of G on itself. This construction defines the functor

$$\overrightarrow{\mathbb{Z}}(-) : \mathsf{Gpds} \to \mathsf{Mod}. \qquad \square$$

Notice that $\overrightarrow{\mathbb{Z}}G$ is 'G-free on G_0', i.e. it is freely generated as G-module by G_0 (embedded in $\overrightarrow{\mathbb{Z}}G$ as the set of identities of G).

There is also a generalisation to groupoids of constructions well known in the case of groups: the augmentation map and the augmentation ideal.

For a fixed groupoid G, is also useful to have a trivial G-module, corresponding to the integers $\mathbb{Z}$.

Definition 7.5.3 Let $\overrightarrow{\mathbb{Z}}$ be the (right) G-module consisting of the constant family $\overrightarrow{\mathbb{Z}}(p) = \mathbb{Z}$ for $p \in G_0$, with trivial action of G (which, as usual, means that each $g \in G(p, q)$ act as the identity $\mathbb{Z}(p) \to \mathbb{Z}(q)$). We shall regard the $\mathbb{Z}(p)$ as distinct, so that $\overrightarrow{\mathbb{Z}}$ depends also on the object set G_0.

The *augmentation map*

$$\varepsilon : \overrightarrow{\mathbb{Z}}G \to \overrightarrow{\mathbb{Z}},$$

given by $\Sigma n_i g_i \mapsto \Sigma n_i$ is a morphism of G-modules and its kernel $\overrightarrow{I}G$ is called the (right) *augmentation module* of G.

Any morphism of groupoids $\phi : H \to G$ induces a module morphism $\overrightarrow{\mathbb{Z}}H \to \overrightarrow{\mathbb{Z}}G$ over ϕ which maps $\overrightarrow{I}H$ to $\overrightarrow{I}G$.

Since the augmentation map is natural, the augmentation module defines also a functor

$$\overrightarrow{I} : \mathsf{Gpds} \to \mathsf{Mod}. \qquad \Box$$

Exercise 7.5.4 Prove that for $q \in \mathrm{Ob}(G)$, the abelian group $\overrightarrow{I}G(q)$ has $\mathbb{Z}$-basis consisting of all $g - 1_q$, for g a non-identity element of G with target q. □

We will prove that both $\overrightarrow{\mathbb{Z}}$ and $\overrightarrow{I}$ preserve colimits by giving right adjoints for them. That for $\overrightarrow{I}$ has a direct construction: the semidirect product.

Definition 7.5.5 Given a module (M, G), the *semidirect product* $G \ltimes M$ of G and M is the groupoid with the same set of objects as G, and

$$(G \ltimes M)(p, q) = G(p, q) \ltimes M(q),$$

i.e. as a set is $G(p, q) \times M(q)$ and the composition is given by $(x, m)(y, n) = (xy, m^y + n)$, when $x \in G(p, q), y \in H(q, r)$, and $m \in M(q), n \in M(r)$. This semidirect product construction gives a functor

$$\ltimes : \mathsf{Mod} \to \mathsf{Gpds}. \qquad \Box$$

For the study of this, it is convenient to have a generalised notion of derivation, which will be used a lot later in connection with homotopies of morphisms of crossed complexes.

Definition 7.5.6 Let $\phi : H \to G$ be a morphism of groupoids, and let M be a G-module. A function $f : H \to M$ is called a ϕ-*derivation* if it maps $H(p, q)$ to $M(\phi q)$ and satisfies

$$f(xy) = (fx)^{\phi y} + fy$$

whenever xy is defined in H. In particular, if $H = G$, then a 1_G-derivation is called simply a *derivation*.

Exercise 7.5.7 Let G be a groupoid. Prove that the mapping $\kappa : G \to \overrightarrow{I} G$ sending $g \mapsto g - 1_{tg}$ is a derivation, and has the universal property: if $f : G \to N$ is a derivation to a G-module N, then there is a unique G-morphism $f' : \overrightarrow{I} G \to N$ such that $f'\kappa = f$. $\square$

Proposition 7.5.8 *The functor* $\ltimes : \mathsf{Mod} \to \mathsf{Gpds}$ *is a right adjoint of* $\overrightarrow{I} : \mathsf{Gpds} \to \mathsf{Mod}$. *Hence* $\overrightarrow{I}$ *preserves colimits.*

Proof Let us begin by studying $\mathsf{Gpds}(H, G \ltimes M)$ for a groupoid H and module (M, G). A morphism

$$H \to G \ltimes M$$

is of the form $x \mapsto (\phi x, fx)$ where

$$\phi : H \to G$$

is a morphism of groupoids and

$$f : H \to M$$

is a ϕ-derivation. (In particular, all sections

$$G \to G \ltimes M$$

are of the form $x \mapsto (x, fx)$ where $f : G \to M$ is a derivation.)

By Exercise 7.5.7, the map $\kappa : H \to \overrightarrow{I} H$, given by $\kappa(x) = x - 1_q$ for $x \in H(p, q)$, is a universal derivation.

On the other hand, if $\phi : H \to G$ is a morphism of groupoids and M is a G-module, then any ϕ-derivation $f : H \to M$ is uniquely of the form $f = \hat{f}\kappa$ where $\hat{f} : \overrightarrow{I} H \to M$ is a morphism of modules over ϕ. Thus we have a natural bijection

$$\mathsf{Mod}((\overrightarrow{I} H, H), (M, G)) \cong \mathsf{Gpds}(H, G \ltimes M). \qquad \square$$

The right adjoint to $\overrightarrow{\mathbb{Z}}$ comes from the pullback of a groupoid along a map defined in Example **??** in conjunction with the adjoint module of a groupoid.

Definition 7.5.9 Given a module (M, G), we consider M as a set UM with the target map $t : UM \to \mathrm{Ob}\, G$. We may therefore form the pullback groupoid $P(M, H) = t^*H$. This construction gives a functor

$$P : \mathsf{Mod} \to \mathsf{Gpds}. \qquad \square$$

The groupoid $P(M, G)$, with its canonical morphism to G, $(m, g, n) \mapsto g$, is universal for morphisms $\phi : H \to G$ of groupoids such that $\mathrm{Ob}\, \phi$ factors through maps $\beta : M \to \mathrm{Ob}\, G$.

Proposition 7.5.10 *The functor* $P : \mathsf{Mod} \to \mathsf{Gpds}$ *is a right adjoint of* $\overrightarrow{\mathbb{Z}} : \mathsf{Gpds} \to \mathsf{Mod}$. *Hence* $\overrightarrow{\mathbb{Z}}$ *preserves colimits.*

Proof By the definition of $P(M, G)$, the groupoid morphisms $H \to P(M, G)$ are naturally bijective with pairs (α, ϕ) where $\alpha : \mathrm{Ob}\, H \to UM$ is a map, $\phi : H \to G$ is a morphism and $\mathrm{Ob}\, \phi$ is of the form $\beta \circ \alpha$.

However, since $\overrightarrow{\mathbb{Z}} H$ is freely generated as H-module by H_0 (embedded in $\overrightarrow{\mathbb{Z}} H$ as the set of identities of H), such pairs (α, ϕ) are naturally bijective with morphisms of modules $(\gamma, \phi) : (\overrightarrow{\mathbb{Z}} H, H) \to (M, G)$. $\square$

These constructions are related as follows:

Proposition 7.5.11 *The inclusion, $\overrightarrow{I}G \to \overrightarrow{\mathbb{Z}}G$, regarded as a natural transformation, is conjugate under the above adjunction to the natural transformation $\theta = \theta_{(M,G)}$ where*

$$\theta_{(M,G)} : P(M,G) \to G \ltimes M$$

is given by $\theta(m,g,n) = (g, m^g - n)$. For each module (M,G), this $\theta_{(M,G)}$ is a covering morphism of groupoids.

Proof Any commutative triangle

$$\begin{tikzcd}
(\overrightarrow{I}H, H) \arrow[rr, "i"] \arrow[dr, "(\alpha,\phi)"'] & & (\overrightarrow{\mathbb{Z}}H, H) \arrow[dl, "(\gamma,\phi)"] \\
& (M,G) &
\end{tikzcd}$$

in Mod corresponds to a commutative triangle

$$\begin{tikzcd}
G \ltimes M & & P(M,G) \arrow[ll, "\theta"'] \\
& H \arrow[ul, "\xi"] \arrow[ur, "\eta"'] &
\end{tikzcd}$$

in Gpds, where θ is natural and, if $h \in H(p,q)$, then

$$\xi h = (\phi h, \alpha(h - 1_q)) \quad \text{and} \quad \eta h = (\gamma 1_p, \phi h, \gamma 1_q).$$

Given $(m,g,n) \in P(M,G)$, we may take $G = H$, $\phi = id$, and choose γ so that $\gamma 1_p = m$, $\gamma 1_q = n$. Then

$$\begin{aligned}
\theta(m,g,n) &= \xi g \\
&= (\phi g, \alpha(g - 1_q)) \\
&= (g, \gamma(g - 1_q)) \\
&= (g, \gamma(1_p h) - \gamma 1_q) \\
&= (g, m^g - n).
\end{aligned}$$

Finally, let $(g,x) \in G \ltimes M$, with $g \in G(p,q)$ and $x \in M(q)$, and let $m \in M(p)$ be an object of $P(M,G)$ lying over the source p of (g,x). Then there is a unique $n \in M(q)$ such that $m^g - n = x$. Hence there is a unique arrow (m,g,n) over (g,x) with source n. □

Note that if one restricts attention to groups, and modules over groups, the restricted functor $\overrightarrow{\mathbb{Z}}(-)$ does not have a right adjoint since, for example, it converts the initial object 1 in the category of groups to the module $(\overrightarrow{\mathbb{Z}}, 1)$ which is not initial in the category of modules over groups. However, the functor $\overrightarrow{I}$, does, when restricted to groups, have a right adjoint given by the split extension as above.

7.5.2 The derived module

Another basic construction used to linearise the theory of groups in homological algebra is the derived module D_ϕ of a group morphism $\phi : H \to G$, usually appearing in the form $D_\phi = IH \otimes_H \mathbb{Z}G$. We extend this construction to the case of groupoids.

Definition 7.5.12 Let $\phi : H \to G$ be a morphism of groupoids. Its *derived module* is a G-module D_ϕ with a universal ϕ-derivation $h_\phi : H \to D_\phi$: that is, for any ϕ-derivation $f : H \to M$ to a G-module M, there is a unique G-morphism $f' : D_\phi \to M$ such that $f' h_\phi = f$. □

Proposition 7.5.13 *Let $\phi : H \to G$ be a morphism of groupoids. If H is a free groupoid on X, then D_ϕ is a free G-module on $h_\phi(X)$.*

Proof Let $Y = h_\phi(X)$. Let $f : Y \to M$ be graph morphism to a G-module M. Let $h' : X \to M$ be determined by h_ϕ and f. Since H is free on X, this graph morphism extends uniquely to a ϕ-derivation $f' : H \to M$. (We see this since a ϕ-derivation $H \to M$ is equivalent to a groupoid section of the projection $H \ltimes M \to H$.) This ϕ-derivation determines uniquely a G-morphism $f'' : D_\phi \to M$ extending f as required. □

Exercise 7.5.14 Give a direct construction of the derived module as follows: for $q \in \mathrm{Ob}\, G$, let $F(q)$ be the free G-module on the family of sets of elements x of H such that $\phi(x)$ has target q. Then $F(q)$ has an additive basis of pairs (x, g) such that $\phi(x)g$ is defined in G, and the action of G is given by

$$(x, g)^{g'} = (x, gg')$$

when gg' is defined in G. There is a natural map

$$i : H \to F,$$

given by $i(x) = (x, 1_q)$, where $\phi(x)$ has target q. Now we impose on F the relations

$$i(xy) = i(x)^{\phi(y)} + i(y)$$

whenever xy is defined in H. This gives a quotient G-module D_ϕ, a quotient morphism $s : F \to D_\phi$ and a ϕ-derivation $h_\phi = si : H \to D_\phi$. □

For any category C we define the category C^2 to have objects the arrows of C and morphisms $(f, g) : a \to b$ to be the commutative squares in C

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "a"'] & \cdot \arrow[d, "b"] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

with composition the obvious horizontal one.

The universal property of the derived module construction shows that it gives a functor

$$D : \mathsf{Gpds}^2 \to \mathsf{Mod}$$

given by $D(H \xrightarrow{\phi} G) = (D_\phi, G)$.

Remark 7.5.15 Alternatively, regarding the category of G-modules as the functor category $(\mathsf{Ab})^G$, any functor $M : H \to \mathsf{Ab}$ has a left Kan extension $\phi_* M : G \to \mathsf{Ab}$ along $\phi : H \to G$. Then the derived module D_ϕ is canonically isomorphic to $\phi_*(\overrightarrow{I} H)$, the G-module *induced* from $\overrightarrow{I} H$ by $\phi : H \to G$. In the case of a group morphism ϕ, this induced module is just $IH \otimes_H \mathbb{Z}G$, where $\mathbb{Z}G$ is viewed as a left H-module via ϕ and left multiplication. □

Now we obtain a right adjoint

Proposition 7.5.16 *The functor* D *has a right adjoint* $\mathsf{Mod} \to \mathsf{Gpds}^2$ *given by*

$$(M, G) \mapsto (G \ltimes M \xrightarrow{\mathrm{pr}_1} G).$$

Proof This is an immediate consequence of the adjointness of $\overrightarrow{I}$ and $\ltimes$ seen in Proposition 7.5.8 and the formula $D_\phi = \phi_*(\overrightarrow{I}H)$. □

Exercise 7.5.17 Verify that:

1. The augmentation module $\overrightarrow{I}G$ is the derived module of the identity morphism $G \to G$.
2. If G is a totally disconnected groupoid on the set X, and $\phi : G \to X$ is the unique morphism over X to the discrete groupoid on X, then the derived module of ϕ is the abelianisation G^{ab} of G. We suggest more on abelianisation of a groupoid in Exercise 7.5.26.
3. Discuss the derived module of a composition of morphisms $G \to H \to K$. □

7.5.3 The derived chain complex of a crossed complex

Now we can construct our functor

$$\nabla : \mathsf{Crs} \to \mathsf{Chn}.$$

Theorem 7.5.18 *Let* C *be a crossed complex, and let* $\phi : C_1 \to G$ *be a cokernel of* δ_2 *of* C*. Then there are* G*-morphisms*

$$C_2^{ab} \xrightarrow{\partial_2} D_\phi \xrightarrow{\partial_1} \overrightarrow{\mathbb{Z}}G$$

such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "\delta_n"] \arrow[d, "="'] & C_{n-1} \arrow[r] \arrow[d, "="'] & \cdots \arrow[r] & C_3 \arrow[r, "\delta_3"] \arrow[d, "="'] & C_2 \arrow[r, "\delta_2"] \arrow[d, "\alpha_2"] & C_1 \arrow[r, "\phi"] \arrow[d, "\alpha_1"] & G \arrow[d, "\alpha_0"] \\
\cdots \arrow[r] & C_n \arrow[r, "\partial_n"'] & C_{n-1} \arrow[r] & \cdots \arrow[r] & C_3 \arrow[r, "\partial_3"'] & C_2^{ab} \arrow[r, "\partial_2"'] & D_\phi \arrow[r, "\partial_1"'] & \overrightarrow{\mathbb{Z}}G
\end{tikzcd}$$

commutes and the lower line is a chain complex over G*. Here* α_1 *is the universal* ϕ*-derivation,* α_0 *is the* G*-derivation* $x \mapsto x - 1_q$ *for* $x \in G(p, q)$*, as a composition* $G \to \overrightarrow{I}G \to \overrightarrow{\mathbb{Z}}G$*, and* $\partial_n = \delta_n$ *for* $n \geqslant 4$.

Proof Let $X = G_0$, and let X also denote the discrete groupoid on X. The functor $\mathsf{D} : \mathsf{Gpds}^2 \to \mathsf{Mod}$, applied to the sequence of morphisms

$$\begin{tikzcd}
\cdots \arrow[r] & C_3 \arrow[r, "\delta_3"] \arrow[d, "\zeta_3"] & C_2 \arrow[r, "\delta_2"] \arrow[d, "\zeta_2"] & C_1 \arrow[r, "\phi"] \arrow[d, "\phi"] & G \arrow[d, "="] \\
\cdots \arrow[r] & X \arrow[r] & X \arrow[r] & G \arrow[r] & G
\end{tikzcd}$$

gives a sequence of module morphisms

$$\ldots \to (D_{\zeta_3}, X) \to (D_{\zeta_2}, X) \to (D_\phi, G) \to (\overrightarrow{I}G, G).$$

Since a derivation $C_n \to M$ over a null map $\zeta_n : C_n \to X$ is just a morphism to an Abelian groupoid, we may identify D_{ζ_n} with C_n^{ab} and its universal derivation with the Abelianisation map. The map ∂_1 is the composition $D_\phi \to \overrightarrow{I}G \to \overrightarrow{\mathbb{Z}}G$. Thus we obtain the stated commutative diagram in which the vertical maps are the corresponding universal derivations (followed by an inclusion, in the case of α_0).

This establishes all the stated properties except the G-invariance of ∂_2 and the relations $\partial_2\partial_3 = 0$, $\partial_1\partial_2 = 0$.

Clearly $\partial_2\partial_3 = \alpha_1\delta_2\delta_3 = 0$.

Also $\partial_1\partial_2\alpha_2 = \alpha_0\phi\delta_2 = 0$ and since α_2 is surjective, this implies $\partial_1\partial_2 = 0$.

Finally, if $x \in C_2^{ab}$, $g \in G$ and x^g is defined, choose $a \in C_2$, $b \in C_1$ such that $\alpha_2 a = x$, $\phi b = g$. Then

$$\begin{aligned}
\partial_2(x^g) &= \alpha_1\delta_2(a^b) & \\
&= \alpha_1(b^{-1}cb), & &\text{where } c = \delta_2 a,\\
&= [(\alpha_1(b^{-1}))^{\phi c} + \alpha_1 c]^{\phi b} + \alpha_1 b, & &\text{since } \alpha_1 \text{ is a } \phi\text{-derivation},\\
&= (\alpha_1(b^{-1}))^{\phi b} + [\alpha_1 c]^{\phi b} + \alpha_1 b, & &\text{since } \phi c = 1,\\
&= -\alpha_1 b + (\alpha_1 c)^{\phi b} + \alpha_1 b & &\text{since } \alpha_1 \text{ is a } \phi\text{-derivation},\\
&= (\alpha_1 c)^{\phi b} & &\text{since } D_\phi \text{ is Abelian},\\
&= (\partial_2 x)^g, & &\text{as required.}
\end{aligned}$$

□

Remark 7.5.19 Suppose $\delta_2 : C_2 \to C_1$ is a crossed module such that C_2 is the free crossed module on R and C_1 is the free groupoid on X. Let $\phi : C_1 \to G$ be the cokernel of δ_2. Then the corresponding G-module morphism $\partial_2 : C_2^{ab} \to D_\phi$ may by the above results be interpreted as the Fox derivative $(\partial r/\partial x)$, [CF77].

Definition 7.5.20 For any crossed complex C, ∇C is the chain complex given in the bottom row of the main diagram of Theorem 7.5.18. This gives a *derived functor*

$$\nabla : \mathsf{Crs} \to \mathsf{Chn}.$$

□

7.5.4 Exactness and lifting properties of ∇

Proposition 7.5.21 *Let $C = \{C_r\}$ be a crossed complex and suppose that the sequence of groupoids*

$$C_3 \xrightarrow{\delta} C_2 \xrightarrow{\delta} C_1 \xrightarrow{\phi} G \to 1$$

is exact. Then the sequence of G-modules in $\nabla' C$:

$$C_3 \xrightarrow{\partial} C_2^{Ab} \xrightarrow{\partial} D_\phi \xrightarrow{\partial'} \vec{I}G \to 0$$

is exact.

Proof The exactness of $C_2 \to C_1 \xrightarrow{\phi} G \to 1$ implies that

$$\begin{tikzcd}
(C_2 \to 1) \arrow[r] \arrow[d] & (C_1 \xrightarrow{\phi} G) \arrow[d] \\
(1 \to 1) \arrow[r] & (G \xrightarrow{=} G)
\end{tikzcd}$$

is a pushout square in the arrow category Gpds^2. Applying $D : \mathsf{Gpds}^2 \to \mathsf{Mod}$, as in the proof of Theorem 7.5.18, and noting that D preserves colimits by Proposition 7.5.16, we obtain a pushout square

$$\begin{tikzcd} C_2^{Ab} \arrow[r, "\partial"] \arrow[d] & D_\phi \arrow[d] \\ 0 \arrow[r] & \overrightarrow{I}G \end{tikzcd} \quad \text{over} \quad \begin{tikzcd} 1 \arrow[r] \arrow[d] & G \arrow[d] \\ 1 \arrow[r] & G \end{tikzcd}$$

in Mod. Since $\partial : C_2^{Ab} \to D_\phi$ is in fact a G-morphism, it follows that

$$C_2^{Ab} \to D_\phi \to \overrightarrow{I}G \to 0$$

is an exact sequence of G-modules.

To prove exactness of $C_3 \to C_2^{Ab} \to D_\phi$, write $N = \operatorname{Ker}\phi = \delta C_2$ and note that the exactness of

$$C_3 \to C_2 \to N \to 1$$

implies the exactness of

$$C_3 \to C_2^{Ab} \to N^{Ab} \to 1.$$

It remains, therefore, to show that the map $\gamma : N^{Ab} \to D_\phi$ induced by $\partial : C_2^{Ab} \to D_\phi$ is injective.

Now $\phi : C_1 \to G$ is a quotient morphism of groupoids with totally intransitive kernel N. In these circumstances the additive groupoid structure of D_ϕ is given by generators $[c] \in D_\phi(q)$ for $c \in C_1(p, q)$, with defining relations

$$[cy] = [c] + [y] \text{ for } c \in C_1(p, q),\ y \in N(q);$$

the groupoid C_1 acts on this additive groupoid by

$$[c]^x = [cx] - [x]$$

and N acts trivially, making D_ϕ a G-module; the canonical ϕ-derivation $\alpha_1 : C_1 \to D_\phi$ is given by $\alpha_1(c) = [c]$.

Choose coset representatives $t(c) \in cN$ of N in C_1 with $t(1_q) = 1_q$. Then for all $c \in C_1$, $c = t(c)s(c)$ where $s(c) \in N$. The map $s : C_1 \to N$ satisfies $s(y) = y$ for $y \in N$ and

$$s(cy) = s(c)y \ \text{ for all } \ c \in C_1(p, q),\ \ y \in N(q).$$

Consequently, there is an additive map $s^* : D_\phi \to N^{Ab}$ defined by $s^*[c] = \alpha s(c)$, where α is the canonical map $N \to N^{Ab}$. Since, for any $u = \alpha y$ in N^{Ab},

$$s^*\gamma u = s^*\gamma\alpha y = s^*\alpha_1 y = \alpha s(y) = \alpha y = u,$$

γ is injective, as required. □

Corollary 7.5.22 *If $\delta : C_2 \to C_1$ is a crossed module with kernel K, and $\phi : C_1 \to G$ is the cokernel of δ, then the sequence $K \to C_2^{Ab} \to D_\phi$ is exact.*

Proof Put $C_3 = K$ in Proposition 7.5.21. □

Definition 7.5.23 The crossed complex C (or crossed module) is *regular* if $K \cap [C_2, C_2] = 0$, where K is the kernel of $\delta : C_2 \to C_1$. □

Corollary 7.5.24 *If $C_2 \to C_1$ is a regular crossed module with kernel K, then the sequence $0 \to K \to C_2^{Ab} \to D_\phi$ is exact.*

Proof This follows from Corollary 7.5.22 and the definition of regular. □

The following is a useful result for applications to free crossed resolutions and to identities among relations.

Proposition 7.5.25 *If in the crossed complex C, the groupoid C_1 is free, then C is regular. In particular, the fundamental crossed complex ΠX_* of a CW-complex X_* is regular.*

Proof Since $N = \delta C_2$ is a subgroupoid of C_1, it is a free groupoid (in fact a family of free groups). Hence the map $\delta : C_2 \to N$ has a homomorphic section s. But the kernel K of δ is in the centre of C_2, since C_2 is a crossed module over C_1. Hence $C_2 = K \times_{C_0} s(N)$ is a groupoid, that is, for each $p \in C_0$, $C_2(p) = K(p) \times sN(p)$. This implies that $[C_2, C_2] = [sN, sN]$ and hence that $K \cap [C_2, C_2] = 0$. □

In the following exercises, we sketch in a special case another description of the derived module which is useful later in section 8.4. We need the notion of abelianisation of a groupoid.

Exercise 7.5.26 If G is a groupoid its *abelianisation* is a morphism $\upsilon : G \to G^{ab}$ which is universal for morphisms to abelian groups. Show abelianisation is defined and gives a left adjoint to the inclusion of the category of abelian groups into the category Gpds of groupoids. Refer to the notion of universal group UG of a groupoid G in [Bro06, Section 8.1], and prove that $G^{ab} \cong (UG)^{ab}$. Calculate G^{ab} in terms of the transitive components of G, and show that if G is a tree groupoid in the sense of [Bro06] then G^{ab} is a free abelian group. Hence calculate G^{ab} for any transitive groupoid in terms of a vertex group and a tree. □

Exercise 7.5.27 Let $\phi : F \to G$ of an epimorphism of groups. Form the universal covering groupoid $p : \widetilde{G} \to G$, [Bro06, Chapter 10], and let $q : \widehat{F} \to F$ be the pullback of p by ϕ. Then q is also a covering morphism of groupoids. There is a function $\upsilon : F \to \widehat{F}$ which sends $a \in F$ to the unique covering element of a which ends at the object $1 \in F$. Prove that $\widehat{F}^{ab}$ admits the structure of G-module and that the composite $F \xrightarrow{\upsilon} \widehat{F} \to \widehat{F}^{ab}$ is a ϕ-derivation Prove that the morphism of G-modules $D_\phi \to \widehat{F}^{ab}$ given by the universal property of $F \to D_\phi$ is an isomorphism by using the 5-lemma on a map from the exact sequence of Proposition 7.5.21 to one derived from an analysis of $\widehat{F}^{ab}$ using the previous exercise. □

7.5.5 The right adjoint of the derived functor

The main task of this subsection is to construct a functor $\Theta : \mathsf{Chn} \to \mathsf{Crs}$ and prove it is right adjoint to ∇. This shows that some information on a crossed complex C can be recovered from the chain complex ∇C, and also has the important consequence that ∇ preserves colimits. We will use ∇ in Chapter 9 to give a convenient description in dimensions > 2 of the tensor product of crossed complexes.

In order to construct Θ we use an intermediate functor Θ'.

Definition 7.5.28 For a chain complex L over a groupoid H, $\Theta' L = \Theta'(L, H)$ is the crossed complex

$$\Theta' L \quad := \quad \cdots \longrightarrow L_n \xrightarrow{\partial} L_{n-1} \longrightarrow \cdots \longrightarrow L_3 \xrightarrow{\partial} L_2 \xrightarrow{(0,\partial)} H \ltimes L_1 \; .$$

Here $H \ltimes L_1$ acts on L_n $(n \geqslant 2)$ via the projection $H \ltimes L_1 \to H$, so that L_1 acts trivially. □

Note that $\Theta' L$ does not involve L_0. To correct this, we use another construction which brings in L_0 in an essential way. Let us begin by defining ΘL and checking that the definition works.

Definition 7.5.29 For any chain complex L, we consider the canonical covering morphism

$$\theta : P(UL_0, H) \to H \ltimes L_0$$

of Proposition 7.5.11. We define

$$\Theta(L) = \theta^* \Theta' L,$$

the pull-back along θ of the crossed complex of Definition 7.5.28.

We obtain a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & E_3 \arrow[r] \arrow[d, "\sigma_3"] & E_2 \arrow[r] \arrow[d, "\sigma_2"] & E_1 \arrow[r] \arrow[d, "\sigma_1"] & P(L_0, H) \arrow[d, "\theta"] \\
\cdots \arrow[r] & L_3 \arrow[r] & L_2 \arrow[r, "{(0,\partial)}"'] & H \ltimes L_1 \arrow[r, "{(1,\partial)}"'] & H \ltimes L_0
\end{tikzcd}$$

in which each E_n is a groupoid over $E_0 = L_0$, and each σ_n is a covering morphism.

For $n \geqslant 2$, the composite map $L_n \to H \ltimes L_0$ is 0 and, since $\operatorname{Ker}\theta$ is discrete, it follows that E_n is just a family of groups each isomorphic to a group of L_n. There is also an action of E_1 on E_n $(n \geqslant 2)$ induced by the action of $H \ltimes L_1$ on L_n; for if $e_1 \in E_1(x, y)$, where $x \in L_0(p)$, $y \in L_0(q)$, and if $e_n \in E_n(x)$, then $\sigma_1 e_1$ acts on $\sigma_n e_n$ to give an element of $L_n(q)$ which lifts uniquely to an element of $E_n(y)$.

It is now easy to see that $E = \{E_n\}_{n \geqslant 0}$ is a crossed complex and that the σ_i form a morphism $\sigma : E \to \Theta' L$ of crossed complexes.

This gives a functor

$$\Theta : \mathsf{Chn} \to \mathsf{Crs}.$$ □

An explicit description of $E = \Theta(L, H)$ can be extracted from the constructions given above. The set of objects of every E_n is L_0.

An arrow of E_1 from x to y, where $x \in L_0(p)$, $y \in L_0(q)$, $p, q \in H_0$, is a triple (h, a, y), where $h \in H(p, q)$, $a \in L_1(q)$, and $x^h = y + \partial a$. Composition in E_1 is given by

$$(h, a, y)(k, b, z) = (hk, a^k + b, z)$$

whenever hk is defined in H and $y^k = z + \partial b$.

For $n \geqslant 2$, E_n is a family of groups; the group at the object $y \in L_0(q)$ has arrows (a, y) where $a \in L_n(q)$, with composition

$$(a, y) + (b, y) = (a + b, y).$$

The boundary map $\delta : E_2 \to E_1$ is given by

$$\delta(a, y) = (1_q, \partial a, y) \text{ for } a \in L_2(q),\ y \in L_0(q).$$

The boundary map $\delta : E_n \to E_{n-1}$ $(n \geqslant 3)$ is given by $\delta(a, y) = (\partial a, y)$ and the action of E_1 on E_n $(n \geqslant 2)$ is given by

$$(a, y)^{(k,b,z)} = (a^k, z),$$

where $k \in H(q, r)$, $a \in L_n(q)$, $y \in L_0(q)$ and $y^k = z + \partial b$.

Proposition 7.5.30 *The functor Θ is a right adjoint of ∇. Hence ∇ preserves colimits.*

Proof A morphism $(\beta, \psi) : (\nabla C, G) \to (L, H)$ in Chn is equivalent to a commutative diagram in Mod:

$$
\begin{tikzcd}
\cdots \arrow[r] & C_3 \arrow[r] \arrow[d, "\beta_3"] & C_2^{ab} \arrow[r] \arrow[d, "\beta_2"] & D_\phi \arrow[r] \arrow[d, "\beta_1"] & \overrightarrow{I}\, G \arrow[r, "i"] \arrow[d, "\beta_0'"] & \overrightarrow{\mathbb{Z}}\, G \arrow[dl, "\beta_0"] \\
\cdots \arrow[r] & L_3 \arrow[r, "\partial"'] & L_2 \arrow[r, "\partial"'] & L_1 \arrow[r, "\partial"'] & L_0 &
\end{tikzcd}
$$

(over some morphism $\psi : G \to H$) and hence, by Propositions 7.5.8, 7.5.16, to a commutative diagram in Gpds:

$$
\begin{tikzcd}
\cdots \arrow[r] & C_3 \arrow[r] \arrow[d, "\beta_3"] & C_2 \arrow[r] \arrow[d, "\overline{\beta_2}"] & C_1 \arrow[r, "\phi"] \arrow[d, "\gamma_1"] & G \arrow[d, "\xi"'] \arrow[dr] & \\
\cdots \arrow[r] & L_3 \arrow[r, "\partial"'] & L_2 \arrow[r, "{(0, \partial)}"'] & H \ltimes L_1 \arrow[r, "{(1, \partial)}"'] & H \ltimes L_0 & P(L_0, H) \arrow[l, "\theta"]
\end{tikzcd}
$$

where $(\ldots, \beta_3, \overline{\beta_2}, \gamma_1)$ is a morphism of crossed complexes, and θ is the canonical covering morphism. This in turn is equivalent to a commutative diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & C_3 \arrow[r] \arrow[d, "\beta_3"] & C_2 \arrow[r, "\delta"] \arrow[d, "\overline{\beta_2}"] & C_1 \arrow[r, "\omega"] \arrow[d, "\gamma_1"] & P(L_0, H) \arrow[d, "\theta"] \\
\cdots \arrow[r] & L_3 \arrow[r, "\partial"'] & L_2 \arrow[r] & H \ltimes L_1 \arrow[r] & H \ltimes L_0
\end{tikzcd}
$$

because, in any such diagram, $\theta\omega\delta = 0$ and θ is a covering morphism, so $\omega\delta = 0$, that is, ω factorises through $\phi : C_1 \to G$.

This diagram is therefore equivalent to a morphism of crossed complexes $C \to E$. Hence (β, ψ) is therefore equivalent to a morphism of crossed complexes $C \to E$. This shows that the functor $\Theta :$ Chn $\to$ Crs is right adjoint to ∇. □

7.5.6 Some colimits in chain complexes.

The fact that $\nabla :$ Crs $\to$ Chn preserves all colimits implies that the Higher Homotopy van Kampen theorem proved in section 8.2 for the fundamental crossed complex ΠX_* of a filtered space X_* can be converted into a similar theorem for the chain complex $CX_* = \nabla \Pi X_*$. The interpretation of this result will be discussed in Section 8.4.

The following simple example illustrates some of the interesting features that arise in computing colimits in Crs and Chn. Note that if all the crossed complexes in a diagram $\{C^\lambda\}$ are reduced then the colimit of $\{C^\lambda\}$ is reduced provided that the diagram is connected, in which case the colimit of $\{\nabla C^\lambda\}$ can be computed in the category of chain complexes over groups instead of groupoids.

Thus we consider a simple connected diagram of reduced crossed modules. Note that in the reduced case, we can abbreviate $\overrightarrow{I}, \overrightarrow{\mathbb{Z}}$ to $I, \mathbb{Z}$.

Example 7.5.31 Let $M \to P$, $N \to P$ be crossed modules over a group P. Their coproduct in the category of crossed modules over P is given by the pushout in Crs:

$$\begin{tikzcd}
 & (\cdots 0 \to M \to P) \arrow[dr] & \\
(\cdots 0 \to 0 \to P) \arrow[ur] \arrow[dr] & & (\cdots 0 \to M \bowtie N \to P) \\
 & (\cdots 0 \to N \to P) \arrow[ur] &
\end{tikzcd}$$

where the group $M \bowtie N$ is the Peiffer product described by Brown in [Bro84]. □

Example 7.5.32 To find the corresponding chain complexes let $G = P/\delta M$, $H = P/\delta N$ and write ϕ, ψ for the quotient maps $P \to G$, $P \to H$. Then the corresponding derived modules are $D_\phi = IP \otimes_P \mathbb{Z}G$ and $D_\psi = IP \otimes_P \mathbb{Z}H$.

Now we wish to compute the pushout in Chn (or in chain complexes over groups) of

$$\begin{tikzcd}
 & (\cdots 0 \to M^{ab} \to IP \otimes_P \mathbb{Z}G \to \mathbb{Z}G, G) \\
(\cdots 0 \to 0 \to IP \to \mathbb{Z}P, P) \arrow[ur] \arrow[dr] & \\
 & (\cdots 0 \to N^{ab} \to IP \otimes_P \mathbb{Z}H \to \mathbb{Z}H, H)
\end{tikzcd}$$

To do this, we first form the pushout K of

$$\begin{tikzcd}
 & G \\
P \arrow[ur] \arrow[dr] & \\
 & H
\end{tikzcd}$$

namely $K = P/(\delta M \cdot \delta N)$; this is the group acting on the pushout chain complex. Next we form the induced modules over K of each module in the diagram and then form pushouts of K-modules in each dimension. This gives the chain complex

$$(\cdots 0 \to (M^{ab} \otimes_P \mathbb{Z}K) \oplus (N^{ab} \otimes_P \mathbb{Z}K) \to IP \otimes_P \mathbb{Z}K \to \mathbb{Z}K, K).$$

Since $K = P/\delta M \delta N$, and δM acts trivially on M^{ab}, we have

$$M^{ab} \otimes_P \mathbb{Z}K = M^{ab}/[M^{ab}, N];$$

similarly $N^{ab} \otimes_P \mathbb{Z}K = N^{ab}/[N^{ab}, M]$. Thus the pushout in dimension 2 is

$$M^{ab}/[M^{ab}, N] \oplus N^{ab}/[N^{ab}, M],$$

which is easily identifiable as $(M \bowtie N)^{ab}$, confirming that ∇ preserves this pushout. □

7.6 Notes

Blakers in [Bla48] defined what he calls a 'group system' associated to a (reduced) filtered space, and which we now call a reduced crossed complex. Thus he gives the definition of ΠX_* in that case.

The techniques are used to relate homology and homotopy. Blakers attributes to S. Eilenberg the suggestion of considering the whole structure.

The idea was also used by J.H.C. Whitehead in his paper [Whi49b], in the case of the skeletal filtration of CW-complexes, and there called the 'homotopy system' of the CW-complex. This paper contains some profound theorems, and was an inspiration for the work of Brown and Higgins. The results of our Section 7.5 (which come largely from [BH90]) are intended to give a more general setting and more detailed analysis of Whitehead's results in this paper. The section also brings together results from [Cro61, Cro71], and for the construction of ∇ from [GR80]. Proposition 7.5.25 for the reduced case is due to Whitehead [Whi49b].

It is interesting that the construction of ∂_2 in Theorem 7.5.18 was given by Whitehead (in [Whi49b] in the group case) well before the publication of work of Fox on his free differential calculus [Fox53], and the relation between the two works seems not to have been generally noticed.

Our Proposition 7.5.21 gives an extension of the exact module sequence of Crowell [Cro61, Cro71]; see also [ML63, p.120].

Chapter 8

The Higher Homotopy van Kampen Theorem (HHvKT) and its applications

Introduction

Now we turn to the first of our series of homotopical applications of crossed complexes and the functor

$$\Pi : \mathsf{FTop} \to \mathsf{Crs},$$

namely the consequences of a Higher Homotopy van Kampen Theorem (HHvKT).

The statement and many of the applications of the theorem are entirely analogous to those of the theorem in dimension 2 given in Part I. The method of proof is also analogous to that in Part I, but is much more complicated algebraically and topologically. So the proof is deferred to Part III.

There are some interesting contrasts between the results of this Part and those in Part I. The applications in Part I involved crossed modules, a nonabelian structure. Hence those results are largely unobtainable by traditional methods of algebraic topology.

The applications of the HHvKT in dimensions > 2 involve modules, rather than crossed modules, over the fundamental group or groupoid, and so are much nearer to traditional results of algebraic topology. Thus, even though the coproduct Theorem 8.3.5, and the homotopical excision Theorem 8.3.7, do not appear in traditional texts, or papers, they are possibly reachable by methods of singular homology and covering spaces, using the latter to bring in the operations of the fundamental group. Handling many base points is less traditional.

However our aim is to show how such results follow in a uniform way by a study of the homotopically defined functor Π. Thus the Relative Hurewicz Theorem, a key result in this borderline between homology and homotopy theory, is seen in a broader context which includes nonabelian results in dimensions 1 and 2. This has been useful to envisage and prove generalisations which are nonabelian in all dimensions. We discuss this further in a final chapter on Further Prospects.

The results of this Chapter on the functor Π are crucial for later applications, such as the notion of classifying space BC of a crossed complex C and the application of this to the homotopy classification of maps of topological spaces, where the fundamental group or groupoid is involved.

In evaluating these results, and comparing with traditional expositions, it should be borne in mind that we use subdivision methods, but only cubically, as these may be modeled algebraically in higher homotopy groupoids. So simplicial approximation is not used, except where we need results from the theory of simplicial sets. Also we do not use homology theory, except to relate our results to traditional ones.

It is hoped that this re-vision of basic algebraic topology will suggest wider applications, since notions of homotopy and deformations are crucial in many areas.

Crossed complexes give in a sense a linear algebraic model of homotopy theory. This limits their rôle for many problems. On the other hand, as in many areas of mathematics, a linear approximation can be useful! More general applications also follow once the tensor product of crossed complexes has been set up and applied in later chapters.

Thus it is hoped that this use of crossed complexes to give a coherent account of this area of algebraic topology will stimulate further developments.

8.1 HHvKT for crossed complexes

The HHvKT gives a mode of calculation of the fundamental crossed complex functor

$$\Pi : \mathsf{FTop} \to \mathsf{Crs}$$

from filtered topological spaces to crossed complexes. This functor is defined *homotopically*, i.e. in terms of certain homotopy classes of certain maps, and not in terms of any other combinatorial model of the filtered space. So it is remarkable that we can calculate in this way, starting with simple information on the trivial values of the functor on simple filtrations of contractible spaces.

An easy consequence of the definition of Π is that it preserves coproducts, which are in these two categories just disjoint union. This is one of the advantages of the groupoid approach. Much more subtle is the application to 'gluing' spaces, and we approach this concept, as in Part I, Chapter 6, through the notion of coequaliser.

As we have seen in Chapter 1, the version of the classical van Kampen theorem for the fundamental groupoid rather than group gives useful results for non connected spaces, but still requires a 'representativity' condition in dimension 0. The corresponding theorem for crossed modules, which computes certain second relative homotopy groups, as discussed in Chapter 6, also needs a "1-connected" condition. It is thus not surprising that our general theorem requires a connectivity condition in all dimensions.

Proposition 8.1.1 *For a filtered space X_* the following conditions (i),(ii) and (iii) are equivalent:*

(i) $(\phi)_0$*: The function $\pi_0 X_0 \to \pi_0 X_r$ induced by inclusion is surjective for all $r \geqslant 0$; and, for all $i \geqslant 1$,*

$$(\phi_i) : \pi_i(X_r, X_i, \nu) = 0 \textit{ for all } r > i \textit{ and } \nu \in X_0.$$

(ii) (ϕ'_0)*: The function $\pi_0 X_s \to \pi_0 X_r$ induced by inclusion is surjective for all $0 = s < r$ and bijective for all $1 \leqslant s \leqslant r$; and, for all $i \geqslant 1$,*

$$(\phi'_i) : \pi_j(X_r, X_i, \nu) = 0 \textit{ for all } \nu \in X_0 \textit{ and all } j, r \textit{ such that } 1 \leqslant j \leqslant i < r.$$

(iii) (ϕ'_0) *and, for all $i \geqslant 1$,*

$$(\phi''_i) : \pi_j(X_{i+1}, X_i, \nu) = 0 \textit{ for all } j \leqslant i, \textit{ and } \nu \in X_0.$$

The proof is a straightforward argument on the exact homotopy sequences of various pairs and triples and is omitted.

Definition 8.1.2 We call a filtered space satisfying any of the conditions (i), (ii), (iii) of the previous proposition *connected*. □

Remark 8.1.3 This condition is satisfied in many important cases. The HHvKT will allow us to construct some new connected filtered spaces as colimits of old ones. In particular, we will prove that the skeletal filtration of a CW-complex X is a connected filtration.

Note also that the condition $\pi_1(X_r, X_1, x) = 0$ means that any path in X_r joining x to a point in X_1 is homotopic in X_r rel end points to a path in X_1. This condition is equivalent to $\pi_1(X_1, x) \to \pi_1(X_r, x)$ is surjective. □

Example 8.1.4 Clearly, a disjoint union of connected filtered spaces is connected.

Now we have set the background to state the HHvKT in the most general form we are going to use. Its algebraic content is that under some connectedness conditions, the fundamental crossed complex functor Π preserves certain colimits. Since Π preserves the coproducts in $\mathsf{FTop}, \mathsf{Crs}$, and colimits can be constructed from coproducts and coequalisers, the meat of the theorem is in the statement on preservation of certain coequalisers.

In order to give background to the statement of the HHvKT, we recall that if the space X is the union of two open sets U, V then we have a pushout diagram of spaces:

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & X \end{tikzcd}$$

If X is the union of three open sets U, V, W then we have a diagram

$$\begin{tikzcd} & & & V \cap W \arrow[dl] \arrow[dd] \\ U \cap V \arrow[rr] \arrow[dd] & & V \arrow[dd] & \\ & U \cap W \arrow[rr] \arrow[dl] & & W \arrow[dl] \\ U \arrow[rr] & & X & \end{tikzcd}$$

and a map $f : X \to Y$ is entirely determined by maps f_U, f_V, f_W defined on U, V, W, with values in Y, and which agree on the two fold intersections $V \cap W, W \cap U, U \cap V$.

The most general situation of this type is expressed by the notion of coequaliser, which we have used already in Chapter 6 of Part I. Suppose given a cover $\mathcal{U} = \{U^\lambda\}_{\lambda \in \Lambda}$ of X such that the interiors of the sets of $\mathcal{U}$ cover X. Then we can form the diagram

$$\begin{tikzcd} \bigsqcup_{\zeta \in \Lambda^2} U^\zeta \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \bigsqcup_{\lambda \in \Lambda} U^\lambda \arrow[r, "c"] & X \end{tikzcd}$$

where c is determined by the inclusions $U^\lambda \to X$ and a, b are determined by the inclusions $U^\zeta \to U^\lambda, U^\zeta \to U^\mu$ for $\zeta = (\lambda, \mu) \in \Lambda^2$. Note that $ca = cb$, and that a map $f : \bigsqcup_{\lambda \in \Lambda} U^\lambda \to Y$ determines uniquely a map $f' : X \to Y$ with $f'c = f$, if and only if $fa = fb$. Thus we say that c is a coequaliser of a, b in the category Top.

Now suppose further that X_* is a filtered space. For each $\zeta = (\zeta_1, \cdots, \zeta_n) \in \Lambda^n$ we set

$$U^\zeta = U^{\zeta_1} \cap \cdots \cap U^{\zeta_n}$$

and consider the *induced filtration*

$$U^\zeta_* := U^\zeta_0 \subseteq U^\zeta_1 \subseteq \cdots \subseteq U^\zeta$$

where $U^\zeta_i = U^\zeta \cap X_i$ for each $i \in \mathbb{N}$. Then we have a coequaliser diagram of filtered spaces

$$\bigsqcup_{\zeta\in\Lambda^2} U^\zeta_* \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda\in\Lambda} U^\lambda_* \xrightarrow{c} X_* \ .$$

Theorem 8.1.5 (Higher Homotopy van Kampen Theorem) *Let X_* be a filtered space, and $\mathcal{U} = \{U^\lambda\}_{\lambda\in\Lambda}$ a family of subsets of X whose interiors cover X. Suppose that for every finite intersection U^ζ of elements of $\mathcal{U}$, the induced filtration U^ζ_* is connected. Then*
(Conn) *X_* is connected, and*
(Iso) *in the following Π-diagram of the cover,*

$$\bigsqcup_{\zeta\in\Lambda^2} \Pi U^\zeta_* \underset{b}{\overset{a}{\rightrightarrows}} \bigsqcup_{\lambda\in\Lambda} \Pi U^\lambda_* \xrightarrow{c} \Pi X_* \ ,$$

c is the coequaliser of a, b in the category Crs of crossed complexes.

Remark 8.1.6 The proof of the theorem will be given in Chapter 15, using the cubical techniques of ω-groupoids. The conclusion of the theorem on connectivity is important and non trivial. It can be proved by the deformation arguments given in Part III, without introducing the algebraic category of ω-groupoids given there. The isomorphism part, which determines ΠX_* in terms of the pieces ΠU^λ_*, should be seen as an all dimensional, nonabelian, local-to-global result in homotopy theory.

8.2 Some immediate consequences of the HHvKT

We obtain many usable applications by specialising the HHvKT.

Definition 8.2.1 A filtered space X_* is *reduced* if X_0 consists of a single point, i.e. $X_0 = \{*\}$; then $*$ is taken as base point of each $X_n, n \geqslant 0$, and the relative homotopy groups of X_* are abbreviated to $\pi_n(X_n, X_{n-1})$. The base point in X_0 is *nondegenerate* if each inclusion $X_0 \to X_n$, is a closed cofibration for all $n \geqslant 1$. □

Here are a few applications.

8.2.1 Coproducts with amalgamation

Let us consider a covering where any two elements intersect along a fixed subspace X_0

Theorem 8.2.2 *Let X_* be a filtered space and suppose:*

(i) *$\mathcal{U} = \{U^\lambda\}_{\lambda\in\Lambda}$ is a family of subsets of X whose interiors cover X;*

(ii) U^0 *is a subset of* X *such that* $U^\lambda \cap U^\mu = U^0$ *for all* $\lambda, \mu \in \Lambda$ *such that* $\lambda \neq \mu$;

(iii) U^0_* *and* $U^\lambda_*, \lambda \in \Lambda$ *are connected filtrations. Then*

(Conn) *the filtration* X_* *is connected, and*
(Iso) *the following is a coequaliser diagram of crossed complexes:*

$$\Pi U^0_* \xrightarrow{(a^\lambda)} \bigsqcup_{\lambda \in \Lambda} \Pi U^\lambda_* \xrightarrow{c} \Pi X_*,$$

where a^λ, c *are induced by inclusions.*

Proof Note that the conditions we give immediately imply the connectivity conditions required for the theorem. □

Another consequence gives the homotopy groups of a wedge of spaces.

Corollary 8.2.3 *Suppose, in addition to the assumptions of the Theorem, that* X_* *is a reduced filtered space (i.e.* X_0 *is a singleton), and* ΠU^0_* *is the trivial crossed complex. Then the morphisms* $\Pi U^\lambda_* \to \Pi X_*$ *induced by inclusions define an isomorphism*

$$*_\lambda \Pi U^\lambda_* \to \Pi X_*$$

from the coproduct crossed complex in Crs_* *to* ΠX_*.

8.2.2 Pushouts

In the case of a covering by two open sets we obtain the Higher Homotopy van Kampen Theorem in the pushout form either directly from Theorem 8.1.5 or as a particular case of Theorem 8.2.2.

Theorem 8.2.4 *Let* X_* *be a filtered space and suppose:*

(i) X *is the union of the interiors of* U^1 *and* U_2;

(ii) $U^0 = U^1 \cap U^2$;

(iii) U^0_*, U^1_*, U^2_* *are connected filtrations. Then*

(Conn) X_* *is connected, and*
(Pushout) *the following diagram of morphisms of crossed complexes*

$$\begin{tikzcd}
\Pi U^0_* \arrow[r] \arrow[d] & \Pi U^1_* \arrow[d] \\
\Pi U^2_* \arrow[r] & \Pi X_*
\end{tikzcd}$$

induced by inclusions, is a pushout diagram in Crs.

This pushout form of HHvKT can be generalised to allow the case when X is the adjunction space of V and a map $f : A \to U$.

Theorem 8.2.5 (The pushout HHvKT for cofibrations) *Suppose that the commutative diagram of filtered spaces*

$$\begin{tikzcd} A_* \arrow[r, "f"] \arrow[d, "i"'] & U_* \arrow[d, "\bar{\imath}"] \\ V_* \arrow[r, "\bar{f}"'] & X_* \end{tikzcd}$$

is such that for $n \geqslant 0$, the maps $i_n : A_n \to V_n$ are closed cofibrations, $A_n = A \cap V_n$, and X_n is the adjunction space $U_n \cup_{f_n} V_n$. Suppose also that the filtrations U_*, V_*, A_* are connected. Then
(Con) *X_* is connected, and*
(Iso) *the induced diagram*

$$\begin{tikzcd} \Pi A_* \arrow[r] \arrow[d] & \Pi U_* \arrow[d] \\ \Pi V_* \arrow[r] & \Pi X_* \end{tikzcd}$$

is a pushout of crossed complexes.

Proof This is a deduction of standard kind from Theorem 8.3.5 using mapping cylinders. □

We are going to illustrate the use of Theorem 8.2.5 in several situations. First a direct application to quotient filtrations.

Theorem 8.2.6 *Let V_* be a filtered space, $A \subseteq V$, and $X = V/A$. We define the filtrations A_*, and X_* by $A_n = V_n \cap A$, and $X_n = V_n/A_n, n \geqslant 0$. Suppose that each $A_n \to V_n$ is a closed cofibration, and both A_*, V_* are connected. Then*
(Con) *X_* is connected, and*
(Iso) *we have a pushout of crossed complexes*

$$\begin{tikzcd} \Pi A_* \arrow[r] \arrow[d, "i_*"'] & 0 \arrow[d] \\ \Pi V_* \arrow[r] & \Pi X_*. \end{tikzcd}$$

Proof All we have to do is to apply Π to the diagram

$$\begin{tikzcd} A_* \arrow[r, "f"] \arrow[d, "i"'] & \{*\} \arrow[d, "\bar{\imath}"] \\ V_* \arrow[r, "\bar{f}"'] & X_* \end{tikzcd}$$

that satisfies the conditions of Theorem 8.2.5. □

Applying to this result the fact that the dimension functors $(-)_n$ preserve colimits, we get some results on homotopy groups. Let us first fix some notation.

Corollary 8.2.7 *Let V_*, A_* and X_* be filtered space as in Theorem 8.2.6. If V_* is reduced, then we have*

$$\pi_n(X_n, X_{n-1}) = \frac{\pi_n(V_n, V_{n-1})}{N}$$

where N is the $\pi_1 V_1$-submodule generated by all elements $\{u - u^a \mid u \in \pi_n(V_n, V_{n-1}), a \in i_*\pi_1 A_1\}$ and $i_*\pi_n(A_n, A_{n-1})$.

8.3 Results on pairs of spaces: induced modules and relative homotopy groups

All this Section relates to the case when the filtration is reduced to two stages. The HHvKT in this setting becomes Theorem 8.3.5 and gives quite easily some computations of homotopy groups of pairs of spaces and, as consequence, some classical results (the Suspension Theorem, the Brouwer degree Theorem, and the Relative Hurewicz Theorem). These are basic theorems in homotopy theory, and it should be noted that we obtain them without the machinery of homology theory.

It will be clear from Part I that a major aspect of this work is to tie in the fundamental group and higher homotopy groups. This contrasts with previous approaches, where the action of the fundamental group is often obtained by passing to the universal covering space. It was an aesthetic objection to this diversion to obtain the fundamental group of the circle which led to the groupoid work in [Bro06] and so to the present work. It is also unclear at present how to obtain the results of Part I by covering space methods.

8.3.1 Specialisation to pairs

Although there are important results on pointed pairs of spaces (X, A) we still have to use the case where A may not be path connected or at any rate has a set of base points, which we will always write A_0. Thus (X, A) with the subset A_0 of A will be called a *based pair* (and a based pair (U, C) will have set of base points C_0).

To relate the homotopy groups of a pair of spaces to the fundamental crossed complex of a filtered space we associate to a based pair of spaces (X, A) a special filtration as follows:

Definition 8.3.1 For any based pair of spaces (X, A) and dimension $n \geqslant 2$, *the filtration* $E_n(X, A)$ *of* X *associated to the based pair* (X, A) is given by

$$\underset{0}{A_0} \subseteq \underset{1}{A} \subseteq \cdots \subseteq \underset{n-1}{A} \subseteq \underset{n}{X} \subseteq \cdots \subseteq \underset{r}{X} \subseteq \cdots,$$

i.e. it is A_0 in dimension 0, A in dimensions $0 < r < n$, and X in dimensions $r \geqslant n$. $\Box$

The fundamental crossed complex of $E_n(X, A)$ has only two non zero stages: the groupoid in dimension 1 is $\pi_1(A, A_0)$ and the n-dimensional module (crossed module if $n = 2$) is $\pi_n(X, A, A_0)$. This is the crossed complex we called $E_n(\pi_n(X, A, A_0), \pi_1(A, A_0))$. The following is clear.

Proposition 8.3.2 *Consider a based pair* (X, A)*, and suppose* $n > 2$*. Then the fundamental crossed complex* $\Pi(E_n(X, A))$ *of its associated filtration is the crossed complex*

$$E_n(\pi_n(X, A, A_0), \pi_1(A, A_0)) = (\ \cdots \longrightarrow 0 \longrightarrow \underset{n}{\pi_n(X, A, A_0)} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{1}{\pi_1(A, A_0)}\ )$$

associated to the $\pi_1(A)$*-module* $\pi_n(X, A)$.

All we need to make that appropriate for use of the HHvKT is to translate the connectivity of $E_n(X, A)$ into conditions on the pair (X, A) and see what form the HHvKT takes in this case. The following is clear.

Proposition 8.3.3 *The filtration $E_n(X, A)$ associated to a based pair of spaces (X, A) is connected if and only the induced maps from $\pi_0 A$ to $\pi_0 A$ and $\pi_0 X$ are surjective, and for all $x \in A_0$, $\pi_i(X, A, x) = 0$ for $1 \leqslant i < n$.*

Note that the condition $\pi_1(X, A, x) = 0$ means that any path in X from x to a point in A is homotopic rel end points to a path in A.

Definition 8.3.4 If the conditions in the previous proposition hold, we say the based pair (X, A) is $(n-1)$-*connected*.

8.3.2 Coproducts with amalgamation

Let us translate Theorem 8.2.2 to the case of pairs.

Theorem 8.3.5 *Let (X, A) be a based pair, and suppose:*

(i) *$\mathcal{X} = \{X^\lambda\}_{\lambda\in\Lambda}$ is a family of subsets of X whose interiors cover X;*

(ii) *X^0 is a subset of X such that $X^\lambda \cap X^\mu = X^0$ for all $\lambda, \mu \in \Lambda$ such that $\lambda \neq \mu$;*

(iii) *for $\lambda = 0$ or $\lambda \in \Lambda$, the based pairs (X^0, A^0), (X^λ, A^λ), formed by intersection with X^0 and X^λ, are $(n-1)$-connected.*

Then:
(Conn) *the based pair (X, A) is $(n-1)$-connected, and*
(Iso) *the following is a coequaliser diagram in XMod if $n = 2$ and in Mod if $n > 2$:*

$$(\pi_n(X^0, A^0, A_0), \pi_1(A^0, A^0_0)) \xrightarrow{(a^\lambda)} \bigsqcup_{\lambda\in\Lambda} (\pi_n(X^\lambda, A^\lambda, A^\lambda_0), \pi_1(A^\lambda, A^\lambda_0)) \xrightarrow{c} (\pi_n(X, A, A_0), \pi_1(A, A_0)),$$

where a^λ, c are induced by inclusions.

Remark 8.3.6 In particular, when $\Lambda = \{1, 2\}$, the theorem produces a pushout diagram:

$$\begin{tikzcd}
(\pi_n(X^0, A^0, A^0_0), \pi_1(A^0, A^0_0)) \arrow[r] \arrow[d] & (\pi_n(X^1, A^1, A^1_0), \pi_1(A^1, A^1_0)) \arrow[d] \\
(\pi_n(X^2, A^2, A^2_0), \pi_1(A^2, A^2_0)) \arrow[r] & (\pi_n(X, A, A_0), \pi_1(A, A_0))
\end{tikzcd}$$

We apply this result in the next subsections to deduce some classical results, including the description of the fundamental crossed complex of the skeletal filtration of a *CW*-complex as a free crossed complex.

8.3.3 Induced modules and homotopical excision

We now specialise the pushout part of the theorem of the previous subsection into an excision result which has many applications:

Theorem 8.3.7 (Homotopical Excision 1) *Let X be the union of the interiors of two subspaces U and V, and $A = U\cap V$. Suppose also that U, V, A are path-connected and (V, A) is $(n-1)$-connected. Then:*

(Con) *the pair* (X, U) *is* $(n-1)$*-connected, and*
(Iso) *for* $n \geqslant 3$, $\pi_n(X, U)$ *as* $\pi_1 U$*-module is isomorphic to the module induced from the* $\pi_1 A$*-module* $\pi_n(V, A)$ *by* $\lambda = i_{1*} : \pi_1 A \to \pi_1 U$ *the map given by the inclusion* $i_1 : A \to U$*, i.e.*

$$\pi_n(X, U) = \lambda_* \pi_n(V, A).$$

Proof This is just Theorem 8.3.5 applied to $E_{n-1}(X, A)$. □

Remark 8.3.8 This should be compared with the excision axiom for relative homology, which in one form simply says that if X is the union of open sets U, V then the map in homology $H_i(U, U \cap V) \to H_i(U \cup V, V)$ induced by inclusion, is an isomorphism for all $i \geqslant 0$. It is this result which makes homology calculable. By contrast, this homotopical excision result has connectivity conditions, it determines only one group, but it also links two groups in separated dimensions, where the lower one is usually nonabelian.

This theorem applies to give a comparable result, but for closed cofibrations:

Theorem 8.3.9 (Homotopical Excision 2) *Suppose that in the commutative square of spaces*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & U \arrow[d, "\bar{\imath}"] \\
V \arrow[r, "\bar{f}"'] & X = U \cup_f V
\end{tikzcd}$$

the map i *is a closed cofibration and* X *is the adjunction space* $U \cup_f V$*. Suppose also that* U, V, A *are path-connected and* (V, A) *is* $(n-1)$*-connected. Then:*
(Con) *the pair* (X, U) *is* $(n-1)$*-connected, and*
(Iso) *for* $n \geqslant 3$, $\pi_n(X, U)$ *as* $\pi_1 U$*-module is isomorphic to the module induced from the* $\pi_1 A$*-module* $\pi_n(V, A)$ *by the map induced by* $\lambda = f_* : \pi_1 A \to \pi_1 U$*, i.e.*

$$\pi_n(X, U) = \lambda_* \pi_n(V, A).$$

Proof This can be obtained either from the previous theorem using mapping cylinder arguments or directly from Theorem 8.2.5 applied to $E_{n-1}(X, A)$. □

Of course the corresponding results for $n = 2$, with 'module' replaced by 'crossed module' are also true; they have been given in Part I, and a number of consequences of a nonabelian type were deduced.

Now we give several applications of Theorem 8.3.7, starting with the pair (CA, A) where CA denotes the cone on the space A. Now CA is contractible and so has zero homotopy groups. By the homotopy exact sequence of the pair, the boundary map $\partial : \pi_r(CA, A, x) \to \pi_{r-1}(A, x)$ is an isomorphism for all $x \in A$. Thus the pair (CA, A) is n-connected if and only if A is $(n-1)$-connected, i.e. if A is connected and $\pi_r(A, x) = 0$ for $1 \leqslant r < n$.

First we derive the first n homotopy groups of the n-sphere S^n, using suspension and induction. Since the suspension is just a quotient of the cone, we can use Theorem 8.3.7 to relate the homotopy groups of the suspension SA to those of the base A.

Theorem 8.3.10 (The Suspension Theorem) *For a space* A*, consider* SA *the (unreduced) suspension of* A*. If* A *is* $(n-2)$*-connected, for* $n \geqslant 3$*, then*
(Con) SA *is* $(n-1)$*-connected and*
(Iso) $\pi_n SA \cong \pi_{n-1} A$.

Proof We define $V = CA$ the cone on A, $U = \{*\}$ a point and f the constant map. Then $X = U \cup_f V$ is the (unreduced) suspension of A, and we can consider the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & \{*\} \arrow[d, "\bar{\imath}"] \\ CA \arrow[r, "\bar{f}"'] & X = SA. \end{tikzcd}$$

Since A is $(n-2)$-connected if and only if (CA, A) is $(n-1)$-connected, we can apply Theorem 8.3.7, getting that $(X, *)$ is $(n-1)$-connected and

$$\pi_n SA \cong \pi_n(CA, A).$$

Using again the homotopy exact sequence of this pair, we have $\pi_n(CA, A) \cong \pi_{n-1} A$. $\square$

Corollary 8.3.11 (Brouwer Degree Theorem) *For $n \geqslant 1$, $\mathbf{S}^n$ is $(n-1)$-connected and*

$$\pi_n(\mathbf{S}^n, 1) \cong \mathbb{Z}.$$

Proof Recall that in Part I we have seen that if $n = 2$ and A is a path-connected space then SA is 1-connected and

$$\pi_2(SA, x) \cong \pi_1(A, x)^{ab}.$$

Given the value of $\pi_1(\mathbf{S}^1, 1)$ as $\mathbb{Z}$ (a result proved in Section 1.7), we deduce $\pi_2(\mathbf{S}^2, 1) \cong \mathbb{Z}$.

The induction step follows easily from the Theorem. $\square$

This is actually a non elementary result: that the sphere $\mathbf{S}^n$ is $(n-1)$-connected means that any map $\mathbf{S}^r \to \mathbf{S}^n$ for $r < n$ is nullhomotopic, while the determination of $\Pi(\mathbf{S}^n, 1)$ includes the Brouwer degree theorem, that the maps $\mathbf{S}^n \to \mathbf{S}^n$ are classified up to homotopy by an integer, called the *degree* of the map. This was one of the early triumphs in homotopy classification results. Proofs of these results have to use some kind of subdivision argument, often through the route of simplicial approximation, which we avoid completely.

Corollary 8.3.12 *Let $\mathbf{E}^n_*$ be the skeletal filtration of the n-cell with cell structure $E^0 = e^0$, $E^1 = e^0_\pm \cup e^1$, and for $n \geqslant 2$, $E^n = e_0 \cup e^{n-1} \cup e^n$. Then $\Pi \mathbf{E}^n_* \cong \mathbb{F}(n)$, the free crossed complex on one generator of dimension n, for all $n \geqslant 0$.*

Proof This follows from the Brouwer Degree Theorem and the homotopy exact sequence of the pair(E^n, S^{n-1}). $\square$

Corollary 8.3.13 *Let X_* be a connected filtration, and let $Y_* = X_* \cup \{e^n_\lambda\}$ be formed by attaching n-cells by filtered maps $f_\lambda : S^{n-1}_* \to X_*$, $\lambda \in \Lambda$. Then Y_* is connected, and has fundamental crossed complex formed from ΠX_* by attaching free generators x^n_λ in dimension n.*

Proof This follows from the previous corollary and the pushout version of the HHvKT. Note that in this application, we are using many base points in the disjoint union of copies of n-cells. $\square$

Corollary 8.3.14 *If X is a CW-complex with skeletal filtration X_*, then X_* is a connected filtration, and ΠX_* is the free crossed complex on the classes of the characteristic maps of X.*

Proof This follows from the previous corollary by induction on the skeleta of X. $\square$

Remark 8.3.15 We note that the use of many base points and so groupoids rather than groups is not a luxury in these applications. Non reduced CW-complexes occur naturally, for example the geometric n-simplex, and a non trivial covering space of reduced CW-complex is no longer reduced.

8.3.4 Attaching a cone, and the Relative Hurewicz Theorem

We see what Theorem 8.3.7 implies in the case when we are attaching a cone CA via a map of the space A.

Proposition 8.3.16 *Let $X = U \cup_f CA$ for some map $f : A \to U$. For any $n \geqslant 3$, if U is path connected and A is $(n-2)$-connected, then*
(Con) *(X, U) is $(n-1)$-connected and*
(Iso) *the $\pi_1(U)$-module $\pi_n(X, U)$ is isomorphic to the induced module $\lambda_*(\pi_{n-1}(A))$, i.e.*

$$\pi_n(X, U) \cong \pi_{n-1}A \otimes \mathbb{Z}(\pi_1 U).$$

□

A consequence is the effect of attaching n-cells on some of the homotopy groups of a space.

Exercise 8.3.17 Let A, B, U be path-connected, based spaces. Let $X = U \cup_f (CA \times B)$ where CA is the (unreduced) cone on A and f is a map $A \times B \to U$. The homotopy exact sequence of $(CA \times B, A \times B)$ gives

$$\pi_i(CA \times B, A \times B) \cong \pi_{i-1}A, i \geqslant 2, \text{ and } \pi_1(CA \times B, A \times B) = 0.$$

Suppose now that $n > 2$ and A is $(n-2)$-connected. Then $\pi_1 A = 0$. We conclude from Theorem 8.3.7 that (X, U) is $(n-1)$-connected and $\pi_n(X, U)$ is the $\pi_1 U$-module induced from $\pi_{n-1}A$, considered as trivial $\pi_1 B$-module, by $\lambda = f_* : \pi_1 B \to \pi_1 U$. Hence $\pi_n(X, U)$ is the $\pi_1 U$-module

$$\pi_{n-1}A \otimes_{\mathbb{Z}(\pi_1 B)} \mathbb{Z}(\pi_1 U).$$

□

Now we deduce a version of the classical relative Hurewicz Theorem.

Theorem 8.3.18 (Relative Hurewicz Theorem) *Let (V, A) be a pair of spaces. Suppose $n \geqslant 3$, A and V are path connected and (V, A) is $(n-1)$-connected. Then*
(Con) *$V \cup CA$ is $(n-1)$-connected, and*
(Iso) *the natural map*

$$\pi_n(V, A, x) \to \pi_n(V \cup CA, CA, x) \xrightarrow{\cong} \pi_n(V \cup CA, x)$$

presents $\pi_n(V \cup CA, x)$ as $\pi_n(V, A, x)$ factored by the action of $\pi_1(A, x)$.

Proof Let $X = V \cup CA$. We would like to apply Theorem 8.3.5 to the diagram of inclusions

$$\begin{tikzcd} A \arrow[r] \arrow[d] & CA \arrow[d, "\bar{\imath}"] \\ V \arrow[r, "\bar{f}"'] & X = V \cup CA \end{tikzcd}$$

but the subspaces do not satisfy the interior condition. We change the subspaces to $A' = A \times [0, \frac{1}{2}[\subseteq CA$ and $V' = V \cup A'$. Those subspaces have the same homotopy type as A and V (moreover the pair (V', A') has the homotopy type of (V, A)) and we can apply Theorem 8.3.5 to the diagram of inclusions

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & CA \arrow[d, "\bar{\iota}"] \\ V' \arrow[r, "\bar{f}"'] & X = V \cup CA \end{tikzcd}$$

This yields that X is $(n-1)$-connected, and that

$$\pi_n(X, CA) = \lambda_* \pi_n(V, A),$$

the module induced from the $\pi_1 A$-module $\pi_n(V, A)$ by $\lambda = f_* : \pi_1 A \to \pi_1 CA = 0$.

It follows, since $\pi_1 CA = 0$, that $\lambda_* \pi_n(V, A)$ is obtained from $\pi_n(V, A)$ by killing the $\pi_1 A$-action. (If $n = 2$ that would give the abelianisation).

To finish, note that using that CA is contractible, we get that $\pi_r(X, CA, x)$ is isomorphic to $\pi_r(X, x)$, by the homotopy exact sequence of the pair. □

The usual forms of the Hurewicz Theorem involve homology groups, which lie outside the main scope of this book, although many readers will be well familiar with them. Here we make a few remarks to give a brief account of the relation between the to approaches.

The homology functors H_n, $n \geqslant 0$, assign to any topological space A or pair of spaces (V, A) abelian groups $H_n(A)$, $H_n(V, A)$ such that: there is a natural exact sequence

$$\cdots \to H_{n+1}(V, A) \xrightarrow{\partial} H_n(A) \to H_n(V) \to H_n(V, A) \to H_{n-1}(A) \to \cdots ;$$

if A is a point then $H_0(A) \cong \mathbb{Z}$; and the excision and homotopy axioms, which we do not state here, hold. These axioms imply that the boundary $\partial : H_{n+1}(\mathbf{E}^{n+1}, \S^n) \to H_n(\S^n)$ is an isomorphism; it follows by induction that $H_n(\S^n) \cong \mathbb{Z}$ for $n > 0$. Choose a generator ι^n of this group, giving a generator, also written ι^{n+1}, of $H_{n+1}(\mathbf{E}^{n+1}, \S^n)$. The Hurewicz morphisms

$$\omega_n : \pi_n(A, x) \to H_n(A), \quad \omega_{n+1} : \pi_{n+1}(V, A, x) \to H_n(V, A)$$

are then defined by sending the class of a map f in such a homotopy group to $f_*(\iota^n), f_*(\iota^{n+1})$ where f_* is the induced map in homology. This leads to a morphism from the exact homotopy sequence of a pair to the exact homology sequence,which we use in the next theorem.

We have the following:

Theorem 8.3.19 (Absolute Hurewicz Theorem) *If X is an $(n-1)$-connected space, then the Hurewicz morphism $\omega_i : \pi_i(X, x) \to H_i(X)$ is an isomorphism for $0 \leqslant i \leqslant n$ and an epimorphism for $i = n + 1$.*

We shall outline a proof of this result in Theorem 14.7.9. The use of filtered spaces is quite appropriate for this proof, and follows the lines of some classical papers.

The usual version of the *Relative Hurewicz Theorem* involves not $\pi_n(V \cup CA, x)$ but the homology $H_n(V, A)$. It is possible to get this more usual version from the one we have just proved in a three stage process.

First, notice that, given the conclusion of our theorem, that $V \cup CA$ is $(n-1)$-connected, then $\pi_n(V \cup CA, x)$ is isomorphic to $H_n(V \cup CA)$ by the absolute Hurewicz Theorem.

Then it is easy to prove that $H_n(V \cup CA)$ is isomorphic to $H_n(V \cup CA, CA)$ by the homology exact sequence, using that CA is acyclic because it is contractible.

Last, we notice that, by excision, the morphism induced by inclusion $H_n(V, A) \to H_n(V \cup CA, CA)$ is an isomorphism.

Here is another corollary of the Relative Hurewicz Theorem, which assumes a bit more on the Hurewicz morphism from homotopy to homology. We call it Hopf's theorem, although he gave only the case $n = 2$.

Proposition 8.3.20 (Hopf's theorem) *Let (V, A) be a pair of pointed spaces such that:*

(i) $\pi_i(A) = 0$ *for* $1 < i < n$*;*

(ii) $\pi_i(V) = 0$ *for* $1 < i \leqslant n$*;*

(iii) *the inclusion $A \to V$ induces an isomorphism on fundamental groups.*

Then the pair (V, A) is n-connected, and the inclusion $A \to V$ induces an epimorphism $H_n A \to H_n V$ whose kernel consists of spherical elements, i.e. of the image of $\pi_n A$ under the Hurewicz morphism $\omega_n : \pi_n(A) \to H_n(A)$.

Proof That (V, A) is n-connected follows immediately from the homotopy exact sequence of the pair (V, A) up to $\pi_n(V)$. We now consider the next part of the exact homotopy sequence and its relation to the homology exact sequence as shown in the commutative diagram:

$$\begin{tikzcd}
\pi_{n+1}(V, A) \arrow[r, "\partial"] \arrow[d, "\omega_{n+1}"'] & \pi_n(A) \arrow[r] \arrow[d, "\omega_n"'] & \pi_n(V) \arrow[r] \arrow[d] & \pi_n(V, A) \arrow[d] \\
H_{n+1}(V, A) \arrow[r, "\partial'"'] & H_n(A) \arrow[r, "i_*"'] & H_n(V) \arrow[r] & H_n(V, A)
\end{tikzcd}$$

The Relative Hurewicz Theorem implies that $H_n(V, A) = 0$, and that ω_{n+1} is surjective. Also ∂ in the top row is surjective, since $\pi_n(V) = 0$. It follows easily that the sequence $\pi_n(A) \to H_n(A) \to H_n(V) \to 0$ is exact. □

8.4 The chain complex of a filtered space and of a CW-complex.

In this section we identify for certain filtered spaces X_* the chain complex $\nabla \Pi X_*$ in terms of chains of universal covers.

All spaces which arise will now be assumed to be Hausdorff and to have universal covers. Recall, [Bro06, 10.5.8], that if X is a topological space and $v \in X$ then the universal coving map $p : \widetilde{X}(v) \to X$, can be constructed by topologising the fundamental groupoid $\pi_1(X)$ and considering the final point map $t : \pi_1 X \to X$, writing

$$\widetilde{X}(v) = t^{-1}(v)$$

and identifying p with the initial point map s. This space has a canonical base point, $1_v \in \pi_1 X$. These spaces form a bundle over X on which $\pi_1 X$ operates by composition, but not preserving the base point.

Let X_* be a filtered space. For $v \in X_0$, $i \geqslant 0$, let $\widehat{X}_*(v)$ denote the filtered space consisting of $\widetilde{X}(v)$ and the family of subspaces

$$\widehat{X}_i(v) = p^{-1}(X_i).$$

Definition 8.4.1 We define for a filtered space X_* the chain complex with operators CX_* to have

$$C_i X_*(\nu) = H_i(\widehat{X}_i(\nu), \widehat{X}_{i-1}(\nu)), \quad C_0 X_* = H_0(\widehat{X}_0(\nu)),$$

for all $, \nu \in X_0, i \geqslant 1$, and to have groupoid of operators $\pi_1(X, X_0)$ with operation induced by the bundle operations given above. This defines the functor *fundamental chain complex of a filtered space*

$$C : \mathsf{FTop} \to \mathsf{Chn}.$$

Proposition 8.4.2 *If X_* is a connected filtered space, then*

$$CX_* = \nabla \Pi X_*.$$

Proof Notice that, in this case, $\widehat{X}_i(\nu)$ is the universal cover of X_i based at ν for $i \geqslant 2$.

We will use the Relative Hurewicz Theorem 8.3.18.

Let $\nu \in X_0$ and let $i \geqslant 3$. The pair $(\widehat{X}_i(\nu), \widehat{X}_{i-1}(\nu))$ is $(i-1)$-connected, and $\widehat{X}_{i-1}(\nu))$ is simply connected, and so

$$\begin{aligned} \pi_i(X_i(\nu), X_{i-1}, \nu) &\cong \pi_i(\widehat{X}_i(\nu), \widehat{X}_{i-1}(\nu), 1_\nu) \text{ since } p \text{ is a covering,} \\ &\cong H_i(\widehat{X}_i(\nu), \widehat{X}_{i-1}(\nu)) \qquad \text{by the relative Hurewicz theorem,} \end{aligned}$$

since $\widehat{X}_i(\nu)$ and $\widehat{X}_{i-1}(\nu)$ are in fact the universal covers at ν of X_i and X_{i-1} respectively. If $i = 2$, a similar argument applies but in this case $\pi_1(\widehat{X}_1, \nu) = \delta\pi_2(\widehat{X}_2(\nu), \widehat{X}_1(\nu), 1_\nu)$. So the relative Hurewicz theorem in dimension 2 (Theorem 5.5.2) now gives

$$\begin{aligned} H_2(\widehat{X}_2(\nu), \widehat{X}_1(\nu)) &\cong \pi_2(\widehat{X}_2(\nu), \widehat{X}_1, 1_\nu)^{\mathrm{ab}} \\ &\cong \pi_2(X_2, X_1, \nu)^{\mathrm{ab}} \\ &= (\nabla \Pi X_*)_2. \end{aligned}$$

The case $i = 1$ is essentially the result of Crowell [Cro71, Section 5]. For another sketch proof, we can use the result of Exercises **??** on the abelianisation of $\pi_1(X, X_0)$, and the result of our Exercise 7.5.27 giving a description of the derived module in terms of an abelianisation of a groupoid. □

Corollary 8.4.3 *Let X_* be a filtered space and suppose that X is the union of a family $\mathcal{U} = \{U^\lambda\}_{\lambda \in \Lambda}$ of open sets such that $\mathcal{U}$ is closed under finite intersection. Let U^λ_* be the filtered space obtained from X_* by intersection with U^λ. Suppose that each U^λ_* is a connected filtered space. Then X_* is connected and the natural morphism in* Chn

$$\operatorname{colim}^\lambda CU^\lambda_* \to CX_*$$

is an isomorphism.

Proof This is an easy consequence of the HHvKT 8.1.5 which gives a similar result for Π rather than C. Then we apply ∇ which has a right adjoint and so preserves colimits. □

We note that results such as this have been used by various workers ([Lom81, PS85]) in the case X_* is the skeletal filtration of a *CW*-complex and the family $\mathcal{U}$ is a family of subcomplexes, although usually in simple cases. The general form of this 'Van Kampen Theorem' for CX_* does not seem to have been noticed, and this is probably due to the unfamiliar form of colimits in the category Chn of chain complexes over varying groupoids. Even in the group case these colimits are not quite what might be expected (see Example 7.5.31).

Chapter 9

Tensor products and homotopies of crossed complexes

This Chapter is built around the notion of *monoidal closed category*, and on the use of such a structure on the category Crs of crossed complexes.

This monoidal closed structure for the category Crs gives a natural 'exponential law' of a natural isomorphism

$$e : \mathsf{Crs}(C \otimes D, E) \cong \mathsf{Crs}(C, \mathsf{CRS}(D, E)),$$

for crossed complexes C, D, E. Here 'monoidal' refers to the 'tensor product' $C \otimes D$ and 'closed' refers to the 'internal hom' $\mathsf{CRS}(D, E)$. The elements of $\mathsf{CRS}(D, E)$ may be written out explicitly – they are morphisms $D \to E$ in dimension 0, homotopies of morphisms in dimension 1, and 'higher homotopies' for $n > 1$. One advantage of this procedure is that we can use crossed complex techniques not only on filtered spaces but also on maps and homotopies of filtered spaces.

For $C \otimes D$ we can give in the first instance only generators $c \otimes d, c \in C_m, d \in D_m, m, n \geqslant 0$ and the structure and axioms on these.

This will raise conceptual difficulties for those not used to the ideas, and in the case of crossed complexes it also raises technical difficulties, since there is an elaborate set of formulae for the so called 'tensor product'. So we shall give some background and introduction in Section 9.1.

Sometimes we use a formal description of this monoidal closed structure on the category of crossed complexes, but the fact that we can if necessary get our hands dirty, that is write down some complex formulae and rules and calculate with them, is one of the aspects of the theory that gives power to the category of crossed complexes.

The complication of the rules for the tensor product is due to their modeling the geometry of the product of cells. It is important to get familiar with these formulae for the tensor product as they will be used frequently in the applications of this and the next few chapters.

The natural way to be sure this structure exists is not to define it directly, but through the equivalence with the category of ω-groupoids and the natural definition of tensor product and internal hom in that category. This ensures that the definitions for the category Crs will work, and this we do in Chapter 15. Here, we state directly the Definition that results from this detour, risking that this could make the rules for the tensor product in Crs seem too awkward.

We give a direct description of $C \otimes D$ first of all in dimensions 1 and 2, in subsections 9.4.1, 9.4.2. Then we use the monoidal closed structure on the category Chn of chain complexes with a groupoid of operators, in order to get a clearer description of $(C \otimes D)_n$ for $n > 2$ as $(\nabla C \otimes \nabla D)_n$.

As an introduction to the crossed complex case, and because we need this example, we describe in Section 9.2 the structure of monoidal closed category for the category Mod of modules over groupoids. This gives a natural equivalence in Mod

$$\mathsf{Mod}((M \otimes N, G \times H), (L, K)) \cong \mathsf{Mod}((M, G), \mathsf{MOD}((N, H), (L, K)).$$

Then in Section 9.3 we define $\mathsf{CRS}(D, E)$ by defining the elements of $\mathsf{CRS}_n(D, E)$ (with special emphasis on the homotopies $\mathsf{CRS}_1(D, E)$). The rules for addition, action and boundaries are related to the geometry associated to the free crossed complexes.

Analogously to the development of the tensor product for R-modules indicated above, the set of morphisms of crossed modules

$$C \to \mathsf{CRS}(D, E)$$

is bijective to the set of 'bimorphisms'

$$(C, D) \to E,$$

where these bimorphisms play for crossed complexes the same role that bilinear maps play for modules.

Then, we can form the tensor product of crossed modules, as in the case of R-modules, by taking free objects and quotienting out by the appropriate relations.

We end the algebraic part of this Chapter by proving in Section 9.6 the important result that the tensor product preserves freeness. This uses crucially the adjoint relation of the tensor product to the internal hom.

The second part of the Chapter deals with the topological applications, namely relations between the monoidal closed category of Crs and the fundamental crossed complex functor

$$\Pi : \mathsf{FTop} \to \mathsf{Crs}.$$

We start by giving in Section 9.7 a structure of monoidal closed category to FTop the category of filtered topological spaces and filtered maps that is a straightforward generalisation of the cartesian closed category structure of Top already mentioned in this introduction.

The way the two structures of monoidal closed category on FTop and Crs are related is explained in Section 9.8. As before, we leave the proofs for Chapter 15 to Part III of the book. The main result is Theorem 9.8.1 stating how the functor Π behaves with respect to tensor products. In particular, if X_*, Y_* are filtered spaces, then there is a natural transformation

$$\theta \ : \ \Pi(X_*) \otimes \Pi(Y_*) \to \Pi(X_* \otimes Y_*)$$

which is an isomorphism if X_*, Y_* are CW-complexes.

The tensor product in the categories FTop and Crs allows homotopies to be interpreted in these categories as maps from a 'cylinder functor' which in FTop is of the form $I_* \otimes -$. Thus an immediate consequence of Theorem 9.8.1 is that the fundamental crossed complex functor Π is a homotopy functor. This, and the analysis of the cone of a crossed complex, leads in Section 9.9 to computations on the fundamental crossed complex of an n-simplex providing a version of the simplicial Homotopy Addition Lemma (Theorem 9.9.4). A similar result is true for n-cubes giving a cubical Homotopy Addition Lemma (Proposition 9.9.9).

9.1 Some exponential laws in topology and algebra

The start of the idea of a monoidal closed category is that a function of two variables $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ can also be regarded as a variable function of one variable. This is the basis of partial differentiation. In general, this transforms into the idea that if Z^Y denotes the set of functions from the set Y to the set Z, then we have a bijection of sets

$$e : Z^{X \times Y} \to (Z^Y)^X$$

given by

$$e(f)(x)(y) = f(x, y), \quad x \in X, y \in Y.$$

This corresponds to the exponential law for numbers $m^{np} = (m^n)^p$, and so the previous law is called the *exponential law for sets*.

Because there is a bijection $X \times Y \to Y \times X$ this also means we can set up bijections between the three set of functions

$$X \to Z^Y, \quad Y \to Z^X, \quad X \times Y \to Z.$$

This becomes particularly interesting in its interpretation when $Y = I = [0, 1]$, the unit interval, since the functions $I \to Z$ can be thought of as paths in Z, and so the set of these functions is a kind of space of paths; in practice we will want to have topologies on these sets and speak only of continuous functions, but let us elide over that for the moment.

The functions $X \to Z$ we can intuitively call 'configurations of X in Z'. A function $X \times I \to Z$ we can think of as a deformation of configurations. This can be seen alternatively as a path in the configuration space Z^X, or as a configuration $X \to Z^I$ in the path space of Z. These alternative points of view have proved strongly useful in mathematics.

It is useful to rephrase the exponential law slightly more categorically, so as make analogies for other categories, so we write it also as a bijection

$$e : \mathsf{Set}(X \times Y, Z) \cong \mathsf{Set}(X, \mathsf{SET}(Y, Z)).$$

Here the distinction between Set and SET, i.e. between external and internal to the category, is less clear than it will be in our other examples.

Now suppose that X, Y, Z are topological spaces, and $\mathsf{Top}(Y, Z)$ denotes the set of continuous maps $Y \to Z$. We would like to make this set into a topological space $\mathsf{TOP}(Y, Z)$ so that the exponential correspondence gives a natural bijection

$$\mathsf{Top}(X \times Y, Z) \cong \mathsf{Top}(X, \mathsf{TOP}(Y, Z)).$$

However this turned out not to be possible for all topological spaces, and in the end a reasonable solution was found by restricting to what are called 'compactly generated spaces', and working entirely in the category of these spaces. In this book Top will mean the category of compactly generated spaces. An account of this category is given by Brown in [Bro06], and we assume this to be known. The existence of the exponential law as above is summarised by saying that the category Top is a *cartesian closed category*. Here 'cartesian' refers to the fact that we use the categorical product in the category, and 'closed' means that there is a space $\mathsf{TOP}(Y, Z)$ for all spaces Y, Z in the category Top. The space $\mathsf{TOP}(Y, Z)$ is also called the *internal hom* in Top.

It is a deduction from the exponential law that there is also a natural homeomorphism

$$\mathsf{TOP}(X \times Y, Z) \cong \mathsf{TOP}(X, \mathsf{TOP}(Y, Z)).$$

We leave the proof of this to the reader.

There are a couple of special characteristics to this example. First, the underlying set of the space $\mathsf{TOP}(Y, Z)$ is the set $\mathsf{Top}(Y, Z)$, but for other exponential laws there is no reason why this should be so. We shall come later to this point.

Second, the product we are using is the categorically defined product in the category. There are analogous laws which do not involve the cartesian product in the category.

For example, if Mod_R denotes the category of left modules over a commutative ring R with morphisms the R-linear maps, then we can for R-modules M, N form an R-module structure on the set $\mathsf{Mod}_R(M, N)$ to form an R-module which we write $\mathsf{MOD}_R(M, N)$. This we call the *internal hom* in the category Mod_R.

For another R-module L we can then consider

$$\mathsf{Mod}_R(L, \mathsf{MOD}_R(M, N)).$$

However this set is bijective with the set of R-bilinear maps $(L, M) \to N$, by which is meant the functions $L \times M \to N$ which are linear in each variable. Then we have an exponential law

$$\mathsf{BiLin}_R((L, M); N) \cong \mathsf{Mod}_R(L, \mathsf{MOD}_R(M, N)),$$

where the left hand side denotes the set of bilinear maps.

A standard construction is the 'universal bilinear map' $(L, M) \to L \otimes_R M$ so as to obtain a natural bijection

$$\mathsf{BiLin}_R((L, M); N) \cong \mathsf{Mod}_R(L \otimes_R M, N)$$

and hence a natural bijection

$$\mathsf{Mod}_R(L \otimes_R M, N) \cong \mathsf{Mod}_R(L, \mathsf{MOD}_R(M, N)).$$

However the *tensor product* construction, i.e. the bifunctor,

$$- \otimes_R - : \mathsf{Mod}_R \times \mathsf{Mod}_R \to \mathsf{Mod}_R,$$

does not give a categorical product in the category Mod_R. To describe this situation, category theorists have developed the notion of *monoidal closed category*.

The exponential law is an adjoint relationship. In the last situation it states that for all M the functor $— \otimes_R M$ is left adjoint to $\mathsf{MOD}_R(M, —)$. This has some immediate consequences on the preservation of colimits and limits by these functors, and these consequences are valuable.

If M, N are left R-modules, then $M \otimes_R N$ can be constructed as the free R-module F on elements $m \otimes n$ for $m \in M, n \in N$ factored by the relations

$$(m + m') \otimes n = (m \otimes n) + (m' \otimes n),$$

$$m \otimes (n + n') = (m \otimes n) + (m \otimes n'),$$

$$rm \otimes n = m \otimes rn,$$

for all $m, m' \in M, n, n' \in N, r \in R$.

Notice that the two first families of relations have some consequences like

$$m \otimes 0 = 0 \otimes n = 0$$

$$(-m) \otimes n = m \otimes (-n) = -(m \otimes n)$$

while the third family of relations can be used to define an structure of R-module by the action

$$r(m \otimes n) = rm \otimes n = m \otimes rn.$$

This gives as a consequence the linearity on both variables of the tensor product

$$(rm + r'm') \otimes n = r(m \otimes n) + r'(m' \otimes n),$$

$$m \otimes (sn + s'n') = s(m \otimes n) + s'(m \otimes n').$$

An important feature is the universal bilinear map

$$M \times N \to M \otimes_R N$$

given by $(m, n) \mapsto m \otimes n$.

By the given construction, an element of $M \otimes_R N$ is an R-linear combination of *decomposable elements* of the form $m \otimes n$. In general, it is not quite so obvious what are the actual elements of $M \otimes_R N$ for specific M, N, R. Nonetheless, the tensor product of R-modules plays an important role in module theory. One reason is that whereas a bilinear map does not have a defined notion of kernel, a morphism $M \otimes_R N \to P$ to an R-module P does have a kernel. This process of using a universal property to replace a function with complicated properties by a morphism is a powerful procedure in mathematics.

Both as a reminder of and as an introduction to the more involved crossed complex case, this process is described in Section 9.2 for the category Mod of modules over groupoids.

The main part of this Chapter revolves around the monoidal closed category structure of Crs. The way to be sure that the definition we gives is natural and convenient, and works, is to define it through the equivalence of the category Crs with the category of ω-groupoids, and the natural definition of monoidal closed structure on that category. That is the procedure of Chapter 15. Here, we state directly the Definition that results from this construction, even if this could make the definition of the internal hom and tensor product in Crs seem somehow artificial. Nevertheless, it is important to get acquainted soon with the formulae for the tensor product because this structure, and the way it reflects certain geometry, is one of the features that gives crossed complexes considerable power. This structure will be used frequently in the applications of the next few chapters.

First a few words relating the closed category structure with homotopy. We have already observed that in any crossed complex C, the set of n-dimensional elements C_n is bijective to the set of morphisms of crossed complexes $\mathsf{Crs}(\mathbb{F}(n), C)$, where $\mathbb{F}(n)$ is the free crossed complex on one generator of dimension n (see Remark following Definition 7.4.8).

A monoidal closed category structure on Crs is given by a tensor product of crossed complexes $-\otimes-$ construction, an internal hom construction $\mathsf{CRS}(-,-)$, which is going to be a crossed complex having $\mathsf{Crs}(-,-)$ as set of objects, and a natural isomorphism

$$\mathsf{Crs}(C \otimes D, E) \;\cong\; \mathsf{Crs}(C, \mathsf{CRS}(D, E)),$$

for all crossed complexes C, D, E. When we take $C = \mathbb{F}(n)$, we have

$$\mathsf{Crs}(\mathbb{F}(n) \otimes D, E) \;\cong\; \mathsf{Crs}(\mathbb{F}(n), \mathsf{CRS}(D, E)) \;\cong\; \mathsf{CRS}_n(D, E)$$

So, the elements of $\mathsf{CRS}_n(D, E)$ can be seen as 'n-fold left homotopies' $D \to E$. In particular, for $n = 1$, we may define the set of homotopy classes of morphisms of crossed complexes and prove

$$[D, E] = \pi_0(\mathsf{CRS}(D, E)).$$

9.2 Monoidal closed structure on Mod.

There are well known definitions of tensor product and internal hom functor for Abelian groups (without operators). If one allows operators from arbitrary groups the tensor product is easily generalised, with the tensor product of a G-module and an H-module being a $(G \times H)$-module. However, the adjoint construction of internal hom functor does not exist, basically because the group morphisms from G to H do not form a group. To rectify this situation we allow operators from arbitrary groupoids, rather than groups, and we give a discussion of the monoidal closed category structure of Mod the category of modules over groupoids introduced in Definition **??**.

As is customary, we write M for the G-module (M, G) when the operating groupoid G is clear from the context. Also, to simplify notation, we will assume throughout this chapter that the Abelian groups $M(x)$ for $x \in G_0$ are all disjoint; any G-module is isomorphic to one of this type.

In many of our categories, it will be easier to describe internal homs explicitly, than the corresponding tensor product. We illustrate this by describing the internal hom structure in the category Mod.

First note that in the Appendix B we describe an internal hom groupoid $\mathsf{GPDS}(G, H)$ in the category Gpds, whose objects are functors $f : G \to H$ and whose morphisms are natural transformations $\phi : f \to f'$. Notice also that these natural transformation ϕ are given by a family $\{\phi(x)\}_{x \in G_0}$ where $\phi(x) \in H(f(x), f'(x))$ and the diagram

$$\begin{tikzcd}
f(x) \arrow[r, "\phi(x)"] \arrow[d, "f(g)"'] & f'(x) \arrow[d, "f'(g)"] \\
f(y) \arrow[r, "\phi(y)"'] & f'(y)
\end{tikzcd}$$

commutes for all $g \in G(x, y)$.

Definition 9.2.1 Let $(M, G), (N, H)$ be modules. To construct the *internal hom* $\mathsf{MOD}((M, G), (N, H))$ we consider the set of morphisms of modules

$$\mathsf{Mod}((M, G), (N, H)) = \{(\theta, f) : (M, G) \to (N; H) \mid (\theta, f) \text{ is a morphism of modules}\}$$

and we have to give this set the structure of module over a groupoid. Notice that θ is given by a family $\{\theta(x)\}_{x \in G_0}$ where

$$\theta(x) : M(x) \to N(f(x))$$

are group morphisms satisfying $\theta(x)(m^g) = \theta(x)(m)^{f(g)}$.

For a fixed functor $f : G \to H$, we define the set of morphisms of modules over f,

$$\mathsf{Mod}((M, G), (N, H))(f) = \{(\theta, f) : (M, G) \to (N; H) \mid (\theta, f) \text{ is a morphism of modules}\}$$

It is easy to see that each $\mathsf{Mod}((M, G), (N, H))(f)$ forms an Abelian group under element-wise addition, so all morphisms $\mathsf{Mod}((M, G), (N, H))$ form a family of Abelian groups indexed by the set of objects of the groupoid $\mathsf{GPDS}(G, H)$.

$$\mathsf{MOD}((M, G), (N, H)) = \{\mathsf{Mod}((M, G), (N, H))(f)\}_{f \in \mathsf{Gpds}(G,H)}.$$

It remains to describe the action of $\mathsf{GPDS}(G, H)$ on $\mathsf{MOD}((M, G), (N, H))$, i.e. for each $f, f' \in \mathsf{Gpds}(G, H)$ we need a map

$$\mathsf{Mod}((M, G), (N, H))(f) \times \mathsf{GPDS}(G, H)(f, f') \to \mathsf{Mod}((M, G), (N, H))(f')$$

So let θ be such that (θ, f) is a morphism of modules and let $\phi : f \to f'$ be a natural transformation. We define

$$(\theta, f)^{\phi} = (\theta^{\phi}, f')$$

where θ^{ϕ} is a family of morphisms

$$\theta^{\phi}(x) : M(x) \to N(f'(x)),$$

where $x \in G_0$ and $\theta^{\phi}(x)$ is defined as the composition

$$M(x) \xrightarrow{\theta(x)} N(f(x)) \xrightarrow{(-)^{\phi(x)}} N(f'(x)),$$

i.e. $\theta^{\phi}(x)(m) = (\theta(x)(m))^{\phi(x)}$. They give a morphism because

$$\theta^{\phi}(x)(m^g) = (\theta(x)(m^g))^{\phi(x)} = (\theta(x)(m))^{f(g)\phi(x)} = (\theta(x)(m))^{\phi(x)f'(g)} = (\theta^{\phi}(x)(m))^{f'(g)}.$$

It is not difficult to prove that this definition satisfies the properties of an action giving a structure of module

$$\mathsf{MOD}(M, N) = (\mathsf{Mod}((M, G), (N, H)), \mathsf{GPDS}(G, H))$$

which is the internal hom functor in Mod. □

It is quite straightforward to see that, as in the group case, we can characterise the elements of this internal hom functor in terms of 'bilinear' maps.

Definition 9.2.2 A *bilinear map of modules over groupoids* $(M, G) \times (N, H) \to (P, K)$ is given by a pair of maps (θ, f) where $f : G \times H \to K$ is a map of groupoids and $\theta : M \times N \to P$ is given by a family of bilinear maps $\theta(x, y) : M(x) \times N(y) \to P(f(x, y))$ which preserve actions, i.e.

$$\theta(x, y)(m^g, n^h) = (\theta(x, y)(m, n))^{f(g,h)}.$$

□

Proposition 9.2.3 *There is a natural bijection between bilinear maps* $M \times N \to P$ *and morphisms of modules from* M *to* $\mathsf{MOD}(N, P)$).

Proof Let us consider an element $(\theta, f) \in \mathsf{Mod}(M, \mathsf{MOD}(N, P))$ then we can define

$$\hat{f}(x, y) = f(x)(y) \quad \text{and} \quad \hat{\theta}(m, n) = \theta(m)(n).$$

It is easy to see that $(\hat{\theta}, \hat{f})$ is a bilinear map and that this assignation is a natural bijection. □

Now the tensor product as just defined is the one that transforms these bilinear maps into morphisms of modules.

Definition 9.2.4 The *tensor product* in Mod of modules $(M, G), (N, H)$ is the module

$$(M \otimes N, G \times H)$$

where, for $x \in G_0$, $y \in H_0$, $(M \otimes N)(x, y) = M(x) \otimes_{\mathbb{Z}} N(y)$ and the action is given by

$$(m \otimes n)^{(g,h)} = m^g \otimes n^h.$$

□

Remark 9.2.5 The module $M \otimes N$ is the $(G \times H)$-module generated by all elements

$$\{m \otimes n \mid m \in M, n \in N\}$$

subject to the relations

$$(m + m') \otimes n = (m \otimes n) + (m' \otimes n),$$
$$m \otimes (n + n') = (m \otimes n) + (m \otimes n'),$$
$$(m \otimes n)^{(g,h)} = m^g \otimes n^h.$$

Thus to define a morphism $M \otimes N \to P$ all we need is a bilinear map $M \times N \to P$. $\Box$

Proposition 9.2.6 *There is a natural bijection between bilinear maps* $M \times N \to P$ *and morphisms of modules from* $M \otimes N$ *to* P.

Proof Let us consider a bilinear map $(\theta, f) : M \times N \to P$. Then we can define

$$\hat{f}(x, y) = f(x)(y) \quad \text{and} \quad \hat{\theta}(m \otimes n) = \theta(m, n).$$

It is easy to see that $(\hat{\theta}, \hat{f})$ is a morphism of modules ant that this assignation is a natural bijection. $\Box$

The tensor product gives the category Mod a symmetric monoidal structure with unit object the module $(\mathbb{Z}, 1)$, where 1 denotes the trivial group seen as a groupoid.

Let us see that both the tensor product and the internal morphisms just defined give Mod the structure of symmetric monoidal closed category.

Proposition 9.2.7 *There is a natural bijection*

$$\mathsf{MOD}(L \otimes M, N) \cong \mathsf{MOD}(L, \mathsf{MOD}(M, N)).$$

Proof It is straightforward to verify the natural bijection

$$\mathsf{Mod}(L \otimes M, N) \cong \mathsf{Mod}(L, \mathsf{MOD}(M, N)),$$

where L is a G-module.

These families of groups are modules over $\mathsf{GPDS}(G \times H, K) \cong \mathsf{GPDS}(G, \mathsf{GPDS}(H, K))$ and the actions agree, giving a natural isomorphism of modules

$$\mathsf{MOD}(L \otimes M, N) \cong \mathsf{MOD}(L, \mathsf{MOD}(M, N)).$$

$\Box$

Should not symmetry be checked?

9.3 Monoidal closed structure on Crs

Analogously to the way the internal morphisms gave a correspondence from morphisms in the internal hom construction to bilinear maps and then to morphisms of the tensor product, as in

$$\mathsf{Mod}_R(C \otimes D, E) \cong \mathsf{Mod}_R(C, \mathsf{MOD}_R(D, E)),$$

so we obtain an internal hom $\mathsf{CRS}(D, E)$ for crossed complexes D, E as part of an exponential law

$$\mathsf{Crs}(C \otimes D, E) \cong \mathsf{Crs}(C, \mathsf{CRS}(D, E)),$$

for crossed complexes C, D, E where the internal hom $\mathsf{CRS}(D, E)$ is of course again a crossed complex. Crossed complexes have structure in a range of dimensions, whereas R-modules have structure in just one dimension, so the description of the internal hom in Crs has to be much more complicated than that in Mod_R, and indeed this complication is part of its value in modeling complicated geometry.

We define the internal hom for crossed complexes as giving a 'home' for the notion of 'higher dimensional homotopy', and then explain the tensor product for crossed complexes. This necessitates defining the notion of *bimorphism*

$$b : (C, D) \to E$$

for crossed complexes C, D, E, so that such bimorphisms correspond exactly to morphisms $C \to \mathsf{CRS}(D, E)$.

The algebraic properties of bimorphisms are quite complicated, but also reflect some important geometric properties, namely the cellular subdivision of products $\mathbf{E}^m \times \mathbf{E}^n$ of cells $\mathbf{E}^m$. Here we have

$$\mathbf{E}^0 = \{1\}, \quad \mathbf{E}^1 = e^0_\pm \cup e^1, \quad \mathbf{E}^m = e^0 \cup e^{m-1} \cup e^m, \ m \geqslant 2,$$

where $e^0_- = -1, e^0_+ = 1$. Thus in general the product of these cells has a cell structure with 9 cells.

The picture for the cylinder $\mathbf{E}^1 \times \mathbf{E}^2$ is as follows.

[Cylinder picture, horizontally for $\mathbf{E}^1$ direction]

We cannot draw the picture for $\mathbf{E}^2 \times \mathbf{E}^2$, but that structure contains two solid tori, one of which is pictured as follows

[torus picture]

and which can be seen as the above cylinder with the two ends identified. Note that the boundary of $\mathbf{E}^2 \times \mathbf{E}^2$ is homeomorphic to a 3-sphere. This can be represented as the set of points $(x, y, z, w) \in \mathbb{R}^4$ such that $x^2 + y^2 + z^2 + w^2 = 1$ and one of the solid tori is represented by the subset of $\mathbf{S}^3$ of points such that

$$x^2 + y^2 \leqslant 1/2, \quad \text{whence } z^2 + w^2 \geqslant 1/2.$$

The corresponding algebraic expression for the boundary of the solid cylinder $e^1 \times e^2$ should involve the cells $e^1 \times e^1, e^0_- \times e^2, e^0_+ \times e^2$. Our conventions set the base point of the cylinder at $(1, 1)$, i.e. at the 'top' end of the cylinder. In the end we take the boundary to be

$$\delta(e^1 \times e^2) = -(e^1 \times e^1) - (1 \times e^2) + (-1 \times e^2)^{e^1 \times 1},$$

where the conventions as to sign and order of the terms come from some other considerations which we explain later. When we come to take the boundary in the solid torus in $\mathbf{E}^2 \times \mathbf{E}^2$ we get a similar formula, except that now $-1 \times e^2, 1 \times e^2$ are identified to $1 \times e^2$ and so the formula becomes

$$\delta(e^1 \times e^2) = -(e^1 \times e^1) - (1 \times e^2) + (1 \times e^2)^{e^1 \times 1},$$

which relates to our picture of the solid torus.

Another complication is when we glue two cylinders together as in

[gluing picture].

The base point of the whole cylinder is at the right hand end, but the base point of the first cylinder is half way along. Thus the algebraic formulae have to reflect this.

Finally, we have to distinguish the formulae for $\mathbf{E}^m \times \mathbf{E}^n$ for m, n odd, even, and equal to 0,1,or $\geqslant 2$.

All these complications are reflected in the notion of a *bimorphism* given in subsection 8.3.3. They imply quite different formulae for the operations on $c \otimes d$ for the dimensions of c or d being $0, 1$ or $\geqslant 2$.

9.3.1 The internal hom structure in Crs

Recall from the introduction to this Chapter, that the elements of $\mathsf{CRS}_n(C, D)$ can be seen as n-fold homotopies

$$\mathbb{F}(n) \otimes C \to D$$

reflecting the geometry of $\mathbb{F}(n)$. In particular, $\mathbb{F}(1) = \mathcal{I}$ is the unit interval, giving a cylinder construction.

An advantage of this viewpoint is that the elements of the internal hom crossed complex $\mathsf{CRS}(B, C)$ in dimension n have a nice interpretation. What is not so clear is that these elements taken altogether can be given the structure of crossed complex.

This difficulty is overcome in Chapter 15 in Part III by working with a different but equivalent structure, that of ω-groupoids, which is based on cubes.

So in this section, our aim is not to give the full justification of the results, but hope to explain their intuitive content. We begin the definition of $\mathsf{CRS}(C, D)$ from the bottom dimension upwards.

In dimension 0, $\mathsf{CRS}(C, D)_0$ is a set defined as

$$\mathsf{CRS}_0(C, D) = \mathsf{Crs}(C, D).$$

For dimension 1, we use the concept of left (1-)homotopy, which has many points of contact with the concept of homotopy between morphisms of chain complexes.

Definition 9.3.1 Let C, D be crossed complexes and let

$$f^0, f^1 : C \to D$$

be morphisms of crossed complexes (i.e. elements of $\mathsf{CRS}_0(C, D)$). A *left (1-)homotopy* from f^0 to f^1

$$H : f^0 \sim f^1$$

is a "map of degree 1 from C to D over f^0_0 starting at f^0 and ending at f^1". Here is the definition of this term.

1.- "a map of degree 1" from C to D

For each $n \geqslant 0$, we have a map

$$H_n : C_n \to D_{n+1}.$$

This sequence of maps can be written

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[dl, "H_{n+1}"'] & C_n \arrow[r, "\delta_n"] \arrow[dl, "H_n"'] & C_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[dl, "H_{n-1}"'] & \cdots \arrow[r, "\delta_3"] & C_2 \arrow[r, "\delta_2"] \arrow[dl, "H_2"'] & C_1 \arrow[r, "s,t"] \arrow[dl, "H_1"'] & C_0 \arrow[d, "f_0"] \arrow[dl, "H_0"'] \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\delta_{n+1}"] & D_n \arrow[r, "\delta_n"] & \cdots \arrow[r, "\delta_4"] & D_3 \arrow[r, "\delta_3"] & D_2 \arrow[r, "\delta_2"] & D_1 \arrow[r, "s,t"] & D_0
\end{tikzcd}$$

2.- All H_n for $n \geqslant 0$ have to be "over f^0_0", i.e. H_n is a family of maps $\{H_n(x)\}_{x \in C_0}$ as follows:

- In dimension 0, for each $x \in C_0$, $H_0(x)$ connects f^0 and f^1, i.e.

$$H_0(x) : f_0^1 x \to f_0^0 x$$

- For $n \geqslant 1$, the map is over f_0^0, i.e., for each $x \in C_0$,

$$H_n(x) : C_n(x) \to D_{n+1}(f_0^0 x).$$

3.- Moreover, for $n \geqslant 1$, we ask the H_n to preserve **operation** and **action** in the best possible way:

- For $n \geqslant 2$,

- H_n preserve action over f_1^0, i.e. if $c \in C_n$ and $c_1 \in C_1$, if c^{c_1} is defined, then

$$H_n(c^{c_1}) = H_n(c)^{f_1^0(c_1)} \quad \text{and}$$

- H_n are linear, i.e. for $c, c' \in C_n$, if $c + c'$ is defined, then

$$H_n(c + c') = H_n(c) + H_n(c').$$

- When $n = 1$, H_1 is a (left) derivation over f^0, i.e. for $c, c' \in C_1$, if cc' is defined, then

$$H_1(cc') = H_1(c)^{f_1^0(c')} + H_1(c').$$

(see Remark 9.3.4 for more details)

To have maps H_n satisfying the preceding properties is just to have a homotopy.

4.- The homotopy H is "from f^0 to f^1" if in the diagram

$$\begin{tikzcd}
 & C_n \arrow[r, "\delta_n"] \arrow[dl, "H_n"'] \arrow[d, shift right, "f_n^0"'] \arrow[d, shift left, "f_n^1"] & C_{n-1} \arrow[dl, "H_{n-1}"] \\
D_{n+1} \arrow[r, "\delta_{n+1}"] & D_n &
\end{tikzcd}$$

the "difference" between the two vertical maps is given by the sum of the two triangles. This requires some care with the base point and to treat as different the cases $n \geqslant 2$ and $n = 1$

- For $n \geqslant 2$,

$$f_n^1(c) = [f_n^0(c) + H_{n-1}(\delta_n c) + \delta_{n+1}(H_n c)]^{-H_0(tc)},$$

where $f_n^0(c) + H_{n-1}(\delta_n c) + \delta_{n+1}(H_n c)$ comes from the diagram and the action is used to change base point from $f_0^0(tc)$ to $f_0^1(tc)$, and

- for $n = 1$,

$$f_1^1(c) = H_0(sc) f_1^0(c) \delta_2(H_1 c) H_0(tc)^{-1},$$

With all these preliminaries, we define the groupoid $\mathsf{CRS}_1(C, D)$ as having $\mathsf{Crs}_0(C, D) = \mathsf{Crs}(C, D)$ as objects, the morphisms from $f^0 : C \to D$ to $f^1 : C \to D$ are the homotopies, i.e.

$$\mathsf{CRS}_1(C, D)(f^0, f^1) = \{H : f^0 \sim f^1 \mid \text{homotopies from } f^0 \text{ to } f^1\}.$$

The composition is given as follows:

Let $H : f^0 \sim f^1$ and $K : f^1 \sim f^2$ be left homotopies, then we define $H + K : f^0 \sim f^2$ by

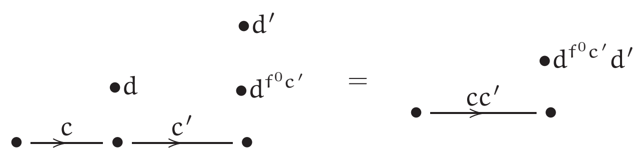
$$(H + K)_n(c) = \begin{cases} K_n(c) + H_n(c)^{K_0(tc)} & \text{if } c \in C_n, n \geqslant 1, \\ H_0(c) + K_0(c) & \text{if } c \in C_0. \end{cases}$$

$\square$

Exercise 9.3.2 Prove that $\mathsf{CRS}_1(C, D)$ is, with this addition, a groupoid. Deduce that homotopy between morphisms of crossed complexes is an equivalence relation. The quotient set is called $[C, D]$. □

Exercise 9.3.3 Define the notion of homotopy equivalence $f : C \to D$ of crossed complexes. Recall that a morphism $f : C \to D$ of crossed complexes induces morphisms of the fundamental groupoids and homology groups. Prove that a homotopy equivalence of crossed complexes induces an equivalence of fundamental groupoids. What can you say about the induced morphism of homology groups? □

Remark 9.3.4 Let us expand a bit on the fact that H_1 is an f^0-*derivation*. Note that C_1 operates on D_2 via f^0 and so we can form the semidirect product groupoid $C_1 \ltimes D_2$ with projection pr_1 to C_1. This groupoid has objects C_0 and arrows pairs $(c, d) \in C_1 \times D_2$, such that $f^0_0\delta_1(c) = t(d)$, with composition $(c, d)(c', d') = (cc', d^{f^0c'}d')$. This can be seen in the picture



It is then easily seen that an f^0-derivation H_1 is determined completely by a morphism $H_1' : C_1 \to C_1 \ltimes D_2$ such that $\mathrm{pr}_1 H_1' = 1_{C_1}$. A corollary is that if C_1 is a free groupoid, then an f^0-derivation is completely determined by its values on a set of free generators of C_1. □

Remark 9.3.5 Notice that the definition of the derivation has been on the left. Sometimes 'right derivations' are useful governed by the rule

$$H_1(cc') = H_1(c') + H_1(c)^{f_1^0(c')}.$$

□

Now we turn to the general structure, defining $\mathsf{CRS}_m(C, D)(f)$ for <u>dimension $m \geqslant 2$</u> and $f \in \mathsf{Crs}(C, D)$; providing it with an action of $\mathsf{CRS}_1(C, D)$ and defining the "boundary" maps.

Definition 9.3.6 Let C, D be crossed complexes and let $m \geqslant 2$. Then an m-*fold homotopy from* C *to* D *over* f is a pair (H, f), where $f : C \to D$ is a morphism of crossed complexes (the *base morphism of the homotopy*) and H is a map of degree m from C to D given by functors $H_n : C_n \to D_{n+m}$ for each $n \geqslant 0$ that are morphism of modules over the morphism f_1 of groupoids, i.e.,

• Relations with **actions** for $n \geqslant 2$

H_n preserve action, i.e. if $c \in C_n$ and $c_1 \in C_1$, then

$$H_n(c^{c_1}) = H_n(c)^{f_1(c_1)}.$$

• Relations with **operations** for $n \geqslant 1$

- H_n are linear for $n \geqslant 2$, i.e. if $c, c' \in C_n$ and $c + c'$ is defined, then

$$H_n(c + c') = H_n(c) + H_n(c').$$

- H_1 is a derivation over f, i.e. if $c, c' \in C_1$ and $c + c'$ is defined, then

$$H_1(cc') = H_1(c)^{f_1(c')} + H_1(c');$$

Thus, in each dimension, H and f preserve structure in the only reasonable way. (However, there is no requirement that H should be compatible with the boundary maps $\delta_n : C_n \to C_{n-1}$ and $\delta'_n : D_n \to D_{n-1}$).

We define

$$\mathsf{CRS}(C, D)_m(f) = \{H \mid (H, f) \text{ are } m\text{-fold homotopies}\}.$$

□

Remark 9.3.7 For $m \geqslant 2$ there is no difference between definition on the left (as given) and on the right, because H_n takes images in abelian groupoids for $n \geqslant 1$. □

Let us see how this family of sets get the structure of a crossed complex.

Definition 9.3.8 The operations, action and boundary maps on $\mathsf{CRS}_m(C, D)$ are given by:

1.- Operations on $\mathsf{CRS}_m(C, D)$.- If $(H, f), (K, f)$ are m-fold homotopies $C \to D$ over the same base morphism f, where $m \geqslant 2$, we define

$$(H + K)(c) = H(c) + K(c)$$

for all $c \in C$.

2.- Actions on $\mathsf{CRS}_m(C, D)$.- If (H, f^0) is an m-fold homotopy $C \to D$ and if $K : f^0 \sim f^1$ is a left homotopy, then we define

$$H^K(c) = H(c)^{K(tc)}$$

for all $c \in C$. Then (H^K, f^1) is a morphisms of modules.

3.- Boundaries on $\mathsf{CRS}_m(C, D)$.- If (H, f) is an m-fold homotopy with $m \geqslant 2$, we define the boundary

$$\delta(H, f) = (\delta H, f)$$

where δH is the $(m - 1)$-fold homotopy given by

$$(\delta H)(c) = \begin{cases} \delta(H(c)) + (-1)^{m+1} H(\delta c) & \text{if } c \in C_n (n \geqslant 2), \\ (-1)^{m+1} H(sc)^{f(c)} + (-1)^m H(tc) + \delta(H(c)) & \text{if } c \in C_1, \\ \delta(H(c)) & \text{if } c \in C_0. \end{cases}$$

For 1-homotopies the boundaries are the source and the target already defined (the initial and final morphisms). □

Theorem 9.3.9 *The above operations give* $\mathsf{CRS}(C, D)$ *the structure of crossed complex.*

Proof This would be somewhat tedious to verify directly, and instead we rely on the fact that this internal hom structure for Crs is derived from the more easily verified internal hom structure on the category of ω-groupoids, given in Chapter 15, and the equivalence between the two categories given in Chapter 13. □

The specific conventions used for constructing the equivalence between crossed complexes and ω-groupoids impose the conventions we use for the internal hom structure on $\mathcal{C}rs$, and hence for the tensor product. The fact that $\mathsf{CRS}(C, D)$ is a crossed complex contains a lot of information.

The formulae for m-fold homotopies are exactly what is needed to express the geometry of the cylinder $\mathbf{I}_* \times \mathbf{E}^n_*$ because a 1-homotopy can be seen as a morphism $\mathcal{I} \otimes C \to D$. We shall now concentrate on 1-homotopies.

Remark 9.3.10 An important observation which we will use later is that if f^0, f^1 are given and $c \in C_n$ then $\delta_{n+1} H_n(c)$ is determined by $H_0 tc$ and $H_{n-1}\delta_n(c)$. This is a key to later inductive constructions of homotopies. □

9.3.2 The bimorphisms as an intermediate step

With the structure of crossed complex on $\mathsf{CRS}(C, D)$ just described, we may study the crossed complex morphisms $\mathcal{C}rs(C, \mathsf{CRS}(D, E))$, see how they are defined and reorganise the data. Such a morphism is given by a family of maps $f_m : C_m \to \mathsf{CRS}_m(D, E)$ commuting with the boundary maps. For each $c \in C_m$, $f_m(c)$ is a homotopy, i.e. a family of maps $f_m(c)_n : D_n \to E_{m+n}$ satisfying some conditions.

We can reorganise these maps, getting a family

$$f_{m,n} : C_m \times D_n \to E_{m+n}$$

and see what the different conditions mean for these maps. That gives the notion of bimorphism.

For the rest of this subsection we use additive notation in all dimensions (including 1 and 2) to reduce the number of formulae.

Definition 9.3.11 A *bimorphism* $\theta : (C, D) \to E$ for crossed complexes C, D, E is a family of maps

$$\theta_{mn} : C_m \times D_n \to E_{m+n}$$

so that, for every $c \in C_m$, the map $\theta_m(c) = \{\theta_{mn}(c, -)\}_{n \in \mathbb{N}}$ is an m-homotopy. That means that the θ_{mn} have to satisfy the following conditions, where $c \in C_m, d \in D_n, c_1 \in C_1, d_1 \in D_1$:

- **Source and target** They preserve target and, whenever appropriate, source

$$\begin{aligned} t(\theta(c, d)) &= \theta(tc, td) \quad \text{for all } c \in C, d \in D\ . \\ s(\theta(x, d)) &= \theta(x, sd) \quad \text{if } m = 0, n = 1\ , \\ s(\theta(c, y)) &= \theta(sc, y) \quad \text{if } m = 1, n = 0\ . \end{aligned}$$

- **Actions** They preserve the action in dimensions $\geqslant 2$

$$\begin{aligned} \theta(c, d^{d_1}) &= \theta(c, d)^{\theta(tc, d_1)} \quad \text{if } m \geqslant 0, n \geqslant 2\ , \\ \theta(c^{c_1}, d) &= \theta(c, d)^{\theta(c_1, td)} \quad \text{if } m \geqslant 2, n \geqslant 0\ . \end{aligned}$$

- **Operations** They preserve compositions in c and d as far as possible:

- For $m \neq 1$ or $n \neq 1$, the θ_{mn} are bimorphisms

$$\begin{aligned} \theta(c, d + d') &= \theta(c, d) + \theta(c, d') \qquad \text{if } m = 0, n \geqslant 1 \text{ or } m \geqslant 1, n \geqslant 2\ , \\ \theta(c + c', d) &= \theta(c, d) + \theta(c', d) \qquad \text{if } m \geqslant 1, n = 0 \text{ or } m \geqslant 2, n \geqslant 1. \end{aligned}$$

- Whenever $m = 1$ or $n = 1$ they behave in each of c, d like derivations

$$\begin{aligned} \theta(c, d + d') &= \theta(c, d)^{\theta(tc, d')} + \theta(c, d') && \text{if } m \geqslant 1, n = 1 \ , \\ \theta(c + c', d) &= \theta(c', d) + \theta(c, d)^{\theta(c', td)} && \text{if } m = 1, n \geqslant 1 \ . \end{aligned}$$

• **Boundaries** The complications here reflect the geometry.

- In high dimensions, the boundary is analogous to that in chain complexes:

$$\delta_{m+n}(\theta(c, d)) = \theta(\delta_m c, d) + (-1)^m \theta(c, \delta_n d) \quad \text{if } m \geqslant 2, n \geqslant 2.$$

- When one of the elements has dimension 1, we have to take account of the action to put elements at the right base point

$$\delta_{m+n}(\theta(c, d)) = \begin{cases} -\theta(c, \delta_n d) - \theta(tc, d) + \theta(sc, d)^{\theta(c, td)} & \text{if } m = 1, n \geqslant 2 \ , \\ (-1)^{m+1}\theta(c, td) + (-1)^m \theta(c, sd)^{\theta(tc, d)} + \theta(\delta_m c, d) & \text{if } m \geqslant 2, n = 1 \ , \\ -\theta(tc, d) - \theta(c, sd) + \theta(sc, d) + \theta(c, td) & \text{if } m = n = 1 \ . \end{cases}$$

- And, whenever one of the elements has dimension 0, we operate only on the other part.

$$\delta_{m+n}(\theta(c, d)) = \begin{cases} \theta(c, \delta_n d) & \text{if } m = 0, n \geqslant 2 \ , \\ \theta(\delta_m c, d) & \text{if } m \geqslant 2, n = 0 \ . \end{cases}$$

□

You should look at these carefully and note (but not necessarily learn!) the way these formulae reflect the geometry and algebra of crossed complexes, which allow for differences between the various dimensions, and also for change of base point.

The bimorphisms are used as an intermediate step in the construction of the tensor product due to the following property

Theorem 9.3.12 *For crossed complexes* C, D, E*, there is a natural bijection from* $\mathsf{Crs}(C, \mathsf{CRS}(D, E))$*, to the set of bimorphisms* $\theta : (C, D) \to E$.

9.3.3 The tensor product of two crossed complexes

Following the pattern in the tensor product of R-modules, we now 'internalise' the concept of bimorphism. That is, we construct a crossed complex, the tensor product $C \otimes D$ of two crossed complexes, and a universal bimorphism

$$\Upsilon : (C, D) \to C \otimes D,$$

so that the bimorphisms $(C, D) \to E$ correspond exactly to the morphisms $C \otimes D \to E$. In effect, this implies that $C \otimes D$ is generated by elements $c \otimes d$, with $c \in C_m$ and $d \in D_n$, $m, n \geqslant 0$, subject to the relations given by the rules of bimorphisms with $\theta(c, d)$ replaced by $c \otimes d$.

We shall also describe $(C \otimes D)_p$ in terms of pieces $(C \otimes D)_{m,n}$ with $m + n = p$, which, from the rules for θ_{mn}, can be given more explicitly in terms of the structures on C_m, D_n.

Let us start by making clear the implication for the groupoid part of $C \otimes D$.

For $p = 0$, we define

$$(C \otimes D)_0 = C_0 \times D_0$$

as sets.

For $p=1$, the groupoid $(C\otimes D)_1$ over $(C\otimes D)_0$ is determined by two parts, namely $(C\otimes D)_{1,0} = C_1\times D_0$ and $(C\otimes D)_{0,1} = C_0\times D_1$. Then $(C\otimes D)_1$ is their coproduct as groupoids over $(C\otimes D)_0$, we write

$$(C\otimes D)_1 = C_1 \# D_1 .$$

This groupoid may be seen also as generated by the symbols

$$\{c\otimes y \mid c\in C_1\}\cup\{x\otimes d\mid d\in D_1\}$$

for all $x\in C_0$ and $y\in D_0$ subject to the relations given by the products in C_1 and on D_1. We shall return to this in Subsection 9.4.1.

Also, we shall prove in Subsection 9.4.2 that the image of $\delta_2(C\otimes D)_2$ in $(C\otimes D)_1$ is generated by all the elements of the set

$$\{\delta c\otimes y\mid c\in C_2\}\cup\{x\times\delta d\mid d\in D_2\}\cup\{(c\otimes y)(x\otimes d)(c\otimes y')^{-1}(x'\otimes d)^{-1}\mid c\in C_1, d\in D_1\}$$

for all $x\in C_0$ and $y\in D_0$ (Notice that the last set consists of the commutators of the generators of $(C\otimes D)_{1,0} = C_1\times D_0$ and $(C\otimes D)_{0,1} = C_0\times D_1$).

Now this has been recorded, we can proceed with the definition of $(C\otimes D)_p$ for $p\geqslant 2$.

Definition 9.3.13 Let C, D be crossed complexes. For any $c\in C_m, d\in D_n$ we consider the symbol $c\otimes d$. Whenever $m+n\geqslant 2$, we define its source and target

$$s(c\otimes d) \;=\; t(c\otimes d) \;=\; tc\otimes td.$$

(Notice that for elements of dimension 0 we define $t(x)=x$ and $t(y)=y$.)

For $p\geqslant 2$, we consider F_p the free $(C\otimes D)_1$-module (or crossed module if $p=2$) on

$$\{c\otimes d\mid c\in C_m, d\in D_n, m,n\in\mathbb{N}, m+n=p\}.$$

To get $(C\otimes D)_p$ we have to quotient out by some relations with respect to the additions and actions. Notice that all relations are "dimension preserving". There are two essentially different cases.

• When both $m,n\neq 1$, we do not have to worry about source and target (both are the same), and the relations are easier:

- Additions: The relations to make $\otimes$ compatible with additions are

$$c\otimes(d+d') = c\otimes d + c\otimes d' \quad \text{if } n\geqslant 2$$

$$(c+c')\otimes d = c\otimes d + c'\otimes d \quad \text{if } m\geqslant 2.$$

- Action: The relations to make $\otimes$ compatible with the actions are

$$\begin{aligned}(c\otimes d)^{(tc\otimes d_1)} &= c\otimes d^{d_1} \text{ if } m\geqslant 0, n\geqslant 2\,,\\ (c\otimes d)^{(c_1\otimes td)} &= c^{c_1}\otimes d \text{ if } m\geqslant 2, n\geqslant 0\,.\end{aligned}$$

what does the following mean? and it is compatible with the relations.

- Cokernel. When $m+n\geqslant 3$, we have to kill the action of $\delta_2(C\otimes D)_2\subseteq(C\otimes D)_1$

• When one element has dimension 1.

- Then the operation has to be related with the action because the groupoid part acts on itself by conjugation.

$$\begin{aligned} c \otimes dd' &= (c \otimes d)^{(tc \otimes d')} + c \otimes d' && \text{if } m \geqslant 1, n = 1 \ , \\ cc' \otimes d &= c' \otimes d + (c \otimes d)^{(c' \otimes td)} && \text{if } m = 1, n \geqslant 1 \ . \end{aligned}$$

- Cokernel. When $m + n \geqslant 3$, we have to kill the action of $\delta_2(C \otimes D)_2 \subseteq (C \otimes D)_1$.

With this, we get $(C \otimes D)_p$ as the quotient of F_p by all these relations.

To finish the structure of $C \otimes D$ as a crossed complex, the **boundaries** are defined on generators with formulae varying according to dimensions.

• When both have dimension $\geqslant 2$

$$\delta_{m+n}(c \otimes d) = \delta_m c \otimes d + (-1)^m (c \otimes \delta_n d)$$

• When one has dimension 1 and the other one has dimension $\geqslant 1$

$$\delta_{m+n}(c \otimes d) \ = \ \begin{cases} -(c \otimes \delta_n d) - (tc \otimes d) + (sc \otimes d)^{(c \otimes td)} & \text{if } m = 1, n \geqslant 2 \ , \\ (-1)^{m+1}(c \otimes td) + (-1)^m (c \otimes sd)^{(tc \otimes d)} + (\delta_m c \otimes d) & \text{if } m \geqslant 2, n = 1 \ , \\ -(tc \otimes d) - (c \otimes sd) + (sc \otimes d) + (c \otimes td) & \text{if } m = n = 1 \end{cases}$$

• When one has dimension 0

$$\delta_{m+n}(c \otimes d) \ = \ \begin{cases} (c \otimes \delta_n d) & \text{if } m = 0, n \geqslant 2 \ , \\ (\delta_m c \otimes d) & \text{if } m \geqslant 2, n = 0 \ . \end{cases}$$

and these definitions are compatible with the relations.

□

Remark 9.3.14 Notice that if we denote by $F_{m,n}$ the free $(C \otimes D)_1$-module on $\{c \otimes d \mid c \in C_m, d \in D_n\}$ for some fixed $m, n \in \mathbb{N}$, F_p is the coproduct of $\{F_{m,n}\}_{m+n=p}$.

Since the relations with respect to the additions and actions we are using to get $(C \otimes D)_p$ preserve the decomposition of F_p as the coproduct of $F_{m,n}$, $(C \otimes D)_p$ also decomposes as coproduct of the quotient of $F_{m,n}$ respect to the corresponding relations. We shall call $(C \otimes D)_{m,n}$ this quotient. □

Remark 9.3.15 There is an alternative way of defining $F_{m,n}$ that works when $m, n \neq 0$, $m + n \geqslant 3$.

We could define $F'_{m,n}$ as the free abelian groupoid on $\{c \otimes d \mid c \in C_m, d \in D_n\}$ and quotient out by the relations on operations included in the previous definition getting an abelian groupoid $C'_{m,n}$. This quotient is isomorphic to $(C \otimes D)_{m,n}$ as abelian groupoid.

Next we define the $(C \otimes D)_1$-action on $C'_{m,n}$ by the formulae in the previous definition (notice that the definition is different when $m = 1$ or $n = 1$). It is not difficult to prove that this gives an action and that $C'_{m,n}$ is isomorphic to $(C \otimes D)_{m,n}$ as $(C \otimes D)_1$-modules □

Exercise 9.3.16 Check the rule $\delta\delta(c \otimes d) = 0$ for some low dimensional cases, such as $\dim(c \otimes d) = 3, 4$, seeing how the crossed module rules come into play. □

Those familiar with the tensor product of chain complexes may note that in that theory the single and simple formula we need is

$$\partial(c \otimes d) = (\partial c) \otimes d + (-1)^m c \otimes (\partial d)$$

where $\dim c = m$. So it is not surprising that the tensor product of crossed complexes has much more power than that of chain complexes, and can handle more complex geometry.

The specific conventions in writing down the formulae for this tensor product of crossed complexes come from another direction, which is explained fully in Chapter 15 in Part III, namely the relation with cubical ω-groupoids with connection. The tensor product there comes out simply, because it is based on the formula $I^m \times I^n \cong I^{m+n}$. The distinction between that formula and that for the product of cells as above lies at the heart of many difficulties in basic homotopy theory. The relation between ω-groupoids and crossed complexes gives an algebraic expression of these geometric relationships.

Using this definition, it can be proved that the tensor product gives a symmetric monoidal structure to Crs the category of crossed modules by defining the maps on generators and checking that they preserve the relations.

Theorem 9.3.17 *With the bifunctor $- \otimes -$, the category* Crs *of crossed complexes has a structure of a symmetric monoidal category, i.e.*

i) *For crossed complexes C, D, E, there are natural isomorphisms of crossed complexes*

$$(C \otimes D) \otimes E \cong C \otimes (D \otimes E),$$

ii) *for all crossed complexes C, D there is a natural isomorphism of crossed complexes*

$$T : C \otimes D \to D \otimes C$$

satisfying the appropriate axioms.

Proof The existence of both isomorphisms could be established directly, giving the values on generators in the obvious way:

i) is given by $(c \otimes d) \otimes e \mapsto c \otimes (d \otimes e)$ and

ii) is given by $T(c \otimes d) = (-1)^{mn} d \otimes c$ if $c \in C_m$ and $d \in D_n$.

and then checking that the relations on generators $c \otimes d$ in Definition (9.3.13) are preserved by both maps. The necessary coherence and naturality conditions are obviously satisfied.

But to check all the cases even for such simple maps seems tedious. An alternative approach is to go via ω-groupoids where the tensor product fits more closely to the cubical context. This shall be done in Chapter 15. □

This proof of commutativity is somehow unsatisfactory because, although it is clear that $c \otimes d \mapsto d \otimes c$ does not preserve the relations in Definition 9.3.13, the fact that $c \otimes d \mapsto (-1)^{mn} d \otimes c$ does preserve them seems like a happy accident. A better explanation is provided by the transposing functor T (see Section 15.4).

Note that while the tensor product can be defined directly in terms of generators and relations and this can sometimes prove useful, such a definition may make it difficult to verify essential properties of the tensor product, such as that the tensor product of free crossed complexes is free. We shall prove that later (Section 9.6), using the adjointness of $\otimes$ and the internal hom functor as a necessary step to prove that $- \otimes C$ preserves colimits.

Nevertheless, this Definition is interesting for its relation to the tensor product of filtered spaces which we shall study in Section 9.8.

Theorem 9.3.18 *For crossed complexes* C, D, E*, there is a natural exponential law giving a natural isomorphism*

$$\mathsf{Crs}(C \otimes D, E) \cong \mathsf{Crs}(C, \mathsf{CRS}(D, E)).$$

This gives the category Crs *of crossed complexes a structure of monoidal closed category. Moreover, they produce isomorphisms of crossed complexes*

$$\mathsf{CRS}((C \otimes D), E) \cong \mathsf{CRS}(C, \mathsf{CRS}(D, E)).$$

It is also important that we have to use crossed complexes of groupoids to make sense of the exponential law in Crs. This is analogous to the fact that the category of groups has no internal hom, while that of groupoids does.

Remark 9.3.19 Consider the groupoid $\mathcal{I}$ having one arrow $\iota : 0 \to 1$ so that $t(\iota) = 1$ (we have seen that this groupoid is $\Pi(E^1)$. A '1-fold left homotopy' of morphisms $f^0, f^1 : C \to D$ is seen to be a morphism $\mathcal{I} \otimes C \to D$ which takes the values of f^0 on $0 \otimes C$ and f^1 on $1 \otimes C$. The existence of this 'cylinder object' $\mathcal{I} \otimes C$ allows a lot of abstract homotopy theory [KP97] to be applied immediately to the category Crs. This is useful in constructing homotopy equivalences of crossed complexes, using for example the gluing lemma [KP97, Lemma 7.3]. □

9.4 Analysis of the tensor product of crossed complexes

The Definition of the tensor product of two crossed complexes $C \otimes D$ is quite complex. We are going to devote this Section to clarify the definition. It happens that at each dimension, the tensor product $(C \otimes D)_p$ decomposes as the coproduct of simpler bits $(C \otimes D)_{m,n}$ that can be identified to (or related with) some better known constructions. As is usual in the crossed complex situation, the description is different (and more complicated) in low dimensions.

So, first we study the groupoid $(C \otimes D)_1$. It is just the coproduct over $C_0 \times D_0$ of the two groupoids $C_1 \times D_0$ and $C_0 \times D_1$. This description is important since all the $(C \otimes D)_p$ are modules (or crossed modules) over $(C \otimes D)_1$

Then, we study the crossed module part of $C \otimes D$. It has three parts, two of them being got from the crossed modules $C_2 \times D_0 \to C_1 \times D_0$ and $C_0 \times D_2 \to C_0 \times D_1$ using the induced crossed module construction for the inclusions $C_1 \times D_0 \to (C \otimes D)_1$ and $C_0 \times D_1 \to (C \otimes D)_1$.

With respect to higher dimensions, $(C \otimes D)_p$ decomposes in many pieces $(C \otimes D)_{m,n}$ for $m+n = p$. When both $m, n \geqslant 3$, $(C \otimes D)_{m,n}$ is just the tensor product as modules studied in Section 9.2. When one (or both) of the dimensions is 2, $(C \otimes D)_{m,n}$ is the tensor product of the abelianisation.

It remain the cases when $m = 0$ or $m = 1$ (and the symmetric cases). To identify them, we shall introduce a couple of constructions associated to a groupoid: the *right regular* H*-module* $\vec{\mathbb{Z}}H$, and the *right augmentation module* $\vec{I}H$.

9.4.1 The groupoid part of the tensor product.

In order to become more familiar with the definition of the tensor product of crossed complexes, in this Section we are going to do the computations with some detail in low dimensions.

Notice first that it is clear from the definition that to construct $(C \otimes D)_p$ we only need to know $\{C_m\}_{m\leqslant p}$ and $\{D_n\}_{n\leqslant p}$ since there are no relations among the generators in $(C \otimes D)_p$ coming from higher dimensions. Let us see what this means for low dimensions.

The case $p = 0$ is immediate. Let us start with the case $p = 1$.

Proposition 9.4.1 *For any pair of crossed complexes* $C, D \in \mathsf{Crs}$ *the groupoid* $(C \otimes D)_1$ *of their tensor product is the following pushout in the category of groupoids*

$$\begin{tikzcd} 0_{C_1} \times 0_{D_1} \arrow[r] \arrow[d] & C_1 \times 0_{D_1} \arrow[d] \\ 0_{C_1} \times D_1 \arrow[r] & (C \otimes D)_1 \end{tikzcd}$$

where, for any groupoid G, 0_G *denotes the trivial sub-groupoid consisting of all identity elements of* G. *It is easy to see that this pushout is the coproduct of* $C_1 \times 0_{D_1}$ *and* $0_{C_1} \times D_1$ *in the category of groupoids over* $C_0 \times D_0$

Let us give a description of this groupoid. By the previous Proposition, it is actually a construction in the category of groupoids. So, let us consider a pair of groupoids G and H and let us define

$$G \# H = G \times 1_H * 1_G \times H,$$

the coproduct in the category of groupoids over $G_0 \times H_0$.

The groupoid $G \# H$ is generated by all elements $(1_x, h), (g, 1_y)$ where $g \in G, h \in H, x \in G_0, y \in H_0$. We will sometimes write g for $(g, 1_y)$ and h for $(1_x, h)$. This may seem to be willful ambiguity, but when composites are specified in $G \# H$, the ambiguity is resolved; for example, if gh is defined in $G \# H$, then g must refer to $(g, 1_y)$, where $y = sh$, and h must refer to $(1_x, h)$, where $x = tg$. This convention simplifies the notation and there is an easily stated solution to the word problem for $G \# H$. Every element of $G \# H$ is uniquely expressible in one of the following forms:

(i) an identity element $(1_x, 1_y)$;

(ii) a generating element $(g, 1_y)$ or $(1_x, h)$, where $x \in G_0, y \in H_0, g \in G, h \in H$ and g, h are not identities;

(iii) a composite $k_1 k_2 \cdots k_n (n \geqslant 2)$ of non-identity elements of G or H in which the k_i lie alternately in G and H, and the odd and even products $k_1 k_3 k_5 \cdots$ and $k_2 k_4 k_6 \cdots$ are defined in G or H.

For example, if $g_1 : x \to y, g_2 : y \to z$ in G, and $h_1 : u \to v, h_2 : v \to w$ in H, then the word $g_1 h_1 g_2 h_2 g_2^{-1}$ represents an element of $G \# H$ from (x, u) to (y, w). Note that the two occurrences of g_2 refer to different elements of $G \# H$, namely $(g_2, 1_v)$ and $(g_2, 1_w)$. This can be represented as a path in a 2-dimensional grid as follows

$$\begin{tikzcd} (x,u) \arrow[d, "g_1"] & (x,v) & (x,w) \\ (y,u) \arrow[r, "h_1"] & (y,v) \arrow[d, "g_2"] & (y,w) \\ (z,u) & (z,v) \arrow[r, "h_2"] & (z,w) \arrow[u, "g_2^{-1}"'] \end{tikzcd}$$

The similarity with the free product of groups is obvious and the normal form can be verified in the same way; for example, one can use 'van der Waerden's trick'. We omit the details (They may be found in [Hig71]).

9.4.2 The crossed module part of the tensor product.

To identify the crossed module in the title for crossed complexes C, D, we need to use two constructions from the theory of crossed modules: the coproduct of crossed modules over the same base and the induced crossed module.

In the case when G is a group, the construction of the coproduct $M \circ_G N$ of crossed G-modules M and N has been studied in Part I. This construction works equally well when G is a groupoid. The family of groups M acts on N via G, so one can form the semidirect product $M \ltimes N$. It consists of a semidirect product of groups $M_p \ltimes N_p$ at each vertex p of G and it is a pre-crossed module over G. One then obtains the crossed G-module $M \circ_G N$ from $M \ltimes N$ by factoring out its Peiffer groupoid.

Now, recall that $(C \otimes D)_2$ as $(C_1 \# D_1)$-crossed module is the coproduct

$$(C \otimes D)_2 = (C \otimes D)_{2,0} \circ (C \otimes D)_{1,1} \circ (C \otimes D)_{0,2}$$

where these last crossed modules have been defined in Remark 9.3.14.

Since C_2 is a crossed module over the groupoid C_1, $C_2 \times D_0$ is a crossed module over $C_1 \times D_0$. Using embedding

$$\mu_1 : C_1 \times D_0 \to C_1 \# D_1$$

we get an induced crossed module

$$\hat{C_2} = \mu_{1*}(C_2 \times D_0).$$

It is not difficult to see that

$$(C \otimes D)_{2,0} \cong \hat{C_2}$$

as $(C_1 \# D_1)$-crossed modules. In the same way, we identify

$$(C \otimes D)_{2,0} \cong \hat{D_2}$$

where $\hat{D_2} = \mu_{2*}(C_0 \times D_2)$.

It remains to identify $(C \otimes D)_{1,1}$. To do this, we need to consider only the case when C and D are just groupoids.

To make things more clear we restrict ourselves to crossed complexes associated to groupoids since the higher dimensional part does not intervene. So $C_n = D_n = \{0\}$ for all $n \geqslant 2$. Then we know $(C \otimes D)_p = \{0\}$ for all $p \geqslant 3$ and we have computed that $(C \otimes D)_0 = C_0 \times D_0$ and $(C \otimes D)_1 = C_1 \# D_1$. Also, to make notation easier, let us write G and H for the groupoids C_1 and D_1.

Notice that there is a canonical morphism

$$\sigma : G \# H \to G \times H$$

induced by the inclusions $1_G \times H \to G \times H$ and $G \times 1_H \to G \times H$. This morphism is defined on a word $k_1 k_2 k_3 \cdots$, by separating the odd and even parts, i.e.

$$\sigma(k_1 k_2 k_3 \cdots) = (k_1 k_3 \cdots, k_2 k_4 \cdots).$$

That is, the map σ introduces a sort of commutativity between G and H.

The kernel of σ will be called the Cartesian subgroupoid of $G \# H$ and will be denoted by $G \square H$, i.e.

$$G \square H = \operatorname{Ker} \sigma.$$

It consists of all identities and all words $k_1 k_2 \cdots k_n$ for which both odd and even products are trivial. Clearly, it is generated by all 'commutators' $[g, h] = g^{-1} h^{-1} g h$, where $g \in G, h \in H$ and g, h are not identities. (Note that $[g, h]$ is uniquely defined in $G \# H$ for any such pair of elements g, h, but the two occurrences of g (or of h) do not refer to the same element of $G \# H$.)

Proposition 9.4.2 *The Cartesian subgroupoid* $G \square H$ *of* $G \# H$ *is freely generated, as a groupoid, by all elements* $[g, h]$ *where* g, h *are non-identity elements of* G, H*, respectively. Thus,* $G \square H$ *is the disjoint union of free groups, one at each vertex, and the group at vertex* (x, y) *has a basis consisting of all* $[g, h]$ *with* $tg = x$ *and* $th = y$ *(*g *and* h *not identity elements).*

Proof In the notation introduced above the 'commutators' $[h, g]$ satisfy the same formal identities as in the group case:

$$[h, g] = [g, h]^{-1},$$

$$[hh_1, g] = [h, g]^{h_1} [h_1, g],$$

$$[h, gg_1] = [h, g_1][h, g]^{g_1}$$

whenever gg_1, hh_1 are defined in G, H. These identities are to be interpreted as equations in $G \# H$, with the obvious meaning for conjugates: $[h, g]^{h_1}$ means $h_1^{-1} h^{-1} g^{-1} h g h_1$, which represents a unique element of $G \# H$.

Now $G \square H$ is an intransitive free groupoid with basis consisting of all $[g, h] (g \in G, h \in H, g, h \neq 1)$ (see Gruenberg [Gru57], Levi [Lev40]). $\square$

Theorem 9.4.3 *The tensor product of the groupoids* G *and* H*, considered as crossed complexes of rank* 1*, is the crossed complex*

$$G \otimes H = (\cdots \to 0 \to \cdots \to 0 \to G \square H \to G \# H)$$

with $g \otimes h = [h, g], x \otimes h = (1_x, h), g \otimes y = (g, 1_y)$ *for* $g \in G, h \in H, x \in G_0, y \in H_0$.

Proof $G \square H$ is a normal subgroupoid of $G \# H$, so

$$\delta : G \square H \to G \# H$$

is a crossed module which we view as a crossed complex C, trivial in dimension $\geqslant 3$. One verifies easily that the equations $\theta(g, h) = [h, g], \theta(g, \cdot) = g, \theta(\cdot, h) = h$ define a bimorphism $\theta : (G, H) \to C$; the equations in Definition 9.3.11(iii) reduce to the standard commutator identities

$$[hh_1, g] = [h, g]^{h_1} [h_1, g],$$

$$[h, gg_1] = [h, g_1][h, g]^{g_1},$$

and the rest are trivial.

It follows that if

$$\phi : (G, H) \to D$$

is any bimorphism, there is a unique morphism of groups $\phi_2 : G \square H \to D_2$ such that $\phi_2([h, g]) = \phi(g, h)$ for all $g \in G, h \in H$. (Note that the definition of bimorphism implies that $\phi(g, h) = 1$ if either $g = 1$ or $h = 1$.) There is also a unique morphism $\phi_1 : G \# H \to D_1$ such that $\phi_1(g) = \phi(g, \cdot)$ and $\phi_1(h) = \phi(\cdot, h)$ for all $g \in G, h \in H$. These morphisms combine to give a morphism

$$\phi : C \to D$$

of crossed complexes as we show below, and this proves the universal property making C the tensor product of G and H, with $g \otimes h = [h, g]$.

We need to verify that $\phi : C \to D$ is a morphism of crossed modules. This amounts to

(i) ϕ is compatible with $\delta : G \square H \hookrightarrow G \# H$. Now

$$\begin{aligned}\delta\phi_2([h, g]) &= \delta\phi(g, h)\\ &= -\phi(\cdot, h) - \phi(g, \cdot) + \phi(\cdot, h) + \phi(g, \cdot) \quad \text{by (9.3.11)(iv)}\\ &= [\phi(\cdot, h), \phi(g, \cdot)] = [\phi_1(h), \phi_1(g)] = \phi_1[h, g]\end{aligned}$$

and

(ii) ϕ preserves the actions of $G \# H$ and D_1. Now

$$\begin{aligned}\phi_2([h, g]^{g_1}) &= \phi_2([h, g_1]^{-1}[h, gg_1])\\ &= -\phi(g_1, h) + \phi(gg_1, h)\\ &= \phi(g, h)^{\phi(g_1, \cdot)} \quad \text{by (9.3.11)(iii)}\\ &= \phi_2([h, g])^{\phi_1(g_1)}.\end{aligned}$$

There is a similar calculation for the action of $h_1 \in H$, and the result follows. □

That gives a useful description of the crossed module part of the tensor product of two crossed complexes C and D.

Theorem 9.4.4 *There is an isomorphism of* $(C_1 \# D_1)$*-crossed modules*

$$(C \otimes D)_2 \cong \mu_{1*}(C_2 \otimes D_0) \circ G \square H \circ \mu_{2*}(C_0 \otimes D_2).$$

This isomorphism maps $c \otimes y$ *and* $x \otimes d$ *to the corresponding generators in* $\mu_{1*}(C_2 \otimes D_0)$ *and* $\mu_{2*}(C_0 \otimes D_2)$ *and* $c \otimes d$ *to the commutator* $(c \otimes y)(x' \otimes d)(c \otimes y')^{-1}(x \otimes d)^{-1}$. *So the subgroupoid* $\delta_2(C \otimes D)_2$ *is generated as a groupoid by the elements*

$$\{\delta c \otimes y \mid c \in C_2\} \cup \{x \otimes \delta d \mid d \in D_1\} \cup \{(c \otimes y)(x \otimes d)(c \otimes y')^{-1}(x' \otimes d)^{-1} \mid c \in C_1(x, x'), d \in D_1(y, y')\}$$

where $x, x' \in C_0, y, y' \in D_0$.

The description in Theorem 9.4.3 is much easier for the case of groups. Any group G can be viewed as a crossed complex $\mathbb{E}_1(G)$ with $\mathbb{E}_1(G)_0 = \{\cdot\}, \mathbb{E}_1(G)_1 = G, \mathbb{E}_1(G)_n = 0$ for $n \geqslant 2$. The tensor product of two such crossed complexes will have one vertex and will be trivial in dimension $\geqslant 3$, that is, it will be a crossed module. We use multiplicative notation for G for reasons which will appear later.

Proposition 9.4.5 *The tensor product of groups* G, H*, viewed as crossed complexes of rank* 1*, is the crossed module* $G \square H \to G * H$*, where* $G \square H$ *denotes the Cartesian subgroup of the free product* $G * H$*, that is, the kernel of the map* $G * H \to G \times H$*. If* $g \in G, h \in H$*, then* $g \otimes h$ *is the commutator* $[h, g] = h^{-1}g^{-1}hg = [g, h]^{-1}$ *in* $G * H$.

Remark 9.4.6 This tensor product of (non-Abelian) groups is related to, but not the same as, the tensor product defined by Brown and Loday and used in their construction of universal crossed squares of groups [BL87a]. The Brown-Loday product is defined for two groups acting compatibly on each other. It also satisfies the standard commutator identities displayed above. The relation between the two tensor products is clarified by Gilbert and Higgins in [GH89]. See also the results of Baues and Conduché in [BC90]. □

9.5 Tensor products and chain complexes

9.5.1 Monoidal closed structure on chain complexes

We have seen that the symmetric monoidal closed structure on the category Crs of crossed complexes, constructed in Section 9.3 from tensor products and homotopies, relies crucially on the consideration of crossed complexes over groupoids as well as over groups. The same is true for chain complexes with operators that we study in this section as an introduction.

There are well known definitions of tensor product and internal hom functor for chain complexes of Abelian groups (without operators). If one allows operators from arbitrary groups the tensor product is easily generalised (the tensor product of a G-module and an H-module being a $(G \times H)$-module) but the adjoint construction of internal hom functor does not exist, basically because the group morphisms from G to H do not form a group. To rectify this situation we allow operators from arbitrary groupoids and we start with a discussion of the monoidal closed category structure of Mod the category of modules over groupoids given in Definition **??**.

The ideas for the monoidal closed category Mod can be extended with little extra trouble to chain complexes over groupoids.

Definition 9.5.1 The *tensor product* of chain complexes C, D over groupoids G, H respectively is the chain complex $C \otimes D$ over $G \times H$ where

$$(C \otimes D)_n = \bigoplus_{i+j=n} (C_i \otimes D_j).$$

Here, the direct sum of modules over a groupoid G is defined by taking the direct sum of the Abelian groups at each object of G. The boundary map

$$\partial : (C \otimes D)_n \to (C \otimes D)_{n-1}$$

is defined on the generators $a \otimes b$ of $(C \otimes D)_n$ by

$$\partial(a \otimes b) = \partial a \otimes b + (-1)^i a \otimes \partial b,$$

where $a \in C_i$, $b \in D_j$, $i + j = n$. □

This tensor product clearly gives a symmetric monoidal structure to the category Chn, with unit object the complex

$$C(\mathbb{Z}, 0) = \cdots \to 0 \to \cdots \to 0 \to \mathbb{Z}$$

over the trivial group. The symmetry map $C \otimes D \to D \otimes C$ is given by

$$x \otimes y \mapsto (-1)^{ij} y \otimes x$$

for $x \in C_i, y \in D_j$.

Definition 9.5.2 The internal hom functor $\mathsf{CHN}(-,-)$ is defined as follows. Let C, D be chain complexes over the groupoids G, H respectively. As in the case of morphisms of modules, it is easy to see that the morphisms of chain complexes $\mathsf{Chn}(M, N)$ form an $\mathsf{GPDS}(G, H)$-module. We write

$$S_0 = \mathsf{Chn}(M, N)$$

for this module and take it as the 0-dimensional part of the chain complex $S = \mathsf{CHN}(C, D)$.

The higher-dimensional elements of S are chain homotopies of various degrees. An *i-fold chain homotopy* ($i \geqslant 1$) from C to D is a pair (s, f) where $s : C \to D$ is a map of degree i (that is, a family of maps $s : C_n \to D_{n+i}$ for all $n \geqslant 0$) which in each dimension is a morphism of modules over $f : G \to H$.

Again the i-fold homotopies

$$S_i = \{s : C \to D \mid i\text{-fold homotopies}\}$$

have a structure of an $\mathsf{GPDS}(G, H)$-module and we define the boundary map

$$\partial : S_i \to S_{i-1} \ (i \geqslant 1)$$

by

$$(\partial s)(x) = \partial(s(x)) + (-1)^{i+1} s(\partial x),$$

the morphism $f : G \to H$ being the same for ∂s as for s.

We observe that ∂s is of degree $i - 1$. Also ∂s commutes or anticommutes with ∂, namely

$$\partial((\partial s)(x)) = (-1)^{i+1}(\partial s)(\partial x).$$

It follows that $\partial\partial : S_i \to S_{i-2}$ is 0 for $i \geqslant 2$. We define $\mathsf{CHN}(C, D)$ to be the chain complex

$$\mathsf{CHN}(C, D) = \cdots \longrightarrow S_i \xrightarrow{\partial} S_{i-1} \longrightarrow \cdots \longrightarrow S_0$$

over $F = \mathsf{GPDS}(G, H)$. □

Proposition 9.5.3 *The functors $\otimes$ and CHN give Chn the structure of symmetric monoidal closed category.*

Proof Again, if L is a chain complex over G, there is an exponential law giving a natural bijection

$$\mathsf{Chn}(L \otimes C, D) \cong \mathsf{Chn}(L, \mathsf{CHN}(C, D))$$

which extends to a natural isomorphism of chain complexes

$$\mathsf{CHN}(L \otimes C, D) \cong \mathsf{CHN}(L, \mathsf{CHN}(C, D))$$

over $\mathsf{GPDS}(G \times H, K) \cong \mathsf{GPDS}(G, \mathsf{GPDS}(H, K))$. □

9.5.2 Abelianisation and the closed category structure

In Subsection 9.3.1 an internal hom functor $\mathsf{Crs}(-,-)$ was defined for crossed complexes similar to that defined in Subsection 9.5.1 for chain complexes over groupoids. The relationship between the two monoidal closed structures is best described in terms of the adjoint functors ∇ and Θ.

Theorem 9.5.4 *For crossed complexes* B, C *and chain complexes* L *there are natural isomorphisms*

(i) $\mathsf{CRS}(C, \Theta L) \cong \Theta\mathsf{CHN}(\nabla C, L)$,

(ii) $\nabla(B \otimes C) \cong \nabla B \otimes \nabla C$.

Proof The two natural isomorphisms are equivalent because

$$\begin{aligned}\mathsf{CHN}(\nabla(B \otimes C), L) &\cong \mathsf{Crs}(B \otimes C, \Theta L) \\ &\cong \mathsf{Crs}(B, \mathsf{CRS}(C, \Theta L)),\end{aligned}$$

while

$$\begin{aligned}\mathsf{Chn}(\nabla B \otimes \nabla C, L) &\cong \mathsf{Chn}(\nabla B, \mathsf{CHN}(\nabla C, L)) \\ &\cong \mathsf{Crs}(B, \Theta\mathsf{CHN}(\nabla C, L)).\end{aligned}$$

The isomorphism (i) is easier to verify than (ii) because we have explicit descriptions of the elements of both sides, whereas in (ii) we have only presentations.

In dimension 0 we have on the left of (i) the set $\mathsf{Crs}(C, \Theta L)$ of morphisms $\hat{f} : C \to \Theta L$; on the right we have the set $\mathsf{Chn}(\nabla C, L)$ of morphisms $(\tilde{f}, \psi) : \nabla C \to L$, where ψ is a morphism of groupoids from $G = \pi_1 C$ to H, the operator groupoid for L. These sets are in one-one correspondence, by adjointness, and their elements are also equivalent to pairs (f, ψ) where $\psi : G \to H$ and f is a family

$$\begin{tikzcd} \cdots \arrow[r, "\delta"] & C_2 \arrow[r, "\delta"] \arrow[d, "f_2"] & C_1 \arrow[r, shift left, "\delta^0"] \arrow[r, shift right, "\delta^1"'] \arrow[d, "f_1"] & C_0 \arrow[d, "f_0"] \\ \cdots \arrow[r, "\partial"] & L_2 \arrow[r, "\partial"] & L_1 \arrow[r, "\partial"] & L_0 \end{tikzcd}$$

such that

(i) $f_0(p) \in L_0(\psi(p))\ (p \in C_0)$,

(ii) f_1 is a $\psi\phi$-derivation, where ϕ is the quotient map $C_1 \to G$,

(iii) f_n is a ψ-morphism for $n \geqslant 2$,

(iv) $\partial f_{n+1} = f_n \delta\ (n \geqslant 1)$,

(v) $\partial f_1(x) = (f_0 \delta^0 x)^{\psi\phi x} - (f_0 \delta^1_x)\ (x \in C_1)$.

Such a family will be called a ψ-derivation $f : C \to L$.

We recall from Definition 9.3.6 that an element in $\mathsf{CRS}_i(C, E)$ is an i-fold homotopy $(\hat{h}, \hat{f}) : C \to E$, where $\hat{f}$ is a morphism $C \to E$ and $\hat{h}$ is a family of maps

$$\begin{tikzcd} \cdots \arrow[r] & C_2 \arrow[r] \arrow[d, "\hat{h}_2"] & C_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\hat{h}_1"] & C_0 \arrow[d, "\hat{h}_0"] \\ \cdots & E_{i+2} & E_{i+1} & E_i \end{tikzcd}$$

satisfying

(i) $\hat{h}_0(p) \in E_i(\hat{f}_0(p))\,(p \in C_0)$;

(ii) $\hat{h}_1$ is a $\hat{f}_1$-derivation;

(iii) $\hat{h}_n$ is a $\hat{f}_1$-morphism for $n \geqslant 2$.

In the case $E = \Theta L$, where L is a chain complex over H, it is easy to see that, if $i \geqslant 2$, such a homotopy is equivalent to the following data: a morphism of groupoids $\psi : G \to H$; a ψ-derivation $f : C \to L$ as in diagram (*); and a family h of maps

$$\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r] \arrow[d, "h_2"] & C_1 \arrow[r, shift left] \arrow[r, shift right, "\delta_1"'] \arrow[d, "h_1"] & C_0 \arrow[d, "h_0"] \\
\cdots & L_{i+2} & L_{i+1} & L_i
\end{tikzcd}$$

satisfying

(i) $h_0(p) \in L_i(\psi p)\,(p \in C_0)$;

(ii) h_1 is a $\psi\phi$-derivation;

(iii) h_j is a ψ-morphism for $j \geqslant 2$.

The maps $\hat{h}_j$ of diagram (**) are then given by

$$\begin{aligned}
\hat{h}_j(x) &= (h_j(x), f_0(q)) \ \text{ if } x \in C_j(q),\ j \geqslant 2,\\
\hat{h}_1(x) &= (h_1(x), f_0(q)) \ \text{ if } x \in C_1(p, q),\\
\hat{h}_0(q) &= (h_0(x), f_0(q)) \ \text{ if } q \in C_0.
\end{aligned}$$

In the case $i = 1$, because of the special form of E_1, we also need a map $\tau : C_0 \to H$ satisfying

(iv) $\tau(q) \in H(\psi'(q), \psi(q))$ for some $\psi'(q) \in \mathrm{Ob}\, H$,

and in this case $\hat{h}_0(q) = (\tau(q), h_0(q), f_0(q))$.

It is now an easy matter to see that these data are equivalent to an element of dimension i in $\Theta\mathsf{CHN}(\nabla C, L)$. In the case $i = 1$, the map τ defines a natural transformation $\tilde{\tau} : \psi' \to \psi$, where $\psi'(g) = \tau(p)\psi(g)\tau(q)^{-1}$ for $g \in G(p, q)$. This $\tilde{\tau}$ is an element of the groupoid $\mathsf{GPDS}(G, H)$ (the operator groupoid for $\mathsf{CHN}(\nabla C, L)$) and provides the first component of the triple $(\tilde{\tau}, \tilde{h}, \tilde{f})$ which is the required element of $\Theta_1\mathsf{CHN}(\nabla C, L)$; the other components are $\tilde{f} : \nabla C \to L$, the morphism of chain complexes induced by f, and $\tilde{h}$, the 1-fold homotopy $\nabla C \to L$ induced by h. Here $\tilde{h}_0(1_p) = h_0(p)$ and $\tilde{h}_n\alpha_n = h_n$ for $n \geqslant 1$, where the α_i are as in the diagram in Theorem (7.5.18). The rest of the proof is straightforward. □

9.6 The tensor product of free crossed complexes is free

The exponential law in Crs of Theorem 9.3.18 has as a consequence that the tensor product of free crossed complex is a free crossed complex.

We start by proving the result for the standard models.

Proposition 9.6.1 *Consider the inclusions* $\mathbb{S}(n-1) \to \mathbb{F}(n)$ *and* $\mathbb{S}(m-1) \to \mathbb{F}(m)$ *then*

$$\mathbb{S}(n-1) \otimes \mathbb{F}(m) \cup \mathbb{F}(n) \otimes \mathbb{S}(m-1) \to \mathbb{F}(m) \otimes \mathbb{F}(n)$$

is of relative free type.

Proof Let us remark that they differ only in dimension $(m+n)$. We have to check that the diagram

$$\begin{tikzcd} \mathbb{S}(n+m-1) \arrow[r] \arrow[d] & \mathbb{S}(n-1) \otimes \mathbb{F}(m) \cup \mathbb{F}(n) \otimes \mathbb{S}(m-1) \arrow[d] \\ \mathbb{F}(n+m) \arrow[r] & \mathbb{F}(m) \otimes \mathbb{F}(n) \end{tikzcd}$$

given on generators by mapping $x_{m+n} \mapsto x_m \otimes x_n$ is a pushout of crossed complexes and this is easily done. $\Box$

The proof of the general theorem builds inductively on the previous case and the next Lemma.

Lemma 9.6.2 *If the following squares are pushouts*

$$\begin{tikzcd} C \arrow[r] \arrow[d] & D \arrow[d] \\ E \arrow[r] & F \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ W \arrow[r] & X \end{tikzcd}$$

then so is the induced square

$$\begin{tikzcd} C \otimes W \cup E \otimes U \arrow[r] \arrow[d] & D \otimes X \cup F \otimes V \arrow[d] \\ E \otimes W \arrow[r] & F \otimes X \end{tikzcd}$$

Theorem 9.6.3 *If* $C' \to C$ *and* $D' \to D$ *are morphisms of relative free type then so also is* $C' \otimes D \cup C \otimes D' \to C \otimes D$*, where* $C' \otimes D \cup C \otimes D'$ *denotes the pushout of the pair of morphisms*

$$C' \otimes D \leftarrow C' \otimes D' \to C \otimes D'.$$

Proof Since the tensor product $- \otimes -$ is symmetric and $- \otimes B$ has a right adjoint, the functors $- \otimes C$ and $D \otimes -$ preserve colimits.

Since $Y \otimes -$ and $- \otimes Z$ preserve coproducts, we deduce the result in the case when $C' \to C$, $D' \to D$ are of the type $\coprod_\lambda \mathbb{S}(n-1) \to \coprod_\lambda \mathbb{F}(n)$ and $\coprod_\lambda \mathbb{S}(m-1) \to \coprod_\lambda \mathbb{F}(m)$.

Putting morphisms of this type in Lemma 9.6.2, and using Lemma 7.4.16, easily follows that the theorem is true for morphisms of simple relative free type, that is for morphisms $C' \to C$, $D' \to D$ obtained as pushouts

$$\begin{tikzcd} \coprod_\lambda \mathbb{S}(n-1) \arrow[r] \arrow[d] & C' \arrow[d] \\ \coprod_\lambda \mathbb{F}(n) \arrow[r] & C \end{tikzcd} \qquad \begin{tikzcd} \coprod_\mu \mathbb{S}(m-1) \arrow[r] \arrow[d] & D' \arrow[d] \\ \coprod_\mu \mathbb{F}(m) \arrow[r] & D \end{tikzcd}$$

Next, using Lemmas 7.4.16, 7.4.15, 9.6.2 we can prove the result for composites of morphisms of relative free type. A general morphism of relative free type is a colimit of simple ones, as in Corollary 7.4.18, and the full result now follows from this Corollary 7.4.15 and Lemma 9.6.2. $\Box$

Corollary 9.6.4 *If $C' \to C$ is a morphism of relative free type and W is a crossed complex of free type, then $C' \otimes W \to C \otimes W$ is of relative free type.*

Corollary 9.6.5 *If C is a free crossed complex and $f : C \to D$ is a morphism of crossed complexes, then a homotopy $H : f^0 \simeq f^1$ of morphisms is entirely determined by its values on the free basis of C.*

9.7 The monoidal closed category of filtered spaces

We proceed a step further and consider the category FTop of filtered spaces and look for a natural structure of closed category.

The categorical product in FTop is given by

$$(X_* \times Y_*)_n = X_n \times Y_n,\ n \geqslant 0.$$

This product is convenient for maps into it, as for any categorical product. However our main example of filtered spaces, that of CW-complexes, suggests a different product as worth consideration, and this will turn out to be convenient for maps from it, to other filtered spaces.

If X_*, Y_* are CW-filtrations, then the product $X \times Y$ of the spaces (in the category of compactly generated spaces) has a natural and convenient CW-structure in which the n-dimensional cells are all products $e^p \times e^q$ of cells of X_*, Y_* respectively where $p + q = n$. This suggests the following definition.

Definition 9.7.1 If X_*, Y_* are filtered spaces, their *tensor product* $X_* \otimes Y_*$ is the filtered space given on $X \times Y$ by the family of subspaces

$$(X \otimes Y)_n = \bigcup_{p+q=n} X_p \times Y_q$$

where the union is simply the union of subspaces of $X \times Y$. □

Exercise 9.7.2 1. We have said that the filtration $X_* \otimes Y_*$ is not the product in the category FTop. Verify that our definition above does define the product $X_* \times Y_*$ in the category FTop.

2. Is there a structure of cartesian closed category on FTop? i.e. is there an internal hom that is adjoint to the cartesian product? □

Notice that, for example, $\mathbf{I}_*^n$ is the n-fold tensor product of $\mathbf{I}_*$ with itself because $\mathbf{I}_*^n$ is the CW-filtered space of the standard n-cube.

With the product $\otimes$, FTop is a monoidal category. The tensor product is also commutative.

We now show how to define an 'internal hom' $\mathsf{FTOP}(Y_*, Z_*)$ in the category FTop so as to make that category a monoidal closed category with an exponential law giving a natural bijection

$$e : \mathsf{FTop}(X_* \otimes Y_*, Z_*) \cong \mathsf{FTop}(X_*, \mathsf{FTOP}(Y_*, Z_*)).$$

To see how this comes about, note that a filtered map $f : X_* \otimes Y_* \to Z_*$ will map $X_p \times Y_q$ to Z_{p+q}, by definition of the filtration on the tensor product of filtered spaces. Under the exponential law for topological spaces we have

$$\mathsf{Top}(X_p \times Y_q, Z_{p+q}) \cong \mathsf{Top}(X_p, \mathsf{TOP}(Y_q, Z_{p+q})).$$

This suggests the definition:

$$\mathsf{FTOP}(Y_*, Z_*)_p = \{g \in \mathsf{Top}(Y, Z) \mid g(Y_q) \subseteq Z_{p+q} \quad \text{for all} \quad q \geqslant 0\}.$$

This gives a filtration on the topological space $\mathsf{TOP}(Y, Z)$ and so defines the filtered space $\mathsf{FTOP}(Y_*, Z_*)_p$. The exponential law in the category Top now gives the exponential law

$$e : \mathsf{FTop}(X_* \otimes Y_*, Z_*) \cong \mathsf{FTop}(X_*, \mathsf{FTOP}(Y_*, Z_*)),$$

from which one can deduce the exponential law

$$e : \mathsf{FTOP}(X_* \otimes Y_*, Z_*) \cong \mathsf{FTOP}(X_*, \mathsf{FTOP}(Y_*, Z_*)).$$

either using the general result in the Appendix B or directly as is left as an exercise.

An advantage of having this internal hom for filtered spaces is that we can apply our fundamental crossed complex functor Π to it. To say more on this, we first discuss the notion of homotopy in FTop.

The convenient definition of homotopy $H : f_0 \simeq f_1 : Y_* \to Z_*$ of maps f_0, f_1 of filtered spaces is that H is a map $I \times Y \to Z$ which is a homotopy $f_0 \simeq f_1$ such that $H(I \times Y_q) \subseteq Z_{q+1}$ for all $q \geqslant 0$. This last condition is equivalent to H being a filtered map $\mathbf{I}_* \otimes Y_* \to Z_*$. Equivalently, we can regard H also as a map

$$\mathbf{I}_* \to \mathsf{FTOP}(Y_*, Z_*), \text{ or } Y_* \to \mathsf{FTOP}(\mathbf{I}_*, Z_*),$$

although the latter interpretation involves the twisting map $\mathbf{I}_* \otimes Y_* \to Y_* \otimes \mathbf{I}_*$.

It is also possible to consider 'higher filtered homotopies' as filtered maps

$$\mathbf{E}^n_* \otimes Y_* \to Z_*$$

or equivalently as maps

$$\mathbf{E}^n_* \to \mathsf{FTOP}(Y_*, Z_*).$$

This will fit with results on crossed complexes.

9.8 Tensor products and the fundamental crossed complex

In order to obtain the homotopy classification Theorem 10.4.17, we need to use tensor products and homotopies of crossed complexes and its relation to homotopies of filtered maps.

We have defined the notion of homotopies for maps of filtered spaces. They give 1-homotopies between the induced morphisms of fundamental crossed complexes. Again, it is possible to prove this directly, but it follows more elegantly from later more general results.

In particular,

Theorem 9.8.1 *If X_* and Y_* are filtered spaces, then there is a natural morphism*

$$\theta : \Pi X_* \otimes \Pi Y_* \to \Pi(X_* \otimes Y_*)$$

such that:

i) θ *is associative;*

ii) *if $*$ denotes a singleton space or crossed complex, then the following diagrams are commutative*

$$\begin{tikzcd} \Pi X_* & (\Pi X_*) \otimes * \arrow[l, "\cong"'] \arrow[d, "\theta"] \\ & \Pi(X_* \otimes *) \arrow[from=1-1, "\cong"'] \end{tikzcd} \qquad \begin{tikzcd} * \otimes \Pi X_* \arrow[r, "\cong"] \arrow[d, "\theta"'] & \Pi X_* \arrow[dl, "\cong"] \\ \Pi(* \otimes X_*) & \end{tikzcd}$$

iii) *θ is commutative in the sense that if $T_c : C \otimes D \to D \otimes C$ is the natural isomorphism of crossed complexes described in Theorem 9.3.17, and $T_t : X_* \otimes Y_* \to Y_* \otimes X_*$ is the twisting map, then the following diagram is commutative*

$$\begin{tikzcd} \Pi X_* \otimes \Pi Y_* \arrow[r, "\theta"] \arrow[d, "T_c"'] & \Pi(X_* \otimes Y_*) \arrow[d, "\Pi(T_t)"] \\ \Pi Y_* \otimes \Pi X_* \arrow[r, "\theta"] & \Pi(Y_* \otimes X_*); \end{tikzcd}$$

iv) *if X_*, Y_* are the skeletal filtrations of CW-complexes, then θ is an isomorphism.*

The proof is deferred to Chapter 15 where we can use the techniques of ω-groupoids. Note that the construction of the natural transformation θ could in principle be proved directly, but this would be technically difficult because of the complications of the relations for the tensor product of crossed complexes.

In fact θ is an isomorphism under more general conditions (see the result by Baues and Brown in [BB93]).

In a similar spirit, let us now prove that the functor $\Pi : \mathsf{FTop} \to \mathsf{Crs}$ is a homotopy functor.

Proposition 9.8.2 *There is a natural morphism of crossed complexes*

$$\psi : \Pi(\mathsf{FTOP}(X_*, Y_*)) \to \mathsf{CRS}(\Pi X_*, \Pi Y_*)$$

which is Π in dimension 0.

Proof It is sufficient to construct the morphism $\hat{\psi}$ as the composition in the following commutative diagram

$$\begin{tikzcd} \Pi(\mathsf{FTOP}(X_*, Y_*)) \otimes \Pi X_* \arrow[rr, "\hat{\psi}"] \arrow[dr, "\theta"'] & & \Pi Y_* \\ & \Pi(\mathsf{FTOP}(X_*, Y_*) \otimes X_*) \arrow[ur, "\Pi e"'] & \end{tikzcd}$$

where $e : \mathsf{FTOP}(X_*, Y_*) \otimes X_* \to Y_*$ is the evaluation morphism, i.e. the adjoint to the identity on $\mathsf{FTOP}(X_*, Y_*)$. □

Corollary 9.8.3 *In particular, a homotopy $F : f_0 \simeq f_1 : X_* \to Y_*$ in FTop induces a (left) homotopy $\Pi F : \Pi f_0 \simeq \Pi f_1 : \Pi X_* \to \Pi Y_*$ in Crs.*

Proof This is an immediate consequence of the information given by ψ in dimension 1. □

Similar statements hold for right homotopies of crossed complexes. A right homotopy $C \to D$ is a morphism $C \otimes \mathcal{I} \to D$, or, equivalently, a morphism $C \to \mathsf{CRS}(\mathcal{I}, D)$. We may also define a right homotopy in FTop to be a map $Y_* \otimes \mathbf{I}_* \to Z_*$. By Theorem 9.8.1, such a map gives rise to a right homotopy $\Pi Y_* \otimes \mathcal{I} \to \Pi Z_*$.

9.9 The homotopy addition lemma for a simplex.

In this section we describe explicitly and algebraically a free basis and the boundary for the crossed complex $\Pi\Delta^n$ where Δ^n is the topological n-simplex with its standard filtration by dimension. This formula is called the homotopy addition lemma for a simplex. We call $\Pi\Delta^n$ the 'n-simplex crossed complex', and its description is used in many places later

It is a feature of our exposition using crossed complexes that the homotopy addition lemma can be seen as an algebraic fact which models accurately the geometry. That happens because crossed complexes model well the geometry, and a key aspect of that is the use of groupoids to handle all the vertices of the simplex.

Definition 9.9.1 First it is useful to write out all the rules for the cylinder $\mathrm{Cyl}(C) = \mathcal{I} \otimes C$, as a reference. Let C be a crossed complex. We apply the relations in the definition of tensor product of crossed complexes (Definition 9.3.13) to this case.

For all $n \geqslant 0$ and $c \in C_n$, $\mathcal{I} \otimes C$ is generated by elements $0 \otimes c, 1 \otimes c$ of dimension n and $\iota \otimes c, \iota^{-1} \otimes c$ of dimension $(n+1)$ with the following defining relations for $a = 0, 1, \iota$:

Source and target

$$\begin{aligned} t(a \otimes c) &= ta \otimes tc \quad \text{for all } a \in \mathcal{I}, c \in C \\ s(a \otimes c) &= a \otimes sc \quad \text{if } a = 0, 1, n = 1\ , \\ s(a \otimes c) &= sa \otimes c \quad \text{if } a = \iota, \iota^{-1}, n = 0\ . \end{aligned}$$

Relations with operations

$$a \otimes c^{c'} = (a \otimes c)^{ta \otimes c'} \quad \text{if } n \geqslant 2,\ c' \in C_1.$$

Relations with additions

$$a \otimes (c + c') = \begin{cases} a \otimes c + a \otimes c', & \text{if } a = 0, 1, n \geqslant 1 \text{ or if } a = \iota, \iota^{-1},\ n \geqslant 2, \\ (a \otimes c)^{ta \otimes c'} + a \otimes c', & \text{if } a = \iota, \iota^{-1}, n = 1 \end{cases}$$

$$(\iota^{-1}) \otimes c = \begin{cases} -(\iota \otimes c) & \text{if } n = 0, \\ -(\iota \otimes c)^{(\iota^{-1}) \otimes tc} & \text{if } n \geqslant 1. \end{cases}$$

Boundaries

$$\delta(a \otimes c) = \begin{cases} -(a \otimes \delta c) - (ta \otimes c) + (sa \otimes c)^{a \otimes tc} & \text{if } a = \iota, \iota^{-1},\ n \geqslant 2, \\ -ta \otimes c - a \otimes sc + sa \otimes c + a \otimes tc & \text{if } a = \iota, \iota^{-1},\ n = 1, \\ a \otimes \delta c & \text{if } a = 0, 1,\ n \geqslant 2. \end{cases}$$

□

These rules simplify if instead of the cylinder, we analyse the cone.

Definition 9.9.2 Let C be a crossed complex. The *cone* Cone(C) is defined by

$$\mathrm{Cone}\,(C) = (\mathcal{I} \otimes C)/(\{1\} \otimes C)\ ,$$

which can alternatively be seen as a pushout

$$\begin{tikzcd} \{1\} \otimes C \arrow[r] \arrow[d] & \{v\} \arrow[d] \\ \mathcal{I} \otimes C \arrow[r] & \mathrm{Cone}\,(C). \end{tikzcd}$$

We call v the *vertex* of the cone. □

Proposition 9.9.3 *So* $\mathrm{Cone}\,(C)$ *is generated by elements* $0 \otimes c, \iota \otimes c$ *of dimensions* $n, n+1$ *respectively, and* v *of dimension* 0 *with the rules*

Source and target

$$t(a \otimes c) = \begin{cases} 0 \otimes tc, & \textit{if } a = 0, \\ v & \textit{otherwise.} \end{cases}$$

Relations with operations

$$a \otimes c^{c'} = a \otimes c \quad \textit{if}\ \ n \geqslant 2,\ c' \in C_1.$$

Relations with additions

$$a \otimes (c + c') = a \otimes c + a \otimes c'.$$

and

$$(\iota^{-1}) \otimes c = \begin{cases} -(\iota \otimes c) & \textit{if } n = 0, \\ -(\iota \otimes c)^{(\iota^{-1}) \otimes tc} & \textit{if } n \geqslant 1. \end{cases}$$

Boundaries

$$\delta(\iota \otimes c) = \begin{cases} -(\iota \otimes \delta c) + (0 \otimes c)^{\iota \otimes tc} & \textit{if } n \geqslant 2, \\ -\iota \otimes sc + 0 \otimes c + \iota \otimes tc & \textit{if } n = 1, \end{cases}$$

$$\delta(0 \otimes c) = 0 \otimes \delta c \qquad \textit{if } n \geqslant 2.$$

The simplicity of the rules for operations and additions is one of the advantages of the form of our definition of the cone, in which the end at 1 *is shrunk to a point.*

We use the above to work out the fundamental crossed complex of the simplex Δ^n in an algebraic fashion. We regard Δ^n topologically as the topological cone

$$\mathrm{Cone}\,(\Delta^{n-1}) = (I \times \Delta^{n-1})/(\{1\} \times \Delta^{n-1}).$$

The vertices of $\Delta^1 = I$ are ordered as $0 < 1$. Inductively, we get vertices $v_0, \ldots, v_n$ of Δ^n with $v_n = v$ being the last introduced in the cone construction, the other vertices v_i being $(0, v_i)$. The fact that our algebraic formula corresponds to the topological one follows from facts stated earlier on the tensor product and on the HHvKT stated in the next section.

We now define inductively top dimensional generators of the crossed complex $\Pi\Delta^n$ by, in the cone complex:

$$\sigma^0 = v,\ \sigma^1 = \iota,\ \sigma^n = (\iota \otimes \sigma^{n-1}),\ n \geqslant 2$$

with σ^0 being the vertex of $\Pi\Delta^0$. convention for v_n?

We give conventions for the faces of σ^n, as illustrated in the following diagram:

$$
\begin{tikzcd}
& 2 = v_2 & \\
& \sigma^2 = \iota\otimes\sigma^1 & \\
0 = 0\otimes 0 \arrow[uur, "\iota\otimes 0 = \partial_1\sigma^2"] \arrow[rr, "\partial_2\sigma^2 = 0\otimes\sigma^1"'] & & 1 = 0\otimes 1 \arrow[uul, "\iota\otimes 1 = \partial_0\sigma^2"']
\end{tikzcd}
$$

$$
0 = \partial_1\sigma^1 \xrightarrow{\quad\sigma^1\quad} 1 = \partial_0\sigma^1
$$

We define inductively

$$
\partial_i\sigma^n = \begin{cases} \iota\otimes\partial_i\sigma^{n-1} & \text{if } i < n, \\ 0\otimes\sigma^{n-1} & \text{if } i = n. \end{cases}
$$

Theorem 9.9.4 (Homotopy Addition Lemma) *The following formulae hold, where* $u_n = \iota\otimes v_{n-1}$*:*

$$
\delta_2\sigma^2 = -\partial_1\sigma^2 + \partial_2\sigma^2 + \partial_0\sigma^2, \tag{9.9.1}
$$
$$
\delta_3\sigma^3 = (\partial_3\sigma^3)^{u_3} - \partial_0\sigma^3 - \partial_2\sigma^3 + \partial_1\sigma^3, \tag{9.9.2}
$$

while for $n \geqslant 4$

$$
\delta_n\sigma^n = (\partial_n\sigma^n)^{u_n} + \sum_{i=0}^{n-1}(-1)^{n-i}\partial_i\sigma^n. \tag{9.9.3}
$$

Proof For the case $n = 2$ we have

$$
\begin{aligned}
\delta_2\sigma^2 &= \delta_2((\iota\otimes\iota)) \\
&= -\iota\otimes 0 + 0\otimes\iota + \iota\otimes 1 \\
&= -\partial_1\sigma^2 + \partial_2\sigma^2 + \partial_0\sigma^2.
\end{aligned}
$$

For $n = 3$ we have:

$$
\begin{aligned}
\delta_3\sigma^3 &= \delta_3(\iota\otimes\sigma^2) \\
&= (0\otimes\sigma^2)^{\iota\otimes v_2} - \iota\otimes\delta_2\sigma^2 \\
&= (0\otimes\sigma^2)^{u_3} - \iota\otimes(-\partial_1\sigma^2 + \partial_2\sigma^2 + \partial_0\sigma^2) \\
&= (\partial_3\sigma^3)^{u_3} - \partial_0\sigma^3 - \partial_2\sigma^3 + \partial_1\sigma^3.
\end{aligned}
$$

We leave the general case to the reader. The key points that make it easy are the rules on operations and additions of Proposition 9.9.3. □

Remark 9.9.5 (i) Notice the formula of δ_2 gives a groupoid formula, and the one of δ_3 gives a formula in a crossed module which is nonabelian.
(ii) There are many possible conventions for the Homotopy Addition Lemma, and that given here is unusual. However, our formula follows naturally from the geometry of the cone and our algebra for the tensor product.

(iii) It is a good exercise to prove that $\delta_2\delta_3 = 0$. It is not so easy to prove directly from the formula that $\delta_3\delta_4 = 0$, and a direct proof (given for example by G.W.Whitehead in his book [Whi78]) does use the second law for a crossed module. Of course we know these composites are 0 since we are working in the category of crossed complexes. □

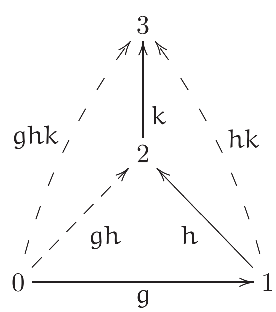
The representation $\mathrm{Cone}\,(\Delta^{n-1}) = \Delta^n$ gives a cellular contracting homotopy of Δ^{n-1}, and so $\Pi\Delta^n$ is a contractible crossed complex. We shall use this fact later.

We can now state the formula in terms of free generators and boundaries for the whole crossed complex $\Pi\Delta^n$. It has a free generator σ^n in dimension n and also free generators $\alpha\sigma^m$ in dimension m for all $0 \leqslant m < n$ and all increasing functions $\alpha : [m] \to [n]$. The boundary of such a $\alpha\sigma^m$ is given by the simplex homotopy addition lemma in dimension m.

Example 9.9.6 The fundamental crossed complex of a simplicial set. Let K be a simplicial set. The fundamental crossed complex ΠK is to have free generators in dimension n given by the elements of K_n and the boundary δk for $k \in K_n$ is given by the homotopy addition lemma in dimension n.

Example 9.9.7 The simplicial nerve of a groupoid Let G be a groupoid. Its *simplicial nerve* $N^\Delta G$ is the simplicial set which in dimension 0 consists of the objects of G and in dimension $n > 0$ consists of the *composable sequences* of arrows of G, i.e. sequences $[g_1, \dots, g_n]$ such that the target of g_i is the source of g_{i+1} for $1 \leqslant i < n$. The face operators ∂_i are defined on these elements so that each face of dimension 2 is commutative. This leads to the following pictures in dimensions 2 and 3:

$$\begin{tikzcd} & 2 \\ 0 \arrow[ur, dashed, "gh"] \arrow[r, "g"'] & 1 \arrow[u, "h"'] \end{tikzcd}$$



and the face operators (∂_i gives the face opposite to the vertex i):

$$\partial_0[g,h,k] = [h,k], \partial_1[g,h,k] = [gh,k], \partial_2[g,h,k] = [g,hk], \partial_3[g,h,k] = [g,h].$$

This tetrahedral picture shows the relation of this construction to associativity of the groupoid operation.

The general formulae are:

$$\partial_i[g_1,\dots,g_n] = \begin{cases} [g_2,\dots,g_n] & \text{if } i = 0, \\ [g_1,\dots,g_i g_{i+1},\dots,g_n] & \text{if } 1 < i < n, \\ [g_1,\dots,g_{n-1}] & \text{if } i = n. \end{cases}$$

We can also define degeneracy operators for $i = 0, \dots, n$ by

$$\varepsilon_i[g_1,\dots,g_n] = [g_1,\dots,g_{i-1},1_i,g_i,\dots,g_n]$$

where 1_i denotes uniquely the identity at the object i for which 1_i gives a composable sequence of length $n+1$. In terms of the notation of the homotopy addition lemma in which $u_n = \partial_0^{n-1}$ we also have $u_n[g_1,\dots,g_n] = g_n$. So we have a formula for $\delta_n[g_1,\dots,g_n]$ which we shall use later.

Remark 9.9.8 From the simplicial nerve $N^\Delta(G)$ of a groupoid G it is natural to form the crossed complex $\Pi N^\Delta(G)$; this has a free generator for each simplex of $N^\Delta(G)$ and boundary given by the simplicial homotopy addition lemma. Then there is a natural isomorphism $\pi_1 \Pi N^\Delta(G) \cong G$, and also $\Pi N^\Delta(G)$ is aspherical, i.e. all homology in dimensions > 1 vanishes. When we discuss resolutions in Chapter 11 we will see $\Pi N^\Delta(G)$ as the standard free crossed resolution of G. However the proof of asphericity is best done using the notion of universal covering crossed complex, which we introduce in Chapter 11, section 11.2.2. □

We can also obtain a cubical homotopy addition lemma using the cube crossed complex $\Pi I^n = \mathcal{I}^{\otimes n}$. In this crossed complex, let $c^n = \iota \otimes \cdots \otimes \iota$ be the n-fold tensor product of ι with itself, and for $\alpha = 0, 1$ let $c_i^\alpha = \partial_i^\alpha c^n$ be the element of dimension $(n-1)$ obtained by replacing in c^n the ι in the ith place by α. The formulae for the boundary in the tensor product then yield by induction, using $\mathcal{I}^{\otimes n} = \mathcal{I} \otimes \mathcal{I}^{\otimes(n-1)}$:

Proposition 9.9.9 [Cubical Homotopy Addition Lemma]

$$\delta_n c^n = \begin{cases} -c_1^1 - c_2^0 + c_1^0 + c_2^1 & \text{if } n = 2, \\ \\ -c_3^1 - (c_2^0)^{u_2 c} - c_1^1 + (c_3^0)^{u_3 c} + c_2^1 + (c_1^0)^{u_1 c} & \text{if } n = 3, \\ \\ \sum_{i=1}^n (-1)^i \{c_i^1 - (c_i^0)^{u_i c}\} & \text{if } n \geqslant 4, \end{cases}$$

(where $c = c^n$ and $u_i = \partial_1^1 \partial_2^1 \cdots \hat{\imath} \cdots \partial_{n+1}^1$).

It should be said that this suggested 'proof' is not quite fair, since we are using a lot of results on crossed complexes the proofs of some of which rely on the cubical homotopy addition lemma established independently. However, this calculation shows how the results tie in, and that once we have these results established they give powerful means of calculation, some of which are inherently nonabelian, and which usually involve module operations not so easily handled by traditional methods.

9.10 Notes

The homotopy addition lemma for a simplex is used in Blakers' 1948 work [Bla48] and was known to be an essential part of proofs of the absolute and relative Hurewicz theorems. However a proof appeared only in [Hu53] in 1955. The proof in G.W. Whitehead's text [Whi78] uses an induction proving at the same time the Hurewicz theorems. It is clear that the algebra of crossed complexes is an essential part of the expression of this lemma. The algebraic derivation given here comes from [BS07].

The notion of monoidal and monoidal closed category can be seen as central to many parts of mathematics, and for the general theory we refer the reader to [ML71]. A full exposition on monoidal categories requires the notion of *coherence*; we avoid dealing with this here because all of the conditions such as associativity on the tensor products with which we deal in the end reduce to associativity of a cartesian product, through the notion of bimorphisms, and so the coherence properties needed follow from the universal properties of categorical products.

Chapter 10

The classifying space of a crossed complex

Introduction

This chapter is one of the most important in this book, since homotopy classification results are among the most difficult in homotopy theory. We define for a crossed complex C, and in a functorial way, a topological space BC, called the *classifying space* of C. The most important property is the following homotopy classification theorem which generalises classical theorems of Eilenberg-Mac Lane:

$$[X, BC] \cong [\Pi X_*, C] \tag{10.0.1}$$

for a CW-complex X with skeletal filtration X_*, and where ΠX_* is the fundamental crossed complex of the filtered space X_*. Here the left hand side gives continuous homotopy classes of maps of spaces and the right hand side gives algebraic homotopy classes of morphisms of crossed complexes. The proof uses a considerable part of the technology of crossed complexes developed in the rest of this book, and the result is a special case of a description of the weak homotopy type of the space of maps $X \to BC$.

Because the crossed complex ΠX_* is free, the right hand side of equation (10.0.1) can be quite explicit, particularly if C is finite. A morphism $\Pi X_* \to C$ is determined by a list of elements of C in various dimensions, subject to boundary conditions. The homotopy classification of these is then an explicit equivalence relation. Of course, because of the nonabelian nature of some of the information in a crossed complex, there are computability questions, and there are also questions of how to analyse this information. We shall find the notion of *fibration of crossed complexes*, and some associated exact sequences, useful in this respect.

Because of the central nature of cubical methods for some of our major results on crossed complexes, it is convenient to define the classifying space BC cubically. So the first sections of this chapter are devoted to an account of cubical sets and related results.

10.1 The cubical site

This Section contains an introductory account of the category Cub of cubical sets and its relationship with the category Top of topological spaces. These basic facts may be found in many places, two quite recent accounts may be found for example in [Cis06, Jar06].

10.1.1 The box category.

The usual definition of cubical set is as a functor from a small category which we call the *site* for cubical sets. We begin by defining the category that we are going to use.

Definition 10.1.1 The *box category* $\square$ is the subcategory of Top having as objects the standard n-cubes $I^n = [0,1]^n$ for $n \geqslant 0$ and the morphisms $\square(I^n, I^m)$ are the maps that can be got by composition of the face inclusions and of projections

$$\delta_i^\alpha : I^n \to I^{n+1} \qquad \text{and of projections} \qquad \sigma_i : I^{n+1} \to I^n$$

defined respectively by

$$\delta_i^\alpha(x_1, \cdots, x_{i-1}, x_i, \cdots, x_n) = (x_1, \cdots, x_{i-1}, l(\alpha), x_i, \cdots, x_n)$$

for $i = 1, 2, \ldots, n$; $\alpha = +, -$ where $l(+) = 1$, $l(-) = 0$, and

$$\sigma_i(x_1, \cdots, x_{i-1}, x_i, \cdots, x_{n+1}) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1})$$

for $i = 1, 2, \ldots, n+1$. □

Proposition 10.1.2 *The morphisms in the category $\square$ are given by all possible composition of inclusions of faces and of projections subject to the relations*

$$\delta_j^\beta \delta_i^\alpha = \delta_i^\alpha \delta_{j-1}^\beta \qquad (i < j), \qquad \text{(A.1)(i)}$$

$$\sigma_j \sigma_i = \sigma_i \sigma_{j+1} \qquad (i \leqslant j), \qquad \text{(A.1)(ii)}$$

$$\sigma_j \delta_i^\alpha = \begin{cases} \delta_i^\alpha \sigma_{j-1} & (i < j) \\ \delta_{i-1}^\alpha \sigma_j & (i > j) \\ \mathrm{id} & (i = j) \end{cases} \qquad \text{(A.1)(iii)}$$

Remark 10.1.3 Using these relations we can easily check that any morphism in $\square$ has a unique expression as $\delta_{i_1}^{\alpha_1} \cdots \delta_{i_k}^{\alpha_k} \sigma_{j_1} \cdots \sigma_{j_l}$ with $i_1 \leqslant \ldots \leqslant i_k$ and $j_1 < \ldots < j_l$. □

10.1.2 The category Cub of cubical sets.

Now cubical sets are just functors from the $\square$ category to the category of sets.

Definition 10.1.4 The category Cub of *cubical sets* is the functor category $\mathsf{FUN}(\square^{\mathrm{op}}, \mathsf{Sets})$. Thus a cubical set is a functor

$$K : \square^{\mathrm{op}} \to \mathsf{Sets},$$

and a map of cubical sets is a natural transformation of functors. □

Remark 10.1.5 A cubical set K is defined by the family of sets $\{K_n = K(I^n)\}_{n \geqslant 0}$, the face maps $\partial_i^\alpha = K(\delta_n^\alpha) : K_n \to K_{n-1}$ $(i = 1, 2, \ldots, n;\ \alpha = +, -)$ and the degeneracy maps $\epsilon_i = K(\sigma_i) : K_{n-1} \to K_n$ $(i = 1, 2, \ldots, n)$ satisfying the usual cubical relations:

$$\partial_i^\alpha \partial_j^\beta = \partial_{j-1}^\beta \partial_i^\alpha \qquad (i < j), \qquad \text{(B.1)(i)}$$

$$\epsilon_i \epsilon_j = \epsilon_{j+1} \epsilon_i \qquad (i \leqslant j), \qquad \text{(B.1)(ii)}$$

$$\partial_i^\alpha \epsilon_j = \begin{cases} \epsilon_{j-1} \partial_i^\alpha & (i < j) \\ \epsilon_j \partial_{i-1}^\alpha & (i > j) \\ \mathrm{id} & (i = j) \end{cases} \qquad \text{(B.1)(iii)}$$

A very important example is the 'free cubical set on one generator in dimension n' which we denote $\mathbb{I}^n$:

Definition 10.1.6 For $n \geqslant 0$ we define $\mathbb{I}^n$ as the cubical set whose m-cells are $\square(I^n, I^m)$ for all $m \geqslant 0$ and whose morphisms are defined by composition. □

Proposition 10.1.7 *Any cubical morphism* $\hat{x} \in \mathsf{Cub}(\mathbb{I}^n, K)$ *corresponds to an element of* K_n *(*$x = \hat{x}(1_{I^n})$*) giving a natural bijection* $\mathsf{Cub}(\mathbb{I}^n, K) \to K_n$. □

Remark 10.1.8 Thus there is an embedding $\square \to \mathsf{Cub}$ which sends $I^n \mapsto \mathbb{I}^n$. This is an example of the *Yoneda embedding* $\Upsilon : C \to \mathsf{FUN}(C^{op}, \mathsf{Sets})$ for any small category C. One of the properties of this embedding is that any object of $\mathsf{FUN}(C^{op}, \mathsf{Sets})$ is a colimit of images under Υ of the objects of the category C. □

Another very important example is the singular cubical set of a topological space:

Definition 10.1.9 For any topological space X, its *singular cubical set* $S^\square X$ is given by all singular cubes, i.e.

$$(S^\square X)_n = \{\sigma : I^n \to X \mid \sigma \text{ a continuous map}\}$$

with faces and degeneracies given by composition with the maps $\delta_i^\alpha : I^{n-1} \to I^n$ and $\sigma_i : I^{n+1} \to I^n$ defined above. This gives a functor

$$S^\square : \mathsf{Top} \to \mathsf{Cub}.$$ □

This definition is a preliminary to the construction of cubical singular homology of a space which we outline in section 14.7 (see also, for example, [Mas80]).

10.1.3 Geometric realisation of a cubical set

There is a left adjoint to this singular cubical set functor:

Definition 10.1.10 For any cubical set $K : \square^{op} \to C$, its *geometric realisation* $|K|$ is the quotient space

$$|K| = \frac{\bigsqcup_n K_n \times I^n}{\equiv}$$

where K_n is given the discrete topology, I^n its standard topology, and the equivalence relation is generated by $(\partial_i^\alpha x, u) \equiv (x, \delta_i^\alpha u)$ and $(\varepsilon_i y, u) \equiv (y, \sigma_i u)$ where $x \in K_{n+1}, y \in K_{n-1}$ and $u \in I^n$. □

This definition comes under the general scheme of a *coend* (see The Appendix). The formal properties of coends and ends are useful for deriving the properties we need for the geometric realisation.

Remark 10.1.11 The realisation of a cubical set $|K|$ can also be interpreted as a coend:

$$|K| = \int^{\square, n} K_n \times I^n. \qquad \square$$

Proposition 10.1.12 *The realisation of a cubical set is a CW-complex having one n-cell for each non-degenerate n-cell.*

Remark 10.1.13 Thus each point of the realisation of a cubical set $|K|$ is an equivalence class $|x, u|$ with $x \in K_n$ and $u \in I^n$ and it has a unique representative $|x, u|$ with x a non-degenerate cube. $\square$

Using this representation it is not difficult to prove that the realisation functor $|\ |$ is left adjoint to the singular cubical functor $S^\square$.

Theorem 10.1.14 *The realisation functor $|\ |$ is left adjoint to the singular cubical functor $S^\square$. That is for each cubical set K and topological space X, there is a natural bijection*

$$\Psi : \mathsf{Top}(|K|, X) \to \mathsf{Cub}(K, S^\square X).$$

Proof For any continuous map $g : |K| \to X$, the cubical map

$$\Psi(g) : K \to S^\square X$$

is given in dimension n by

$$\Psi_n(g)(x)(u) = g(|x, u|),$$

for any n-cube $x \in K_n$ and point $u \in I^n$. The maps Ψ defines a natural transformation, whose inverse is given by sending a cubical map f to the continuous map defined by mapping any class $|x_n, u|$ to $f_n(x_n)(u)$. $\square$

Our aim is to define homotopy theory for cubical sets and to relate this to homotopy theory for topological spaces. This is essential for our main result on homotopy classification.

10.2 Monoidal closed structure on Cub

A *monoidal closed structure* on the category of cubical sets gives for cubical sets M, L, M natural constructions of a *tensor product* $K \otimes L$ and an *internal hom* or *morphism object* $\mathsf{CUB}(L, M)$ which satisfy an *exponential law* in the form of a natural isomorphism

$$\mathsf{Cub}(K \otimes L, M) \cong \mathsf{Cub}(K, \mathsf{CUB}(L, M)). \qquad \text{(expcub)}$$

Since there is also a 'unit interval object' $\mathbb{I}$ as a cubical set, this enables homotopies between cubical sets L and M to be studied, as in any monoidal closed category with a unit interval object, using either maps from the product $\mathbb{I} \otimes L$ to M or maps from $\mathbb{I}$ to the internal morphism object $\mathsf{CUB}(L, M)$ from L to M. However a special condition on the cubical set M (the Kan extension condition) is need to ensure homotopy between maps $L \to M$ is an equivalence relation.

These results will be used in Chapter 15.

10.2.1 Tensor product of cubical sets

We first give the tensor product, which gives the monoidal structure and is an intermediate step in the construction of the internal morphisms functor. The tensor product is defined by a universal property with respect to bicubical maps (rather like the usual tensor product of modules has with respect to bilinear maps).

The tensor product is associative (Proposition 10.2.6), but not symmetric; the failure of symmetry can be controlled by a 'transposition' functor which will be given in Proposition 10.2.20 and Remark 10.2.22.

An n-cube in the tensor product $K \otimes L$ is going to be the 'product' of a p-cube $k \in K$ and a q-cube $l \in L$ for $p + q = n$. We just take care that the last degeneracy in the first factor agrees with the first in the second factor (the reason becomes clear in the geometric example).

Definition 10.2.1 If K, L are cubical sets, their *tensor product* $K \otimes L$ is defined by

$$(K \otimes L)_n = \frac{\left(\bigsqcup_{p+q=n} K_p \times L_q\right)}{\sim}$$

where $\sim$ is the equivalence relation generated by $(\varepsilon_{r+1}x, y) \sim (x, \varepsilon_1 y)$ for $x \in K_r, y \in L_s$ $(r+s = n-1)$. We write $x \otimes y$ for the equivalence class of (x, y). The maps $\partial_i^\alpha, \varepsilon_i$ are defined for $x \in K_p, y \in L_q$ by

$$\partial_i^\alpha(x \otimes y) = \begin{cases} (\partial_i^\alpha x) \otimes y & \text{if } 1 \leqslant i \leqslant p, \\ x \otimes (\partial_{i-p}^\alpha y) & \text{if } p+1 \leqslant i \leqslant p+q, \end{cases}$$

$$\varepsilon_i(x \otimes y) = \begin{cases} (\varepsilon_i x) \otimes y & \text{if } 1 \leqslant i \leqslant p+1, \\ x \otimes (\varepsilon_{i-p} y) & \text{if } p+1 \leqslant i \leqslant p+q+1 \end{cases}$$

and make $K \otimes L$ a cubical set. □

Remark 10.2.2 We note that in $K \otimes L$, we have

$$(\varepsilon_{p+1}x) \otimes y = x \otimes (\varepsilon_1 y)$$

when $x \in K_p$. □

The realisation functor has a strong and simple relation to the tensor product. This is one of the reasons for the utility of cubical methods in contrast to simplicial methods.

Proposition 10.2.3 *Let* K, L *be cubical sets. Then there is a cellular isomorphism*

$$\chi : |K| \otimes |L| \to |K \otimes L|.$$

Proof The bracketing homeomorphism $I^n \cong I^r \times I^s$ whenever $r + s = n$ yields a homeomorphism

$$K_r \times L_s \times I^n \cong K_r \times I^r \times L_s \times I^s$$

whenever $r + s = n$. One now checks that the identifications to give the realisations are on both sides obtained from

$$\bigsqcup_{r+s=n} K_r \times L_s \times I^n$$

by the same identifications. □

To give Cub the structure of a monoidal closed category, we have to construct not only a tensor product, but also an internal hom functor $\mathsf{CUB}(L, M)$ for cubical sets L, M and a natural equivalence of the form (expcub).

First, we are going to interpret the left part of this equivalence in terms of bicubical maps. This procedure resembles the use of bilinear maps as an intermediate step between the tensor product of R-modules and the R-module of homomorphisms.

Definition 10.2.4 A family of maps

$$f_{pq} : K_p \times L_q \to M_{p+q}$$

is called a *bicubical map* $f : (K, L) \to M$ if it satisfies for all p, q and $\alpha = \pm$:

$$\partial_i^\alpha f_{pq}(x, y) = \begin{cases} f_{p-1,q}(\partial_i^\alpha x, y) & \text{if } 1 \leqslant i \leqslant p \\ f_{p,q-1}(x, \partial_{i-p}^\alpha y) & \text{if } p+1 \leqslant i \leqslant p+q, \end{cases} \tag{i}$$

$$\varepsilon_i f_{pq}(x, y) = \begin{cases} f_{p+1,q}(\varepsilon_i x, y) & \text{if } 1 \leqslant i \leqslant p+1 \\ f_{p,q+1}(x, \varepsilon_{i-p} y) & \text{if } p+1 \leqslant i \leqslant p+q+1. \end{cases} \tag{ii}$$

(Notice that this last part gives further vindication of the rule $(\varepsilon_{p+1} x) \otimes y = x \otimes (\varepsilon_1 y)$). □

We now check that the tensor product is the universal construction with respect to bicubical maps. This fact is used as an intermediate step in our route to the internal hom functor CUB.

Proposition 10.2.5 *The projections*

$$\chi_{pq} : K_p \times L_q \to (K \otimes L)_{p+q}$$

defined by $\chi_{pq}(x, y) = x \otimes y$ *form a bicubical map which is universal with respect to all bicubical maps from* (K, L).

Proof Any cubical map $f : K \otimes L \to M$ defines a family of functions $\hat{f}_{pq} : K_p \times L_q \to M_{p+q}$ (given by $\hat{f}_{pq}(x, y) = f_{p+q}(x \otimes y)$) that clearly form a bicubical map.

Conversely, given a bicubical map $f : (K, L) \to M$, there is a unique cubical map $\hat{f} : K \otimes L \to M$ defined by $\hat{f}_{p+q}(x \otimes y) = f_{pq}(x, y)$. The uniqueness is clear. The map f is well defined because the defining equations (ii) for a bicubical map imply that, for $x \in K_p$ and $y \in L_q$

$$f_{p+1,q}(\varepsilon_{p+1} x, y) = \varepsilon_{p+1} f_{pq}(x, y) = f_{p,q+1}(x, \varepsilon_1 y).$$

It is an easy exercise to prove that the resulting map $K \otimes L \to M$ is cubical. □

Proposition 10.2.6 *For cubical sets* K, L, M *there is a natural isomorphism*

$$(K \otimes L) \otimes M \cong K \otimes (L \otimes M).$$

Proof Both sides of the above equation may be defined as universal with respect to *tricubical maps* from (K, L, M). We leave details to the reader. □

10.2.2 Homotopies of cubical maps

Let us move on to the construction of the internal hom CUB. Recall From Proposition 10.1.7 that, for any cubical set K, we have $K_n \cong \mathsf{Cub}(\mathbb{I}^n, K)$ where $\mathbb{I}^n$ is the cubical set freely generated by one element c_n in dimension n.

Thus the internal morphism construction $\mathsf{CUB}(K, L)$ has to be a cubical set satisfying

$$\mathsf{CUB}_n(K, L) \cong \mathsf{Cub}(\mathbb{I}^n, \mathsf{CUB}(K, L)) \cong \mathsf{Cub}(\mathbb{I}^n \otimes K, L)$$

i.e. the n-dimensional elements of $\mathsf{CUB}(K, L)$ are 'n-fold left homotopies'.

Using Proposition 10.2.5 any element $h \in \mathsf{CUB}_n(K, L)$ may be considered also as a bicubical map

$$\hat{h} : (\mathbb{I}^n, K) \to L.$$

Let us begin with the case $n = 1$: then $\mathbb{I}^1 = \mathbb{I}$ is the cubical set generated by c_1 in dimension 1. We denote its vertices by $0 = \partial^+ c_1$ and $1 = \partial^- c_1$. The cubical set $\mathbb{I}$ plays the role of the unit interval in homotopy theory. It is clear that a homotopy

$$h : \mathbb{I} \otimes K \to L$$

would be given by the images of all $h(c_1, x) \in L_{n+1}$ for all $x \in K_n$. Essentially it should be a 'degree one' cubical morphism that forgets about the $\partial_1^\pm$ (which are used to give the images of 0 and 1). Let us make this precise.

Definition 10.2.7 For any cubical set K we define the *left path complex* PK to be the cubical set with

$$(PK)_r = K_{r+1}$$

and cubical operations

$$\begin{aligned} \partial_2^\alpha, \partial_3^\alpha, \dots, \partial_{r+1}^\alpha &: (PK)_r \to (PK)_{r+1}, \quad \text{and} \\ \varepsilon_2, \varepsilon_3, \dots, \varepsilon_{r+1} &: (PK)_{r-1} \to (PK)_r \end{aligned}$$

(that is, we ignore the first operations $\partial_1^-, \partial_1^+, \varepsilon_1$ in each dimension r.)

This construction gives a functor

$$P : \mathsf{Cub} \to \mathsf{Cub}.$$

□

Proposition 10.2.8 *The functor P is right adjoint to $\mathbb{I} \otimes -$, i.e. there is a natural one-one correspondence between*

1. *Cubical maps $\tilde{f} : K \to PL$ and*

2. *Cubical maps $f : \mathbb{I} \otimes K \to L$.*

Proof The proposition follows because both are clearly equivalent to bicubical maps $(\mathbb{I}, K) \to L$.
□

Remark 10.2.9 Here corresponding maps $f, \tilde{f}$ are related by $\tilde{f}(x) = f(c_1 \otimes x)$ and either of them is termed a *left homotopy* from f_0 to f_1, where $f_\alpha : K \to L$ is given by

$$f_\alpha x = f(\alpha \otimes x) = \partial_1^\alpha \tilde{f} x \qquad (\alpha = 0, 1). \qquad \Box$$

The iteration of the left path complex gives a cubical set that is right adjoint to the tensor product with respect to $\mathbb{I}^n$, and thus classifies n-fold left homotopies.

Definition 10.2.10 We define the *n-fold left path complex* $P^n M$ inductively by $P^n M = P(P^{n-1}M)$, so that

$$(P^n M)_r = M_{n+r}$$

with cubical operations

$$\partial_{n+1}^\alpha, \partial_{n+2}^\alpha, \ldots, \partial_{n+r}^\alpha : (P^n M)_r \to (P^n M)_{r-1}$$

$$\varepsilon_{n+1}, \varepsilon_{n+2}, \ldots, \varepsilon_{n+r} : (P^n M)_{r-1} \to (P^n M)_r$$

(that is, we ignore the first n operations $\partial_i^\alpha, \varepsilon_i$ for $i = 1, \cdots n$ in each dimension.) $\Box$

As before, this functor is a special case of the right adjoint to the tensor product.

Proposition 10.2.11 *The functor P^n is right adjoint to $\mathbb{I}^n \otimes -$, i.e. there is a natural one-one correspondence between*

1. *Cubical maps $\tilde{f} : L \to P^n M$ and*
2. *Cubical maps $f : \mathbb{I}^n \otimes L \to M$.*

Proof As before, we can check that both are equivalent to bicubical maps $(\mathbb{I}^n, L) \to M$. $\Box$

Remark 10.2.12 Here corresponding maps $f, \tilde{f}$ are related by $\tilde{f}(x) = f(c_n \otimes x)$ and either of them is termed a *n-fold left homotopy*. $\Box$

That gives the following relation between free cubical sets.

Corollary 10.2.13 *There are natural (and coherent) isomorphisms of cubical sets*

$$\mathbb{I}^m \otimes \mathbb{I}^n \cong \mathbb{I}^{m+n}.$$

Proof This follows from Proposition 10.2.11 since $P^m \circ P^n = P^{m+n}$. $\Box$

10.2.3 The internal hom functor on Cub

Using homotopies, we have constructed the sets $\mathsf{CUB}_n(L, M)$ for cubical sets L and M and any $n \geqslant 0$. To define the cubical set $\mathsf{CUB}(L, M)$ it remains to define faces and degeneracies.

Notice that the omitted operations

$$\partial_1^\alpha, \ldots, \partial_n^\alpha \quad \text{and} \quad \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$$

in each dimension induce morphisms of cubical sets

$$\partial_1^\alpha, \ldots, \partial_n^\alpha : P^n M \longrightarrow P^{n-1}M, \quad \text{and} \quad \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n : P^{n-1}M \longrightarrow P^n M.$$

These morphisms satisfy the cubical laws.

Definition 10.2.14 We now define the *cubical internal hom*

$$\mathsf{CUB}_n(L, M) = \mathsf{Cub}(L, P^n M)$$

and observe that the family $\mathsf{CUB}(L, M)$ of sets $\mathsf{CUB}_n(L, M)$ for $n \geqslant 0$ gets a cubical structure. Its cubical operations

$$\partial_1^\alpha, \partial_2^\alpha, \ldots, \partial_n^\alpha : \mathsf{CUB}_n(L, M) \to \mathsf{CUB}_{n-1}(L, M);$$
$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n : \mathsf{CUB}_{n-1}(L, M) \to \mathsf{CUB}_n(L, M)$$

are induced by those of M. □

Remark 10.2.15 Thus a typical $f \in \mathsf{CUB}_n(L, M)$ is a family of maps $f_r : L_r \to M_{n+r}$ satisfying

$$f_{r-1}\partial_i^\alpha = \partial_{n+i}^\alpha f_r, \qquad f_r \varepsilon_j = \varepsilon_{n+j} f_{r-1} \quad (i, j = 1, 2, \ldots, r)$$

and its faces and degeneracies are given by

$$(\partial_i^\alpha f)_r = \partial_i^\alpha f_r \qquad (\varepsilon_j^\alpha f)_r = \varepsilon_j^\alpha f_r \quad (i, j = 1, 2, \ldots, n, \alpha = 0, 1).$$

In geometric terms, the elements of $\mathsf{CUB}_0(L, M)$ are the cubical maps $L \to M$, the elements of $\mathsf{CUB}_1(L, M)$ are the (left) homotopies between such maps, the elements of $\mathsf{CUB}_2(L, M)$ are homotopies between homotopies, etc. □

Proposition 10.2.16 *The functor* $\mathsf{CUB}(L, -) : \mathsf{Cub} \to \mathsf{Cub}$ *is right adjoint to* $- \otimes L$. *Moreover, the bijections*

$$\mathsf{Cub}(K \otimes L, M) \cong \mathsf{Cub}(K, \mathsf{CUB}(L, M))$$

giving the adjointness are natural with respect to K, L, M.

Proof As before the bijections can be obtained via bicubical maps $(K, L) \to M$. □

As a special case:

Corollary 10.2.17 *The functor* $- \otimes \mathbb{I}^n$ *is left adjoint to* $\mathsf{CUB}(\mathbb{I}^n, -) : \mathsf{Cub} \to \mathsf{Cub}$.

Corollary 10.2.18 *For cubical sets* K, L, M *there is a natural isomorphism of cubical sets*

$$\mathsf{CUB}(K \otimes L, M) \cong \mathsf{CUB}(K, \mathsf{CUB}(L, M)).$$

Proof It is easy to use associativity of the tensor product and the exponential law repeatedly to give for any cubical set E a natural bijection

$$\mathsf{Cub}(E, \mathsf{CUB}(K \otimes L, M)) \cong \mathsf{Cub}(E, \mathsf{CUB}(K, \mathsf{CUB}(L, M))).$$

The result follows. □

The tensor product is not symmetric because $(x, y) \mapsto y \otimes x$ is not a bicubical map. We have also seen that the functors $- \otimes \mathbb{I}^n$ and $\mathbb{I}^n \otimes -$ have different left adjoints. Nevertheless, we can get some symmetry via a 'transposition' functor.

Definition 10.2.19 We define a 'transposition' functor

$$T : \mathsf{Cub} \to \mathsf{Cub},$$

where TK has the same elements as K in each dimension but has its face and degeneracy operations numbered in reverse order, that is, the cubical operations

$$d_i^\alpha : (TK)_n \to (TK)_{n-1} \quad \text{and} \quad e_i : (TK)_{n-1} \to (TK)_n$$

are defined by $d_i^\alpha = \partial_{n+1-i}^\alpha, e_i = \varepsilon_{n+1-i}$.

There are some immediate consequences

Proposition 10.2.20 *The functor* T *satisfies*

1.- T *is an involution, i.e.* $\mathsf{T}^2\mathsf{K}$ *is naturally isomorphic to* K*;*

2.- $\mathsf{T}(\mathsf{K}\otimes\mathsf{L})$ *is naturally isomorphic to* $\mathsf{T}(\mathsf{L})\otimes\mathsf{T}(\mathsf{K})$*; and*

3.- *there is an obvious cubical isomorphism* $\mathsf{T}\mathbb{I}^n \cong \mathbb{I}^n$. □

Instead of an isomorphism of $\mathsf{CUB}(\mathbb{I}^n,\mathsf{L})$ with $\mathsf{P}^n\mathsf{L}$, we have:

Corollary 10.2.21 *There is a natural isomorphism of cubical sets*

$$\mathsf{CUB}(\mathbb{I}^n,\mathsf{L}) \cong \mathsf{T}\mathsf{P}^n\mathsf{T}\mathsf{L}.$$

Proof By Corollary 10.2.11, P^n is right adjoint to $\mathbb{I}^n\otimes -$, so $\mathsf{P}^n\mathsf{T}$ is right adjoint to $\mathsf{T}(\mathbb{I}^n\otimes -)$. Hence $\mathsf{T}\mathsf{P}^n\mathsf{T}$ is right adjoint to $\mathsf{T}(\mathbb{I}^n\otimes\mathsf{T}-)\cong(-\otimes\mathsf{T}\mathbb{I}^n)$ that is naturally isomorphic to $(-\otimes\mathsf{T}\mathbb{I}^n)\cong(-\otimes\mathbb{I}^n)$. Hence $\mathsf{T}\mathsf{P}^n\mathsf{T}$ is naturally isomorphic to the right adjoint $\mathsf{Cub}(\mathbb{I}^n,-)$ of $-\otimes\mathbb{I}^n$. □

Remark 10.2.22 A simpler argument shows that for any cubical complex K the functor $\mathsf{K}\otimes - : \mathsf{Cub}\to\mathsf{Cub}$ has right adjoint $\mathsf{T}(\mathsf{CUB}(\mathsf{TK},\mathsf{T}-))$ and hence that the monoidal closed category Cub is *biclosed*, in the sense of Kelly [Kel36], even though it is not symmetric. □

Corollary 10.2.23 *For cubical sets* K,L *the functors* $\mathsf{Cub}\to\mathsf{Cub}$ *given by* $\mathsf{K}\otimes -$ *and* $-\otimes\mathsf{L}$ *preserve colimits.* □

10.3 Homotopy theory of cubical sets

In this Section we sketch how to develop a homotopy theory of cubical sets directly from the cubical structure.

10.3.1 Kan cubical sets

In arguing with the extension condition which is known as the Kan condition on cubical sets, it is often easier to work with geometric models. These are easier to see as real cubes made from the geometric I^n, where $I=[0,1]$ is the unit interval, and subcomplexes of I^n, but the same arguments can be given for the models $\mathbb{I}^n$ of these complexes in the category Cub, which we call 'formal cubes' and their subcomplexes 'formal subcomplexes'. By a 'cell' in $\mathbb{I}^n$ we mean a non degenerate element. We have to be careful in this section because we are thinking in terms of geometric cubes and their union, but for cubical sets we have elements of various dimensions. Thus in $C\cup a$ as given below where C is a subcomplex and a is a cell, the $\cup$ means union in the sense of subcomplexes generated by C, a. We will use the results of this section in Part III.

Definition 10.3.1 Let B, C be subcomplexes of $\mathbb{I}^n$ such that $C\subseteq B$. We say that C is an *elementary collapse* of B, written $B\searrow^e C$, if for some $s\geqslant 1$ there is an s-cell a of B and $(s-1)$-face b of a such that

$$B = C\cup a, \qquad C\cap a = \partial a\setminus b$$

(where $\partial a\setminus b$ denotes the set of the proper faces of a except b). The face b is called the *free face* of the collapse.

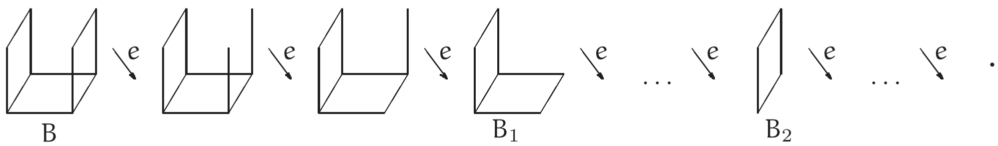
If there is a sequence

$$B_1 \overset{e}{\searrow} B_2 \overset{e}{\searrow} \cdots \overset{e}{\searrow} B_r$$

of elementary collapses, then we write $B_1 \searrow B_r$ and say B_1 *collapses* to B_r. □

Example 10.3.2 If a is a cell then $a \otimes \mathbb{I}$ collapses to $a \otimes \{0\} \cup \partial a \otimes \mathbb{I}$. Here the free face of the collapse is $a \otimes \{1\}$.

We will next define the notion of a 'partial box'. The following picture gives three examples B, B_1, B_2 as part of a choice of a sequence of collapsings $B \searrow \mathbf{0}$ through B_1, B_2.



The formal definition of 'partial box' allows us to give a more widely applicable formulation of the usual Kan extension condition on a cubical set.

Definition 10.3.3 Let C be an r-cell in the n-cube $\mathbb{I}^n$. Two $(r-1)$-faces of C are called *opposite* if they do not meet (except possibly in degenerate elements). A *partial* $(r-1)$*–box* in C is a subcomplex B of C generated by one $(r-1)$-face b of C (called a *base* of B) and a number, possibly zero, of other $(r-1)$-faces of C none of which is opposite to b. The partial box is a *box* if its $(r-1)$-cells consist of all but one of the $(r-1)$-faces of C. □

Proposition 10.3.4 *If* B *is a partial box in* $\mathbb{I}^m$ *then* (i) $B \otimes \mathbb{I}^n$*, and* (ii) $B \otimes \mathbb{I}^n \cup \mathbb{I}^m \otimes \partial\mathbb{I}^n$, *are partial boxes in* $\mathbb{I}^m \otimes \mathbb{I}^n \cong \mathbb{I}^{m+n}$.

Proof Let b be a base for B. Then $b \otimes c^n$ is a base for $B \otimes \mathbb{I}^n$. This proves (i). Further, $\partial(\mathbb{I}^m \otimes \mathbb{I}^n) = (\partial\mathbb{I}^m) \otimes \mathbb{I}^n \cup \mathbb{I}^m \otimes \partial\mathbb{I}^n$, and so (ii) follows. □

We now come to a key theorem on the existence of chains of partial boxes; this applies to give many examples of collapsing, even as a kind of algorithm, and is also essential in the work of Chapter 14.

Theorem 10.3.5 (Chains of partial boxes) *Let* B, B' *be partial boxes in an* r*-cell* C *of* $\mathbb{I}^n$ *such that* $B' \subseteq B$. *Then there is a chain*

$$B = B_s \searrow B_{s-1} \searrow \cdots \searrow B_1 = B'$$

such that

(i) *each* B_i *is a partial box in* C*;*

(ii) $B_{i+1} = B_i \cup a_i$ *where* a_i *is an* $(r-1)$*-cell of* C *not in* B_i*;*

(iii) $a_i \cap B_i$ *is a partial box in* a_i.

Proof We first show that there is a chain

$$B' = B_1 \subset \cdots \subset B_{s-1} \subset B = B_s$$

of partial boxes and a set of $(r-1)$-cells $a_1, a_2, \cdots, a_{s-1}$ such that $B_{i+1} = B_i \cup a_i$, $a_i \not\subseteq B_i$.

If B and B' have a common base this is clear, since we may adjoin to B' the $(r-1)$-cells of $B \setminus B'$ one at a time in any order. If B and B' have no common base, choose a base b for B and let b' be its opposite face in C. Then neither b nor b' is in B'. Hence $B_2 = B' \cup b$ is a partial box with base b and we are reduced to the first case.

Now consider the partial box $B_{i+1} = B_i \cup a_i$, $a \not\subseteq B_i$. We claim that $a_i \cap B_i$ is a partial box in a_i. To see this, choose a base b for B_{i+1} with $b \neq a_i$; this is possible because if a_i were the only base for B_{i+1}, then B_i would consist of a number of pairs of opposite faces of C and would not be a partial box. We now have $a_i \neq b$, $a_i \neq b'$, so $a_i \cap b$ is an $(r-2)$-face of a_i. Its opposite face in a_i is $a_i \cap b'$ and this is not in B_i because the only $(r-1)$-faces of C which contain it are a_i and b'. Hence $a_i \cap B_i$ is a partial box with base $a_i \cap b$.

The proof is now completed by induction on the dimension r of C. If $r = 1$, the theorem is trivial. If $r > 1$, choose B_i, a_i as above. Since $a_i \cap B_i$ is a partial box in a_i, there is a box J in a_i containing it. The elementary collapse $a_i \searrow^{e} J$ gives $B_{i+1} \searrow^{e} B_i \cup J$. But by the induction hypothesis, J can be collapsed to the partial box $a_i \cap B_i$ in a_i, and this implies $B_{i+1} \searrow B_i$. □

Corollary 10.3.6 *If C is a partial $(n-1)$–box in $\mathbb{I}^n$ then $\mathbb{I}^n$ collapses to C.*

Proof We extend C to a box B. By definition, $\mathbb{I}^n$ collapses to B. By the previous theorem, B collapses to C. □

Corollary 10.3.7 *$\mathbb{I}^n$, and any box in $\mathbb{I}^n$, collapses to any of its vertices.*

Proof It is sufficient to prove collapsing to the vertex **0**. We know $\mathbb{I}^n$ collapses to the partial box $\{0\} \otimes \mathbb{I}^{n-1}$. Similarly, any partial box in $\mathbb{I}^n$ collapses to any of its faces. Now proceed by induction. □

Definition 10.3.8 Let K be a cubical set. We say K satisfies the *Kan extension condition*, or *is Kan*, if for every $r \geqslant 1$ and any partial $(r-1)$–box in $\mathbb{I}^r$, any map $B \to K$ extends over I^r. □

Proposition 10.3.9 *A cubical set is Kan if and only if for every $n \geqslant 1$ and any $(n-1)$–box in $\mathbb{I}^n$, any map $B \to K$ extends over $\mathbb{I}^n$.*

Proof The implication one way is trivial. Suppose then the extension over boxes condition is fulfilled, and C is a partial $(n-1)$–box in $\mathbb{I}^n$.Then C is contained in a box B. By assumption and theorem 10.3.5, $\mathbb{I}^n \searrow B \searrow C$. By repeated application of the Kan condition, any map $C \to K$ extends over $\mathbb{I}^n$. □

Example 10.3.10 For any space X the singular cubical set $S^{\square}X$ is a Kan complex. Thus there exists a retraction from I^n to the box $I \times I^{n-1} \to \{0\} \times I^{n-1} \cup I \times \partial I^n$ and indeed to any other box in a similar manner.

Theorem 10.3.11 *If L, M are cubical sets such that M is Kan, then $\mathsf{CUB}(L, M)$ is also Kan.*

Proof Let B be an $(m-1)$–box in $\mathbb{I}^m$. We have to prove that any map $f : B \to \mathsf{CUB}(L, M)$ extends over $\mathbb{I}^m$. But f is equivalent to a map $\hat{f} : B \otimes L \to M$. So it is equivalent to extend $\hat{f}$ over $\mathbb{I}^m \otimes L$.

Let $L^{[n]}$ be the subcubical set of L generated by L_i for $i \leqslant n$. We construct the extension by induction over $\mathbb{I}^m \otimes L^{[n]}$. The case $n = 0$ is easy.

The inductive hypothesis implies we have an extension of the restriction of $\hat{f}$ over $\mathbb{I}^m \otimes L^{[n-1]}$. Let $k \in L_n$, and let $\hat{k} : \mathbb{I}^n \to L$ be the corresponding map. Let h be the composite

$$B \otimes \mathbb{I}^n \xrightarrow{1 \otimes \hat{k}} B \otimes L \xrightarrow{\hat{f}} M.$$

Then we have a map

$$B \otimes \mathbb{I}^n \cup \mathbb{I}^m \otimes \partial\mathbb{I}^n \to M.$$

But $B \otimes \mathbb{I}^n \cup \mathbb{I}^m \otimes \partial\mathbb{I}^n$ is a partial box in $\mathbb{I}^m \otimes \mathbb{I}^n \cong \mathbb{I}^{m+n}$. Since M is Kan, this extends over $\mathbb{I}^m \otimes \mathbb{I}^n$. The image of the top dimensional cell gives the required value of the extension on k. □

10.3.2 Kan fibrations of cubical sets

The applications of the classifying space of a crossed complex require the notion of Kan fibration of cubical set, so we give the theory here. This is also useful in developing the homotopy theory, as we shall see.

Definition 10.3.12 Let $p : L \to M$ be a cubical map. We say p is a *Kan fibration* if for all $n \geqslant 0$ and inclusion $i : B \to \mathbb{I}^n$ of a partial $(n-1)$-box B in $\mathbb{I}^n$, any diagram such as the following

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "i"'] & L \arrow[d, "p"] \\ \mathbb{I}^n \arrow[r, "g"'] \arrow[ur, dashed, "F"] & M \end{tikzcd} \qquad (10.3.1)$$

has a regular completion F; that is given f, g such that $pf = gi$, there is a cubical map F such that $Fi = f$ and $pF = g$. □

Example 10.3.13 A cubical set L is a Kan cubical set if and only if the constant map $L \to *$, where $*$ is a point, is a Kan fibration.

Exercise 10.3.14 If $p : L \to M$ is a Kan fibration, then for each $\nu \in M_0$, $p^{-1}(\nu)$ is a Kan complex. More generally, the pullback of a Kan fibration by any map is also a Kan fibration. □

Exercise 10.3.15 If $p : L \to M$ is a Kan fibration, then so also is

$$\mathsf{CUB}(K, p) : \mathsf{CUB}(K, L) \to \mathsf{CUB}(K, M)$$

for any cubical set K. □

Example 10.3.16 If X_* is a filtered space then there is defined a 'filtered' cubical singular complex RX_* and a quotient morphism $p : RX_* \to \rho X_*$ by taking homotopy classes through filtered maps and relative to the vertices. A key result of Chapter 15, whose proof uses theorem 10.3.5, is that p is a Kan fibration of cubical sets. □

Proposition 10.3.17 *If* $j : A \to K$ *is the inclusion of a subcubical set, and* M *is Kan, then the induced map*

$$\mathsf{CUB}(i, M) : \mathsf{CUB}(K, M) \to \mathsf{CUB}(A, M)$$

is a Kan fibration.

Proof Let $n \geqslant 0$ and let B be a partial $(n-1)$-box in $\mathbb{I}^n$. We have to prove that any diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "i"'] & \mathsf{CUB}(K, M) \arrow[d, "p"] \\ \mathbb{I}^n \arrow[r, "g"'] \arrow[ur, dashed, "F"] & \mathsf{CUB}(A, M) \end{tikzcd}$$

has a completion F. By the exponential law for cubical sets, this is equivalent to any map

$$\mathbb{I}^n \otimes A \cup B \otimes K \to M$$

extends over $\mathbb{I}^n \otimes K$. The proof of this is by induction on the cells of $K \setminus A$, analogously to the proof of theorem 10.3.11. □

In applying the homotopy classification theorem of later sections, we will need to use the realisation of Kan fibrations. The following result will be sufficient for these applications.

Theorem 10.3.18 *Let $p : L \to M$ be a Kan fibration of cubical sets such that M, and hence also L, is Kan. Then $|p| : |L| \to |M|$ is homotopy equivalent over $|M|$ to a fibration of spaces.*

Proof Let $f = |p|$ and choose a factorisation of f

$$|L| \xrightarrow{e} E_f \xrightarrow{\psi} |M|$$

through a homotopy equivalence e and a fibration ψ. Since p is a Kan fibration it has a long homotopy exact sequence for each base point $m \in M_0$ and for the corresponding fibre. Because of the equivalence of homotopy categories given below (theorem 10.3.34), this long exact sequence is mapped isomorphically to the long exact sequence of the fibration ψ. □

10.3.3 Homotopy

In this Section we introduce the basic concepts in homotopy for cubical sets giving a definition of weak equivalence between cubical sets. We also introduce the homotopy groups for Kan cubical sets.

Definition 10.3.19 Let M be a cubical set. For $x, y \in M_0$, we say $x \sim y$ if there is an $a \in M_1$ such that $\partial_1^- a = x$, $\partial_1^+ a = y$.

Proposition 10.3.20 *If M is a Kan cubical set, then the relation $\sim$ on M_0 is an equivalence relation.*

Proof Reflexivity $x \sim x$ is easy by taking $a = \epsilon_1 x$. For the other conditions we need the Kan condition. In the following two diagrams:

$$\begin{tikzcd} x \arrow[r, dashed] \arrow[d, "a"'] & z \\ y \arrow[r, "b"'] & z \arrow[u, "\epsilon_1 z"'] \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "\epsilon_1 x"] \arrow[d, "a"'] & x \arrow[d, "\epsilon_1 x"] \\ y \arrow[r, dashed] & x \end{tikzcd}$$

the first shows the box to fill to obtain transitivity, and the second shows the box to fill to obtain symmetry. □

Definition 10.3.21 Two cubical maps $f, g : L \to M$ are said to be *homotopic* if $f \sim g$ as elements of $\mathsf{CUB}(L, M)_0$.

Remark 10.3.22 It is quite clear that two maps are homotopic if and only if there exists a homotopy

$$H : \mathbb{I} \otimes L \to M$$

from one to the other.

Proposition 10.3.23 *If* M *is a Kan complex, then homotopy is an equivalence relation on maps* $L \to M$.

Definition 10.3.24 If L, M are cubical sets and M is Kan, we define the set of homotopy classes of cubical maps as the quotient

$$[L, M] = \mathsf{Cub}(L, M)/\sim .$$

Let $i : A \to L$ be the inclusion of a subcubical set of the cubical set L. Then the fibre of the map

$$\mathsf{CUB}(i, M) : \mathsf{CUB}(L, M) \to \mathsf{CUB}(A, M)$$

over a map $u : A \to M$, which is just a vertex of $\mathsf{CUB}(A, M)$, is written $\mathsf{CUB}(L, M; u)$. By previous and later results, if M is Kan, so also is $\mathsf{CUB}(L, M; u)$.

Definition 10.3.25 If M is Kan, the set $\pi_0\mathsf{CUB}(L, M; u)$ is also written $[L, M; u]$ and called the set of *homotopy classes of maps* $L \to M$ *rel* u. The disjoint union of these sets is the set of homotopy classes $L \to M$ rel A and is written $[L, M; -]$. □

We need this notion to define the fundamental groupoid $\pi_1 M$ of a Kan complex M.

Let M be a Kan complex. We write $\pi_1 M$ for the set of homotopy classes $\mathbb{I} \to M$ rel $\{0, 1\}$. We know this is well defined. We now introduce a composition on these classes in the usual way, using the ideas of the proof of Proposition 10.3.20. This leads to:

Proposition 10.3.26 *If* M *is a Kan complex, then* $\pi_1 M$ *may be given the structure of groupoid.*

10.3.4 Relation with simplicial sets

The equivalence of the homotopy categories of simplicial sets and and topological spaces was proved a long time ago, see for example [May67], or [GJ99] for a more recent account. There was a general feeling that the same result was true for cubical sets and that the proof should follow the same lines once the proper definitions were given. This has proved very elusive and has been proven recently but going through the simplicial case. We define in this subsection the functors that relate the cubical and simplicial sets.

Definition 10.3.27 For any simplicial set Y we define $S^{\square}(Y)$ its *cubical singular set* as the cubical set given by

$$S^{\square}(Y)_n = \mathsf{Simp}((\Delta^1)^n, Y)$$

with faces and degeneracies coming from the simplicial set structure of $(\Delta^1)^n$. □

Definition 10.3.28 For any cubical set $K : \square^{op} \to C$, its *triangulation* $|K|_S$ is the coend

$$|K|_S = \int^{S^{\square}(\mathbb{I}^n, X)} (\Delta^1)^n .$$

□

Proposition 10.3.29 *There is an adjoint relation*

$$\mathsf{Cub}(K, S^{\square}(Y)) \cong \mathsf{Simp}(|K|_S, Y)$$

for any cubical set K *and simplicial set* Y.

Proposition 10.3.30 *This needs making explicit. When composed with the simplicial singular set and the simplicial realisation they give the cubical singular set and the cubical realisation.*

10.3.5 The equivalence of homotopy categories

The equivalence of the homotopy categories of Kan simplicial sets and the homotopy category of CW-complexes was proved a long time ago, see for example [May67], but the corresponding cubical theory has waited a long time for a detailed, published exposition. We sketch some results that we are going to use. References for this are [Ant02, Jar06, Cis06].

Recall that the homotopy category of Kan simplicial set is equivalent to the category $\mathrm{Ho}(\mathsf{Simp})$ got from the category Simp by adding formal inverses to all weak equivalences. Jardine in [Jar06, Theorem 8.8] proves the following result:

Theorem 10.3.31 *There is an adjoint equivalence of homotopy categories*

$$\mathrm{Ho}(\mathsf{Cub}) \sim \mathrm{Ho}(\mathsf{Simp})$$

given by the "triangulation" functor $|\ \ | : \mathsf{Cub} \to \mathsf{Simp}$ *in one direction and by the "singular cubical set"* $\mathsf{S} : \mathsf{Simp} \to \mathsf{Cub}$ *in the other one.*

In particular, both the unit and the counit are weak equivalences.

Remark 10.3.32 Actually, Jardine proves that both functors preserve weak equivalences. His main technical point consists in defining three different Quillen model structures on the category Cub of cubical sets and using the results of Cisinski in [Cis06] to get that in these three the weak equivalences are actually the same. □

Composing with the classical adjoint equivalence of homotopy categories $\mathrm{Ho}(\mathsf{Simp}) \sim \mathrm{Ho}(\mathsf{Top})$ we get:

Theorem 10.3.33 *There is an adjoint equivalence of the homotopy categories*

$$\mathrm{Ho}(\mathsf{Cub}) \sim \mathrm{Ho}(\mathsf{Top})$$

given by the "realisation" functor $|\ \ | : \mathsf{Cub} \to \mathsf{Top}$ *in one direction and by the "singular cubical set"* $\mathsf{S}^{\square} : \mathsf{Top} \to \mathsf{Cub}$ *in the other one.*

Both the unit and the counit $\varepsilon : |\mathsf{S}^{\square}X| \to X$ *given in Theorem 10.1.14 are weak equivalences.*

As in the simplicial case we can get around the technicalities of adding formal inverses to weak equivalences by restricting ourselves to Kan cubical complexes and CW-complexes getting:

Theorem 10.3.34 *There is an equivalence between the homotopy categories of Kan cubical sets and that of* CW*-complexes.*

Corollary 10.3.35 *There is a natural bijection*

$$[M, N] \cong [|\,M|, |\,N|]$$

for Kan cubical sets $M,\ N$.

Now we proceed to a first step for our homotopy classification theorem.

Theorem 10.3.36 *If* L, M *are cubical sets such that* M *is Kan, then there is a weak homotopy equivalence*

$$\phi : |\mathsf{CUB}(L, M)| \to \mathsf{Top}(|L|, |M|).$$

Proof There is a map

$$\begin{aligned}|\mathsf{CUB}(L,M)| \otimes |L| &\to |\mathsf{CUB}(L,M) \otimes L| \\ &\xrightarrow{|eval|} |M|.\end{aligned}$$

The topological adjoint of this is the map ϕ.

To prove that ϕ is a weak homotopy equivalence we look at the effect of this map on homotopy classes from $|K|$ for an arbitrary cubical set K. We have natural bijections of homotopy classes

$$\begin{aligned}[\,|K|, |\mathsf{CUB}(L,M)|\,] &\cong [K, \mathsf{CUB}(L,M)] && \text{by Corollary 10.3.35} \\ &\cong [K \otimes L, M] && \text{by the exponential law in } \mathsf{Cub} \\ &\cong [\,|K \otimes L|, |M|\,] && \text{by Corollary 10.3.35} \\ &\cong [\,|K| \times |L|, |M|\,] && \text{by Proposition 10.2.3} \\ &\cong [\,|K|, \mathsf{Top}(|L|, |M|)] && \text{by the exponential law in } \mathsf{Top}.\end{aligned}$$

Since these maps are natural, the composite is induced by ϕ. These bijections imply that ϕ is a weak equivalence of spaces. □

We need to move from this to an equivalence of relative, and indeed filtered, theories. Thus in the standard homotopy theory of spaces the relative homotopy group $\pi_n(X, A, a)$ is defined as homotopy classes of maps $I^n, \partial_1^- I^n, J^{n-1}_{(-,1)} \to (X, A, a)$ where I^n is the standard n-cube, $\partial_1^- I^n$ is the $(-,1)$-face and $J^{n-1}_{(-,1)}$ is the union of the other faces of I^n. It is then proved that this set has for $n \geqslant 2$ a group structure induced by composition of cubes in direction 2, and that this structure is abelian for $n \geqslant 3$. For a filtered space X_* various relative homotopy groups may be combined to give a crossed complex ΠX_* where $(\Pi X_*)_n$ is the family of groups $\pi_n(X_n, X_{n-1}, x)$ for $x \in X_0$, for $n \geqslant 2$, $(\Pi X_*)_1$ is the fundamental groupoid $\pi_1(X_1, X_0)$, and $(\Pi X_*)_0$ is X_0.

Such a theory can also be formulated for the relative and indeed filtered homotopy theory of Kan cubical sets. The use of the Kan extension condition for this kind of purpose has been analysed by Kamps and is explained in [KP97]. One of the facts we will use is also that in the cubical set situation we can identify the n-th relative homotopy group of a Kan pair (K, L) as given by elements k of K_n such that $\partial_1^- k \in L_{n-1}$ and $\partial_i^\alpha k \in \operatorname{Im} \epsilon_1^{n-1}$ for $(\alpha, i) \neq (-,1)$. This is the way we wish to define the fundamental crossed complex ΠK_* of a filtered Kan cubical set. In these terms we have the corollary of the equivalence of homotopy categories:

Corollary 10.3.37 *If K_* is a filtration of Kan cubical sets, then the realisation functor gives an isomorphism*

$$\Pi K_* \to \Pi |K_*|. \qquad \square$$

10.4 Cubical sets and crossed complexes

We proceed now a step further and relate the category Cub of cubical sets (or the equivalent of topological spaces) to that of crossed complexes. We assume the monoidal closed structure on the category Crs discussed in Chapter 9.

10.4.1 The fundamental crossed complex of a cubical set

The fundamental crossed complex ΠK of a cubical set K is basic to our work on the classifying space of a crossed complex. First we define $\Pi \mathbb{I}^1$ as the groupoid $\mathcal{I}$, with generator $\iota : 0 \to 1$. Then we define $\Pi\mathbb{I}^r = \mathcal{I}^{\otimes r}$, the r-fold tensor product of $\mathcal{I}$ with itself. We obtain the following:

Theorem 10.4.1 (Homotopy Addition Lemma) *In $\Pi\mathbb{I}^n$ we have a free generator $c^n = \iota^{\otimes n}$ in dimension n with boundary given by:*

$$\delta(c) = \begin{cases} \sum_{i=1}^{n}(-1)^i\{(\partial_i^+ c) - (\partial_i^- c)^{(u_i c)}\} & (n \geqslant 4), \\ -(\partial_3^+ c) - (\partial_2^- c)^{(u_2 c)} - (\partial_1^+ c) + (\partial_3^- c)^{(u_3 c)} + (\partial_2^+ c) + (\partial_1^- c)^{(u_1 c)} & (n = 3), \\ -(\partial_1^+ c) - (\partial_2^- c) + (\partial_1^- c) + (\partial_2^+ c) & (n = 2), \end{cases} \qquad \text{(HAL)}$$

where $u_i = \partial_1^+ \partial_2^+ \cdots \hat{\imath} \cdots \partial_{n+1}^+$ and for $n = 1$, $\delta^\alpha c = \partial_1^\alpha c,\ \alpha = -, +$.

Proof The proof is by induction using the explicit description of the tensor product, analogously to that for the HAL for the simplex Δ^n. □

Remark 10.4.2 The HAL shows that if a is a face of c^n then a can be expressed uniquely in terms of δc^n and the other faces of c^n. We use this fact later. □

Definition 10.4.3 The fundamental crossed complex ΠK of a cubical set K is defined as the coend in Crs:

$$\Pi K = \int^{\square, n} K_n \times \Pi\mathbb{I}^n.$$

Thus ΠK is freely generated by the non degenerate cubes of K with boundaries given by the HAL.

Theorem 10.4.4 *For any cubical sets K, L there is a natural isomorphism*

$$\Pi K \otimes \Pi L \cong \Pi(K \otimes L).$$

Proof It is immediate from the definition that there is an isomorphism

$$\Pi\mathbb{I}^n \otimes \Pi\mathbb{I}^m \cong \Pi\mathbb{I}^{n+m}.$$

Now the coend definition of ΠK yields the result, analogously to the proof of Proposition 10.2.3, using the isomorphism of crossed complexes for $p + q = n$

$$(K_p \times L_q) \otimes \mathcal{I}^n \cong (K_p \times \mathcal{I}^p) \otimes (L_q \times \mathcal{I}^q)$$

where K_p, L_q are discrete crossed complexes. □

Remark 10.4.5 Another view of this result is given in Chapter 15 where the tensor product is set up using the properties of cubical ω-groupoids and the monoidal closed structure on those objects. □

Remark 10.4.6 An application of the Higher Homotopy van Kampen Theorem 8.1.5 gives an isomorphism $\Pi K \cong \Pi|K|_*$ for a cubical set K where $|K|_*$ is the skeletal filtration of the realisation. □

10.4.2 The cubical nerve of a crossed complex

Let us construct an adjoint to the fundamental crossed complex of cubical sets just studied.

Definition 10.4.7 We define the *cubical nerve* NC of a crossed complex C to be in dimension n the set

$$(NC)_n = \mathsf{Crs}(\Pi\mathbb{I}^n, C).$$

Remark 10.4.8 In Chapter 14, Remark 14.6.5, this definition is related to the fundamental algebraic equivalence between the category Crs of crossed complexes and that of cubical ω-groupoids with connections. $\Box$

Proposition 10.4.9 *For a cubical set* K *and crossed complex* C *there is a natural isomorphism*

$$\mathsf{Cub}(K, NC) \cong \mathsf{Crs}(\Pi K, C)$$

making $\Pi : \mathsf{Cub} \to \mathsf{Crs}$ *left adjoint to* $N : \mathsf{Crs} \to \mathsf{Cub}$.

Proof It is based in the fact that one side may be described as a colimit and the other one as a limit:

$$\begin{aligned}
\mathsf{Cub}(K, NC) &\cong \int_{\Box, n} \mathsf{Set}(K_n, (NC)_n) \\
&\cong \int_{\Box, n} \mathsf{Set}(K_n, \mathsf{Crs}(\Pi\mathbb{I}^n, C)) \\
&\cong \int_{\Box, n} \mathsf{Crs}(K_n \times \Pi\mathbb{I}^n, C) \\
&\cong \mathsf{Crs}(\left(\int^{\Box, n} K_n \times \Pi\mathbb{I}^n\right), C) \\
&\cong \mathsf{Crs}(\Pi K, C)
\end{aligned}$$
$\Box$

Proposition 10.4.10 *The cubical nerve* NC *of a crossed complex* C *is a Kan cubical set.*

Proof Let $n \geqslant 0$ and let B be a box in $\mathbb{I}^n$. We us the last proved adjointness relation. So a map $B \to NC$ corresponds to a morphism $f : \Pi B \to C$. Let c^n be the top cell in $\mathbb{I}^n$. We extend f to $g : \Pi\mathbb{I}^n \to C$ by mapping c^n to 0, with the value of g on the omitted $(n-1)$-cell of B being given by the HAL. $\Box$

Remark 10.4.11 Actually the result may be strengthened to say that NC is a T-complex, and indeed N gives an equivalence between the category Crs and that of cubical T-complexes.

Proposition 10.4.12 *Let* C, D *be crossed complexes. There is a natural transformation of cubical sets*

$$\eta : N(C) \otimes N(D) \to N(C \otimes D).$$

Proof It is easy to verify that the function

$$\begin{aligned}
b : N(C), N(D) &\to N(C \otimes D) \\
f, g &\mapsto f \otimes g
\end{aligned}$$

is bicubical. $\Box$

Corollary 10.4.13 *The nerve functor* $\mathsf{N} : \mathsf{Crs} \to \mathsf{Cub}$ *preserves homotopy.*

Proof A homotopy $F : \mathcal{I} \otimes D \to E$ in Crs determines a cubical homotopy as the composition

$$\mathbb{I}^1 \otimes ND \xrightarrow{\eta} N(\mathcal{I} \otimes D) \xrightarrow{N(F)} NE. \qquad \square$$

Now we come to our key theorem which is the basis of the homotopy classification theorem.

Theorem 10.4.14 *For a cubical set* K *and a crossed complex* C *there is a natural isomorphism of cubical sets*

$$\mathsf{CUB}(K, NC) \cong N(\mathsf{CRS}(\Pi K, C)).$$

Proof Let L be a cubical set. Then we have natural bijections

$$\begin{aligned} \mathsf{Cub}(L, \mathsf{CUB}(K, NC)) &\cong \mathsf{Cub}(L \otimes K, NC) \\ &\cong \mathsf{Crs}(\Pi(L \otimes K), C) \\ &\cong \mathsf{Crs}(\Pi L \otimes \Pi K, C) \\ &\cong \mathsf{Crs}(\Pi L, \mathsf{CRS}(\Pi K, C)) \\ &\cong \mathsf{Cub}(L, N(\mathsf{CRS}(\Pi K, C)). \end{aligned}$$

Since this natural bijection holds for all cubical sets L, the theorem follows. $\square$

Remark 10.4.15 Notice the power of the combination of various adjunctions in the proof of the last theorem.

10.4.3 The homotopy classification theorem

Definition 10.4.16 The (cubical) *classifying space* BC of a crossed complex C is defined to be the realisation $|NC|$ of the nerve of C. $\square$

Our aim is the following theorem:

Theorem 10.4.17 (Homotopy Classification Theorem) *Let* X *be a CW-complex and* C *a crossed complex. Then there is a weak equivalence*

$$\chi : B(\mathsf{CRS}(\Pi X_*, C)) \to \mathsf{TOP}(X, BC).$$

Hence there is a bijection

$$[\Pi X_*, C] \cong [X, BC],$$

where the left hand side is homotopy classes of crossed complex maps, and the right hand side is homotopy classes of maps of spaces.

Proof We first assume X is $|K|$ where K is a cubical set. Then the theorem with $X = |K|$ follows from Theorems 10.3.36, 10.4.14. In particular this applies to the case $K = S^{\square}X$. Now by Theorem 10.3.33 X has the homotopy type of $|S^{\square}X|$. $\square$

Remark 10.4.18 This theorem generalises many classical results. It is important that it includes information on fundamental groups and their actions. □

10.5 Fibrations of crossed complexes

The notion of fibration of crossed complexes has an important role in analysing the set $[F, C]$ of homotopy classes of morphisms from a free crossed complex F to a crossed complex C. The notion also allows for relating the homotopy theory of crossed complexes to homotopy theories in other structures, for example that of cubical sets, and as indicated in the Notes at the end of his Chapter.

Definition 10.5.1 A morphism $p : E \to D$ of crossed complexes is a *fibration* if

(i) the morphism $p_1 : E_1 \to D_1$ is a *fibration* of groupoids;

(ii) for each $n \geqslant 2$ and $x \in E_0$, the morphism of groups $p_n : E_n(x) \to D_n(px)$ is surjective.

The morphism p is a *trivial fibration* if it is a fibration, and also a weak equivalence, by which is meant that p induces a bijection on π_0 and isomorphisms $\pi_1(E, x) \to \pi_1(D, px)$, $H_n(E, x) \to H_n(D, px)$ for all $x \in E_0$ and $n \geqslant 2$. □

Remark 10.5.2 It is known that a fibration of groupoids gives rise to a family of exact sequences, [Bro70, Bro06]. There are longer exact sequences for a fibration of crossed complexes which will be stated in Theorem 12.4.1 and applied to the homotopy classification of maps to BC. □

We now analyse cofibrations.

Definition 10.5.3 Consider the following diagram.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ C \arrow[r] \arrow[ur, dotted] & D. \end{tikzcd}$$

If given i the dotted completion exists for all morphisms p in a class $\mathcal{F}$, then we say that i has the *left lifting property (LLP)* with respect to $\mathcal{F}$. We say a morphism $i : A \to C$ is a *cofibration* if it has the LLP with respect to all trivial fibrations. We say a crossed complex C is *cofibrant* if the inclusion $\emptyset \to C$ is a cofibration.

We shall also need the definition that p has the *right lifting property (RLP)* with respect to a class $\mathcal{F}$ if in the above diagram, given p, then the dotted completion exists for all i in the class $\mathcal{F}$. □

Here is an important example of a cofibration. The proof is analogous to standard methods in the usual homological algebra, and to results for CW-complexes.

Proposition 10.5.4 *Let $i : A \to F$ be a relatively free crossed complex. Then i is a cofibration.*

Proof We consider the following diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ F \arrow[r, "f"'] \arrow[ur, dotted, "g"] & B \end{tikzcd}$$

in which p is supposed a trivial fibration, and the morphisms f, a satisfy $fi = pa$. We construct the regular completion g on a relatively free basis X of F by induction, the cases $n = 0, 1$ being easy.

Suppose $n \geqslant 2$ and g is defined on X^{n-1}. Consider an element x of the free basis in dimension n. Then $g\delta x$ is defined and $pg\delta x = f\delta x$.

By the fibration condition, we can choose $y \in E_n$ such that $py = fx$. Let $w = g\delta x - \delta y \in E_{n-1}$. Then $pw = 0, \delta w = 0$. By the triviality condition, w is a boundary, i.e. $w = \delta z$ for some $z \in E_n$. Then $\delta(z + y) = g\delta x$. So we can extend g by defining it on x to be $z + y$. □

Now we can characterise fibrations of crossed complexes in terms of the RLP. First we define $\mathbb{F}(n)$ to be the free crossed complex on one generator c^n of dimension n. Thus $\mathbb{F}(0)$ is a singleton, $\mathbb{F}(1)$ is essentially the groupoid $\mathcal{I}$ and for $n \geqslant 2$ $\mathbb{F}(n)$ consists the integers in dimensions $n, n-1$ with boundary the identity map. We define $\mathbb{S}(n-1)$ to be the subcomplex of $\mathbb{F}(n)$ generated by the part in dimension $n-1$.

Proposition 10.5.5 *Let* $p : E \to D$ *be a morphism of crossed complexes. Then the following conditions are equivalent:*

(i) p *is a fibration;*

(ii) *(covering homotopy property)* p *has the RLP with respect to the inclusion* $C \otimes 1 \to C \otimes \mathbb{F}(m)$ *for all cofibrant crossed complexes* C *and* $m \geqslant 1$*; and*

(iii) *the covering homotopy property (ii) holds for* $m = 1$.

Proof (i)⇒(ii) We verify the covering property by constructing a lifting in the left hand of the following diagrams, where $1 \to \mathbb{F}(m)$ is the inclusion. Let p' in the right hand

$$\begin{tikzcd} C \otimes 1 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ C \otimes \mathbb{F}(m) \arrow[r] & D \end{tikzcd} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathsf{CRS}(\mathbb{F}(m), E) \arrow[d, "p'"] \\ C \arrow[r] & \mathsf{CRS}(1, E) \times_D \mathsf{CRS}(\mathbb{F}(m), D) \end{tikzcd}$$

diagram be induced by p and the inclusion $1 \to \mathbb{F}(m)$. Then a lifting in the left hand diagram is equivalent to a lifting in the right hand diagram. Since C is cofibrant, such a lifting exists if p' is a trivial fibration. But by the exponential law, for this it is sufficient to show that p has the RLP with respect to the inclusion

$$\mathbb{S}(n) \otimes \mathbb{F}(m) \cup \mathbb{F}(n+1) \otimes 1 \to \mathbb{F}(n+1) \otimes \mathbb{F}(m).$$

For $n = -1$, this corresponds precisely to the fibration property of p. In general, a lifting of the image of the top basis element of $\mathbb{F}(n+1) \otimes \mathbb{F}(m)$ is chosen, and the value of the lifting on the remaining basis element of $\mathbb{F}(n+1) \otimes \mathbb{F}(m)$, namely $c_{n+1} \otimes \delta c_m$ if $m \geqslant 2$, $c_n \otimes 0$ if $m = 1$, is determined by the boundary formula for $c_n \otimes c_m$ and the values on $\delta c_{n+1} \otimes c_m$ if $n \geqslant 1$ and $0 \otimes c_m$ and $1 \otimes c_m$ if $n = 0$.

(ii)⇒(iii) is immediate

(iii)⇒(i) This is easily proved on taking C to be the crossed complex of free type on one generator of dimension n. □

This gives a similar characterisation using free crossed complexes.

Proposition 10.5.6 *Let* $p : E \to D$ *be a morphism of crossed complexes. Then* p *is a fibration if and only if for any cofibrant crossed complex* C*, the induced morphism* $p : \mathsf{CRS}(C, E) \to \mathsf{CRS}(C, D)$ *is a fibration.*

Proof It is clear that if p is a fibration and C is cofibrant, the induced morphism $p : \mathsf{CRS}(C, E) \to \mathsf{CRS}(C, D)$ is a fibration.

To prove it in the other direction, one takes again C to be the crossed complex of free type on one generator of dimension n. □

In a similar manner we have:

Proposition 10.5.7 *The following are equivalent for a morphism* $f : E \to B$ *in* Crs*:*

(i) f *is a trivial fibration:*

(ii) f_0 *is surjective; if* $p, q \in E_0$ *and* $b \in B_1(f_0 p, f_0 q)$*, then there is* $e \in E_1$ *such that* $f_1 e = b$*; if* $n \geqslant 1$ *and* $d \in E_n$ *satisfies* $\delta^0 d = \delta_1 d$ *for* $n = 1, \delta d = 0$ *for* $n \geqslant 2$*, and* $b \in B_{n+1}$ *satisfies* $\delta b = f_n d$*, then there is*
$$e \in E_{n+1} \quad \textit{such that} \quad f_{n+1} e = b \quad \textit{and} \quad \delta e = d;$$

(iii) f *has the* RLP *with respect to* $\mathbb{S}(n-1) \to \mathbb{C}(n)$ *for all* $n \geqslant 0$*;*

(iv) *if* C *is a crossed complex of free type then* f *has the* RLP *with respect to* $\mathbb{S}(n-1) \otimes C \to \mathbb{C}(n) \otimes C$ *for all* $n \geqslant 0$*;*

(v) *if* C *is a crossed complex of free type then the induced morphism* $f_* : \mathsf{CRS}(C, E) \to \mathsf{CRS}(C, B)$ *is a trivial fibration.*

Corollary 10.5.8 *Let* F *be a reduced free crossed complex with base point* p *and let* C *be an aspherical reduced crossed complex, i.e.* C *is connected and* $\pi_n(C, q) = 0$ *for all* $n \geqslant 2$, $q \in C_0$*. Then there is a bijection*
$$[F, C] \cong [\pi_1(F, p), \pi_1(C, q)]$$
where the right hand side is conjugacy classes in the category of groups.

Proof Let $G = \pi_1(C, q)$ and let $\mathbb{K}(G, 1)$ be the crossed complex which is G in dimension 1 and trivial elsewhere. Then the natural morphism $p : C \to \mathbb{K}(G, 1)$ is a trivial fibration. Hence so also is $p_* : \mathsf{CRS}(F, C) \to \mathsf{CRS}(F, \mathbb{K}(G, 1)$. But it is easy to check that $\pi_0(\mathsf{CRS}(F, \mathbb{K}(G, 1)) \cong [\pi_1(F, p), G]$. The result follows. □

Remark 10.5.9 This type of argument replaces an inductive argument of lifting morphisms and homotopies which is traditional in homological algebra. Of course the inductive procedure is hidden in the proof we have given.

Finally, there is a relation between a map of crossed complexes being a fibration and its nerve being a Kan fibration.

Proposition 10.5.10 *Let* $p : E \to D$ *be a morphism of crossed complexes. Then* p *is a fibration if and only if the induced map of nerves* $Np : NE \to ND$ *is a Kan fibration.*

Proof Let B be an $(n-1)$-box in $\mathbb{I}^n$. To say that $Np : NE \to ND$ is a Kan fibration is equivalent to saying that any diagram in Cub:
$$\begin{tikzcd} B \arrow[r] \arrow[d] & NE \arrow[d, "Np"] \\ \mathbb{I}^n \arrow[r] \arrow[ur, dotted] & ND \end{tikzcd}$$

has a regular completion given by the dotted arrow. By adjointness, this is equivalent to the existence of a regular completion in Crs of the following diagram:

$$\begin{tikzcd} \Pi B \arrow[r, "k"] \arrow[d, "j"'] & E \arrow[d, "p"] \\ \Pi(\mathbb{I}^n) \arrow[r, "k'"'] \arrow[ur, dotted, "g"] & D \end{tikzcd} \qquad (*)$$

The argument depends on the fact that $\Pi\mathbb{I}^n$ has a free generator c^n in dimension n and the boundary δc^n is determined by the Homotopy Addition Lemma, in terms of all the faces of c^n. But B misses one of the faces of c^n, the so called free face. The value of g on this free face is therefore determined by $g(c^n)$ and the HAL, see remark 10.4.2.

If $n = 0$, this existence is equivalent to $E_1 \to D_1$ being a fibration of groupoids.

If $n \geqslant 2$, let us see that this existence is equivalent to each $E_n(x) \to D_n(px)$ being surjective. To see this, note that if these maps are surjective, and ν is the usual base point of $\mathbb{I}^n$, then we can choose $a \in E_n(p\nu)$ such that $pa = k'c^n$. If we now define $g(c^n) = a$ and $g(x) = k(x)$ for each non-degenerate element x of B, then there is a unique value for g on the free face of B, determined by the homotopy addition lemma, and this with the other values on B defines a morphism $g : \Pi(\mathbb{I}^n) \to E$. This g is a regular completion of (*).

On the other hand, suppose each diagram (*) has a regular completion. Let $b \in D_n(px)$. Define $k : \Pi B \to E$ to be the trivial morphism with value 0_x. Define $k' : \Pi(\mathbb{I}^n) \to D$ by $k'(c^n) = b$, $k'(B) = 0_{px}$ and k' on the free face of B is δb . Then $pk = k'j$. Let g be a regular completion. Then $pg(c^n) = b$. □

Corollary 10.5.11 *Let* $p : E \to D$ *be a fibration of crossed complexes and let* $x \in D_0$*. Let* $F = p^{-1}(x)$*. Then the sequence of classifying spaces* $BF \to BE \to BD$ *is homotopy equivalent to a fibration sequence.*

Proof This follows from theorem 10.3.18. □

10.6 The pointed case

We are going to consider briefly the modifications needed get a pointed, or base point, based version of Theorem 10.4.17.

First, recall that we have defined Crs_* the category of pointed crossed complexes, that has objects the crossed complexes C having a distinguished element $* \in C_0$ and only morphisms preserving this basepoint are included.

Next, we need the notions of tensor product and homotopy in Crs_*. They are the same notions that in crossed complex but adding the good behaviour with respect to the base point. let us make the conditions explicit.

For any pointed crossed complexes C and D, we define an m*-fold pointed left homotopy* from C to D to be an m-fold left homotopy (H, f) satisfying $f(*) = *$ and $H(*) = 0_* \in D_m$. The collection of all these is a sub-crossed complex $\mathsf{CRS}_*(C, D) \subseteq \mathsf{CRS}(C, D)$ with basepoint the zero morphism $c \mapsto 0_*$. This defines the *pointed internal hom* for crossed complexes.

A *pointed bimorphism* $\theta : (C, D) \to E$ is a bimorphism satisfying

$$\begin{cases} \theta(c, *) = 0_* & \text{for } c \in C, \\ \theta(*, d) = 0_* & \text{for } d \in D. \end{cases}$$

The *pointed tensor product* $C \otimes_* D$ is the pointed crossed complex generated by all $c \otimes_* d$ with defining relations those for the tensor product and

$$\begin{cases} c \otimes_* * = 0_* & \text{for } c \in C, \\ * \otimes_* d = 0_* & \text{for } d \in D. \end{cases}$$

It is quite clear that the associativity and the symmetry of the tensor product preserves the relations in the definition of the pointed tensor product, giving as a consequence the following theorem.

Theorem 10.6.1 *The pointed tensor products and internal hom functors described above define a symmetric monoidal closed structure on the pointed category* Crs_*. □

We denote by $[X, Y]_*$ the set of pointed homotopy classes of pointed maps $X \to Y$ of pointed spaces X, Y. Similarly, for pointed crossed complexes D, C, we denote by $[C, D]_*$ the set of pointed homotopy classes of pointed morphisms $C \to D$.

Also, notice that if C is a pointed crossed complex, then BC is naturally a pointed space.

We have all the ingredients to state the pointed version of theorem 10.4.17.

Theorem 10.6.2 *If* X *is a pointed* CW*-complex and* C *is a pointed crossed complex, there is a commutative diagram*

$$\begin{tikzcd} {[X, BC]_*} \arrow[rr, "\alpha"] \arrow[dr] & & {[\Pi X_*, C]_*} \arrow[dl] \\ & \mathrm{Hom}(\pi_1(X,*), \pi_1(C,*)), & \end{tikzcd}$$

in which α *is a bijection of sets of pointed homotopy classes, natural with respect to pointed morphisms of* C *and pointed, cellular maps of* X*, and in which we have identified* $\pi_1(BC, *)$ *with* $\pi_1(C, *)$*,* $\pi_1(X, *)$ *with* $\pi_1(X_*, *)$.

Proof The proof of the existence of the horizontal bijection α of sets of pointed homotopy classes follows the same pattern as the proof of Theorem 10.4.17, but using the pointed constructions $\otimes_*$ and CRS_* described before. We leave the details as an exercise.

The slanting map on the left is induced by the functor $\pi_1(-, *)$ and the first identification indicated in the statement.

The slanting map on the right comes from the second identification indicated in the statement.

To prove commutativity, it is sufficient to assume that $X = |L|$ for some Kan simplicial set L. Then we have to check that maps transformed by the following arrows induce the same map of fundamental groups:

$$\mathsf{Top}(|L|, BC) \longleftarrow \mathsf{Cub}(L, NC) \to \mathsf{Crs}(\Pi L, C).$$

But this is clear on checking the values of these maps on 1-dimensional elements. □

Proposition 10.6.3 *Let* F *be a pointed free crossed complex and let* $p : E \to B$ *be a pointed map and trivial fibration of pointed crossed complexes. Then* $p_* : \mathsf{CRS}_*(F, E) \to \mathsf{CRS}_*(F, B)$ *is also a trivial fibration.*

Proof Let $i : A \to C$ be a relatively free morphism of pointed crossed complexes. Then a regular completion of the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & \mathsf{CRS}_*(F, E) \arrow[d, "p_*"] \\ C \arrow[r] & \mathsf{CRS}_*(F, B) \end{tikzcd}$$

is equivalent to a regular completion of the diagram

$$\begin{tikzcd} A \otimes_* F \arrow[r] \arrow[d, "i \otimes_* 1_F"'] & E \arrow[d, "p"] \\ C \otimes_* F \arrow[r] & B \end{tikzcd}$$

Since F is free, the morphism $i \otimes_* 1_F$ is relatively free, and the result follows. □

Corollary 10.6.4 *Let F, C be reduced crossed complexes such that F is free and C is aspherical. Let $G = \pi_1(C, *)$. Then there is a bijection*

$$[F, C]_* \cong \mathrm{Hom}(\pi_1(F), G),$$

and for $n \geqslant 2$, and any $f : F \to C$, $\pi_n(\mathsf{CRS}_*(F, C), f) = 0$.

Proof These results are clear when $C = \mathbb{K}(G, 1)$. We then use the trivial fibration $p : C \to \mathbb{K}(G, 1)$ and apply $\mathsf{CRS}_*(F, -)$ to p. □

10.7 Applications

In this Section we give some of the many consequences that can be drawn from the bijection

$$[\Pi X_*, C] \cong [X, BC]$$

proved in Theorem 10.4.17.

We first get some results on reduced CW-complexes whose n-type can be realised by a classifying space of a crossed complex.

Then we obtain a key homotopy classification result, Corollary 10.7.6, expressing the topological homotopy set $[X, Y]$ as an algebraic homotopy set $[\Pi X_*, \Pi Y_*]$ when Y is n-aspherical and X is of dimension $\leqslant n$.

We end by looking at the algebraic part $[\Pi X_*, C]$ of Theorem 10.4.17 in some particular cases.

Our first applications of Theorem 10.4.17 gives sufficient conditions on a homotopy n-type to be realisabe as BC for some crossed complex C.

Theorem 10.7.1 *Let $n \geqslant 1$, and let X be a reduced CW-complex with $\pi_i X = 0$, $1 < i < n$. (Notice that this condition is vacuous if $n = 1, 2$.) Then there is a crossed complex C with $C_i = 0$, for all $i > n$ together with a map*

$$f : X \to BC$$

inducing an isomorphism of homotopy groups $f_* : \pi_i X \to \pi_i BC$ for $1 \leqslant i \leqslant n$.

Proof Let X_* be the skeletal filtration of X, let $X_0 = \{x\}$, and let $D = \Pi X_*$. We define C be the crossed complex such that

$$C_i = \begin{cases} D_i & 0 \leqslant i < n \\ \operatorname{Cok} \partial_{n+1} & i = n \\ 0 & i > n \end{cases}$$

Then there is a unique morphism $g : D \to C$ which is the identity in dimensions $< n$ and is the quotient morphism in dimension n. Clearly, this morphism g induces an isomorphism of fundamental groupoids, and of homology groups $H_i(D, x) \to H_i(C, x)$ for $2 \leqslant i \leqslant n$.

By Theorem 10.6.2 there is a pointed morphism

$$f : X \to BC$$

whose homotopy class corresponds to $g : \Pi X_* \to C$. Without loss of generality we may assume f is cellular. Then for all $i \geqslant 1$, the following diagram is commutative, where $\mathbf{S}^i = e^0 \cup e^i$ is the i-sphere:

$$\begin{tikzcd}
{[\mathbf{S}^i, X]_*} \arrow[r, "f_*"] \arrow[d] & {[\mathbf{S}^i, BC]_*} \arrow[d, "\cong"] \arrow[dr, "\cong"] & \\
{[\Pi \mathbf{S}^i_*, \Pi X_*]_*} \arrow[r, "(\Pi f)_*"] \arrow[d, "\cong"'] & {[\Pi \mathbf{S}^i_*, \Pi B\mathbf{C}_*]_*} \arrow[r, "\cong"'] & {[\Pi \mathbf{S}^i_*, C]_*} \arrow[d, "\cong"] \\
H_i(\Pi X_*, x) \arrow[rr, "\cong", "H_i g"'] & & H_i(C, x)
\end{tikzcd}$$

The assumptions on X imply that the map $[\mathbf{S}^i, X]_* \to [\Pi \mathbf{S}^i_*, \Pi X_*]_*$ is bijective for $1 \leqslant i \leqslant n$. So the result on Π_i follows. □

Remark 10.7.2 This Theorem shows that if $\pi_i X = 0$, $1 < i < n$, then the n-type of X is described completely by a crossed complex. For $n = 1$, this is well known (the Eilenberg-Mac Lane spaces do this), and for $n = 2$ it is essentially due to Mac Lane and Whitehead [MLW50]. Indeed, they prove that the 2-type (for which they use the term 3-type) of a reduced CW-complex X is described by the crossed module $\pi_2(X, X^1) \to \pi_1 X^1$, which is the same crossed module as arises for $n = 2$ in the proof of Theorem 10.7.1.

□

In the crossed module case, there is an additional result that is sometimes useful for giving an explicit presentation of a crossed module representing the 2-type of a space. It was first proved by Loday [Lod82].

Proposition 10.7.3 *Let X be a reduced CW-complex and let P be a group such that there is a map $f : BP \to X$ which is surjective on fundamental groups. Let $F(f)$ be the homotopy fibre of f and let $M = \pi_1 F(f)$, so that we have a crossed module $M \to P$. Then there is a map $X \to B(M \to P)$ inducing an isomorphism of π_1 and π_2.*

Proof Let $f : BP \to X$ be a cellular map which is surjective on fundamental groups. Let Y be the reduced mapping cylinder $M(f)$ of f, and let $j : BP \to Y$ be the inclusion. Then the crossed module $\pi_2(Y, BP) \to \pi_1 BP$ is isomorphic to $\mu : M \to P$.

Also j is surjective on fundamental groups, and it follows that the inclusion $X^1 \to Y$ is deformable by a homotopy to a map g', say, with image in BP. This homotopy extends to a homotopy of the inclusion $X \to Y$ to a map $g : X \to Y$ extending g'.

Let Y_* be the filtered space in which Y_0 is the base point of Y, $Y_1 = BP$, $Y_i = Y$ for $i \geqslant 2$. Then $C = \Pi Y_*$ is the trivial extension by zeros of the crossed module $M \to P$. The map $g : X \to Y$ induces a morphism $g_* : \Pi X_* \to \Pi Y_*$ which is realised by a map $X \to B(M \to P)$ inducing an isomorphism of π_1 and π_2. □

Example 10.7.4 We now give an application of the last Proposition which uses the HHvKT for crossed modules. Let X be a CW-complex which is the union of connected subcomplexes Y and Z such that $A = Y \cap Z$ is a $K(P, 1)$, i.e. is a space BP. Suppose that the inclusions of A into Y and Z induce isomorphisms of fundamental groups. Then, as in Proposition 10.7.3, the 2-types of Y and Z may be described by crossed modules $M \to P$ and $N \to P$ respectively, say.

By results of Part I, the crossed module describing the 2-type of X is the coproduct $M \circ N \to P$ of the crossed P-modules M and N. □

We now give another application to the homotopy classification of maps. It also concerns n-aspherical spaces and says that the homotopy classes of maps from a CW-complex of dimension $\leqslant n$ to an n-aspherical space are classified by the homotopy classes of morphisms of their fundamental crossed complexes.

Proposition 10.7.5 *For any CW-complex Y with skeletal filtration Y_*, there is a homotopy fibration*

$$F \to Y \to B\Pi Y_* .$$

Thus if $\pi_i(Y, y) = 0$ for $1 < i < n$, then the fibre F is n-connected.

Proof Results of Chapter 14, particularly Theorem 14.2.7, give a Kan fibration

$$RY_* \to N\Pi Y_* .$$

Also for a CW-complex Y_* the inclusion of RY_* into the singular complex of Y is a homotopy equivalence. So when realising, we have a homotopy fibration sequence

$$F \to Y \to B\Pi Y_* .$$

The results on n-connectedness come from the homotopy exact sequence of this fibration.

□

Corollary 10.7.6 *If Y is a connected CW-complex such that $\pi_i Y = 0$ for $1 < i < n$, and X is a CW-complex with dim $X \leqslant n$, then there is a natural bijection of homotopy classes*

$$[X, Y] \cong [\Pi X_*, \Pi Y_*].$$

Proof The assumptions imply that the fibration

$$Y \to B\Pi Y_*$$

induces a bijection $[X, Y] \to [X, B\Pi Y_*]$.

The fact that the map $[X, B\Pi Y_*] \to [\Pi X_*, \Pi Y_*]$ is a bijection follows from Theorem 10.4.17. □

This Corollary may also be obtained as a concatenation of results proved by J.H.C.Whitehead in [Whi49b]. It is also proved in general circumstances by Baues in his book [Bau89].

By Theorem 10.4.17, we get that the homotopy classes of maps are bijective with the set $[\Pi X_*, C]$. We are going to consider some cases where this algebraic set is computable.

The first case applies to the crossed complex $E_1(G)$ associated to a groupoid G. Recall that $E_1(G)$ is the crossed complex which is G in dimension 1 and trivial elsewhere. Then

$$\mathsf{Crs}(C, E_1(G)) \cong \mathsf{Gpds}(\pi_1 C, G),$$

with a bijection that carries over to homotopy classes, with crossed complexes on the left and groupoids on the right:

$$[C, E_1(G)] \cong [\pi_1 C, G].$$

Proposition 10.7.7 *If C is a crossed complex and G is a groupoid, then there is a homotopy equivalence of crossed complexes*

$$\mathsf{CRS}(C, E_1(G)) \simeq E_1(\mathsf{GPDS}(\pi_1 C, G)).$$

Proof Let D be a crossed complex. Then there are natural bijections

$$\begin{aligned}
[D, \mathsf{CRS}(C, E_1(G))] &\cong [D \otimes C, E_1(G)] && \text{because Crs is a closed category}\\
&\cong [\pi_1(D \otimes C), G] && \text{as indicated above}\\
&\cong [\pi_1 D \times \pi_1 C, G] && \text{because } \pi_1 \text{ preserves products}\\
&\cong [\pi_1 D, \mathsf{GPDS}(\pi_1 C, G)] && \text{because Gpds is a closed category}\\
&\cong [D, E_1(\mathsf{GPDS}(\pi_1 C, G))] && \text{as before.}
\end{aligned}$$

The result follows directly. □

If G is connected, $x \in G_0$, and $f : G \to H$ is a morphism, then the vertex group $\mathsf{GPDS}(G, H)(f)$ is isomorphic to the centraliser of $f(G(x))$ in $H(fx)$. So the previous result with Theorem 10.4.17 yields a result of Gottlieb [Got69] on the fundamental group of spaces of maps into an Eilenberg-Mac Lane space $K(H, 1)$.

In the pointed case the Proposition gives an even simpler result.

Proposition 10.7.8 *If C is a pointed, connected crossed complex and G is a pointed groupoid, then the crossed complex $\mathsf{CRS}_*(C, E_1(G))$ has its set of components bijective with $\mathsf{Gpds}(\pi_1(C, *), G(*))$ the set of morphisms of groups $\pi_1(C, *) \to G(*)$, and all components of $\mathsf{CRS}_*(C, E_1(G))$ have trivial π_1 and H_i for $i \geqslant 2$.*

Proof An argument similar to that in the proof of the previous proposition yields

$$[*, \mathsf{CRS}_*(C, E_1(G))] \cong [*, \mathsf{GPDS}_*(\pi_1 C, G)],$$

which gives the first result. The second result follows since for any pointed crossed complex Z and morphism $f : C \to E_1(G)$ that we shall take as base point, we have

$$\begin{aligned}
[(Z, *), (\mathsf{CRS}_*(C, E_1(G), f)] &\cong [Z \otimes C, E_1(G)]_{\#}\\
&\cong [\pi_1 Z \times \pi_1 C, G]_{\#}\\
&\cong [\pi_1 C, G|f] \cong *.
\end{aligned}$$

where the sets of homotopy classes marked $_{\#}$ are of maps satisfying that restricted to some space are the appropriate ones, i.e. the conditions that come from duality, $|1 \otimes f : * \otimes C \to E_1(G), * : Z \otimes * \to E_1(G)$ in the first case and $|1 \times f : * \times \pi_1 C \to G, * : \pi_1 Z \times * \to G$ in the second one. □

There is another interesting special case of the homotopy classification. Let $n \geqslant 2$, M an abelian group and $\mathsf{Aut}\, M$ the group of automorphisms of M. Then we define

$$\chi(M,n) = \ \cdots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots\cdots \longrightarrow 0 \longrightarrow \mathsf{Aut}\, M,$$

the pointed crossed complex which is: $\mathsf{Aut}\, M$ in dimension 1; M in dimension n; has the given action of $\mathsf{Aut}\, M$ on M; and has trivial boundaries.

Let C be a crossed complex; in useful cases, C will be of free type. We suppose C reduced and pointed. Let $\alpha : \pi_1(C,*) \to \mathsf{Aut}\, M$ be a morphism. The set of pointed homotopy classes of morphisms $C \to \chi(M,n)$ which induce α on fundamental groups is written $[C, \chi(M,n)]^{\alpha}_{*}$. This set is easily seen to have an Abelian group structure, induced by the addition on operator morphisms $C_n \to M$ over α. So we obtain the homotopy classification:

Proposition 10.7.9 *If X is a pointed reduced CW-complex, and $\alpha : \pi_1(X,*) \to \mathsf{Aut}\, M$, then there is a natural bijection*

$$[X, B\chi(M,n)]^{\alpha}_{*} \cong [\Pi X_*, \chi(M,n)]^{\alpha}_{*}$$

where the former set of homotopy classes denotes the set of pointed homotopy classes of maps inducing α on fundamental groups.

Proof The proof is immediate from Theorem 10.6.2. □

This result can be related to the case of local coefficients (see Section **??**).

Notes

The cubical classifying space of a crossed module is used in [FRS95].

The main applications of the cubical classifying space of a crossed complex follow as for the simplicial version in [BH91].

There are generalisations of this work to the equivariant case in [BGPT97, BGPT01], but using the simplicial classifying space, which fits better with published studies on homotopy coherence, [CP97].

Chapter 11

Resolutions.

The notion of 'resolution' of an algebraic object is one way of trying to describe an infinite object and its properties in finitary terms, or in some way other than attempting to list its elements, which might be a foolhardy endeavour. The same methods are used to describe very large objects in manageable ways.

In Chapter 3 we showed how the notions of 'syzygy' and 'resolution' by free modules arose from invariant theory, in trying to deal with algebras of polynomials. There we also showed how the analogous notion of 'identity among relations' for a presentation of a group led to the notion of free crossed module.

In this chapter, we extend the latter ideas to all dimensions using crossed complexes, and so have the notion of *free crossed resolution* of a group, or groupoid. Surprisingly, the extension to groupoids rather than just groups turns out also to be useful for the purposes of calculation, as we shall see in Section 11.2. The reason is that our method is to construct what we call 'a home for a contracting homotopy' and to this end we need to pass to the universal covering groupoid of a group.

11.1 Free crossed resolutions of groups and groupoids

In this Section we first introduce the concept of free crossed resolution of a groupoid G, prove that any two resolutions of the same groupoid are homotopy equivalent and give some direct examples. Then, we study some more complex examples requiring extra theoretical background.

11.1.1 Existence, examples and uniqueness

Definition 11.1.1 A crossed complex C is called *aspherical* if for all $n \geqslant 2$ and $x \in C_0$, we have $H_n(C, x) = 0$. It is *acyclic* if it is aspherical, connected and in addition $\pi_1(C, x) = 0$ for all $x \in C_0$.

Given a groupoid G, a *crossed resolution of* G is an aspherical crossed complex C such that $C_0 = G_0$ together with a groupoid morphism $\phi : C_1 \to G$ over C_0 such that ϕ induces an isomorphism of groupoids $\phi : \pi_1 C \to G$. A *free crossed resolution of* G is a crossed resolution C where each C_n is free in the sense that

- C_1 is a free groupoid;
- $\delta_2 : C_2 \to C_1$ is a free crossed module; and

- for $n \geqslant 3$, C_n is a free G-module.

□

Theorem 11.1.2 *Any group(oid)* G *admits a free crossed resolution.*

Proof We first choose a presentation $\mathcal{P} = \langle X \mid R\rangle$ of G and so we get a free crossed module over a free groupoid

$$C(R) \xrightarrow{\delta_2} F(X)$$

together with $\phi : G \to 1$ inducing an isomorphism $\mathrm{Cok}\,(\delta_2) \to G$. Now $A = \mathrm{Ker}\,\delta_2$ is a G-module and we proceed as in classical homological algebra as outlined in Chapter 3. □

This follows a traditional method of constructing complexes, either CW-complexes or forms of resolutions, by 'killing kernels'; at stage 1 this requires a free crossed module to map onto the normal subgroupoid of a free groupoid normally generated by the relations; at stage $n \geqslant 2$ the kernel of δ_n is a G-module and we choose a graph X_n of generators for this and map the free G-module on X_n onto this kernel.

Of course what this outline construction does not show is how to get hold of a convenient graph of generators X_n for the kernel; some such graph exists, for instance we could take $X_n = \mathrm{Ker}\,\delta_{n-1}$, but this is not at all constructive or convenient. This problem of construction is addressed in Section 11.2, in the case G is a group, using the idea of constructing inductively a free crossed resolution with a contracting homotopy not of G, but of the universal covering groupoid $\widetilde{G}$ of G.

If G is itself free, we need go no further.

Example 11.1.3 If G is a free groupoid $F(X_1)$, then G has a free crossed resolution which is $F(X_1)$ in dimension 1 and is trivial in higher dimensions. □

We can also state a small free crossed resolution of finite cyclic groups, which is a modification of a classical chain complex resolution of these groups.

Example 11.1.4 (A small crossed resolution of finite cyclic groups) A cyclic group C_q of order q with generator c has a free crossed resolution $F = F(C_q)$ as follows:

$$F(C_q) \quad = \quad \cdots \to \mathbb{Z}[C_q] \xrightarrow{\delta_4} \mathbb{Z}[C_q] \xrightarrow{\delta_3} \mathbb{Z}[C_q] \xrightarrow{\delta_2} C_\infty \xrightarrow{\phi} C_q$$

where C_∞ is the infinite cyclic group with free generator x_1; $\mathbb{Z}[C_q]$ is the free C_q-module with free generator x_n for $n \geqslant 2$; and the maps are defined by $\phi(x_1) = c$, the generator of C_q; $\delta_2(x_2) = x_1^q$ and

$$\delta_n(x_n) \;=\; \begin{cases} x_{n-1}\,(1-c) & \text{if } n \text{ is odd;}\\ x_{n-1}\,(1+c+c^2+\cdots+c^{q-1}) & \text{if } n \text{ is even} \end{cases}$$

for $n > 2$. □

Exercise 11.1.5 Prove directly that the preceding example gives a free crossed resolution of C_q. □

Now we give a free crossed resolution for any group, called the *standard crossed resolution*.

Example 11.1.6 There is a *standard* free crossed $F^{st}_*(G)$ resolution of a groupoid G given by:

$$F^{st}(G) \quad = \quad \cdots \longrightarrow F^{st}_n(G) \xrightarrow{\delta_n} F^{st}_{n-1}(G) \xrightarrow{\delta_{n-1}} \cdots\cdots \xrightarrow{\delta_3} F^{st}_2(G) \xrightarrow{\delta_2} F^{st}_1(G) \xrightarrow{\phi} G$$

which is G_0 in dimension 0 and in which for $n \geqslant 1$, $F^{st}_n(G)$ is free on the set $(N^\Delta G)_n$ of composable sequences $[g_1, g_2, \dots, g_n]$, $g_i \in G$ of elements of G, with boundary

$$\delta_n : F^{st}_n(G) \to F^{st}_{n-1}(G)$$

given by

$$\begin{aligned}\delta_2[g,h] &= [gh]^{-1}[g][h],\\ \delta_3[g,h,k] &= [g,h]^k[h,k]^{-1}[g,hk]^{-1}[gh,k],\end{aligned}$$

and for $n \geqslant 4$

$$\begin{aligned}&\delta_n[g_1,g_2,\dots,g_n]\\ &\quad=[g_1,\dots,g_{n-1}]^{g_n} + (-1)^n[g_2,\dots,g_n] + \sum_{i=1}^{n-1}(-1)^{n-i}[g_1,g_2,\dots,g_{i-1},g_ig_{i+1},g_{i+2},\dots,g_n].\end{aligned}$$

The base point $\beta[g_1, g_2, \dots, g_n]$ is the final point tg_n of g_n. In fact $F^{st}_*(G)$ is just $\Pi N^\Delta(G)$ where $N^\Delta(G)$, the simplicial nerve of G, is in dimension n just $\mathsf{Crs}(\Pi\Delta^n, G)$. See also the pictures in Example 9.9.7.

In order to see that the standard resolution *is* aspherical, we restrict to the case G is connected and even further by Proposition 7.1.21 to the case G is a group, considered as a groupoid with object x_0. Consider, as described in the later Section 11.2.2, the universal cover of $F^{st}_*(G)$ based at x_0. This has vertices the elements $g \in G$ and its elements in dimension n are given by pairs $([g_1, g_2, \dots, g_n], g)$ where $g_i, g \in G$. A contracting homotopy on this universal cover $h : \mathrm{id} \simeq x_0$, where x_0 denotes the constant map to x_0, is then given by

$$([g_1, g_2, \dots, g_n], g) \mapsto ([g_1, g_2, \dots, g_n, g], 1_{x_0}). \qquad \Box$$

Remark 11.1.7 One finds in the literature on extensions of a group M by a group G, i.e. an extension $1 \to M \to E \to G \to 1$ the notion of *factor set* of G in M. This consists of a pair of functions

$$k^1 : G \to \mathsf{Aut}(M), \quad k^2 : G \times G \to M$$

satisfying a number of conditions. We will see later that these conditions can be interpreted as saying that a factor set is equivalent to a morphism of crossed complexes from $F^{st}_*(G)$ to the crossed complex extension of the crossed module $\chi : M \to \mathsf{Aut}(M)$. An *equivalence of factor sets* is just homotopy of morphisms. Thus all the complications necessary to describe a factor set are embedded in the standard free crossed resolution of G. $\Box$

Example 11.1.8 Let $\langle X \mid R\rangle$ be a one relator presentation of a group G, that is R consists of a single element $r \in F(X)$, and suppose r is not a proper power. It is a theorem that the kernel of $C(r) \to F(X)$ is trivial, so that this crossed module itself is in essence a free crossed resolution of G. However, the proof of this triviality is by no means easy. A proof of a generalisation of this fact to the case r is a proper power may be found in [DV73]. $\Box$

The following two theorems imply that free crossed resolutions of a groupoid are determined up to homotopy; this motivates the desire to find those free crossed resolutions useful for various aims.

Theorem 11.1.9 *Let C, D be crossed complexes such that C is free and D is aspherical. Let $\alpha : \pi_1 C \to \pi_1 D$ be a morphism of groupoids. Then there is a morphism $f : C \to D$ of crossed complexes such that $\pi_1(f) = \alpha$.*

Such a morphism f is said to be a *lift* of α.

Proof We consider the diagram

$$\begin{tikzcd}
\cdots & C_n \arrow[r, "\delta_n"] & C_{n-1} \arrow[r] & \cdots \arrow[r] & C_2 \arrow[r, "\delta_2"] & C_1 \arrow[r, "\phi"] \arrow[d, dashed, "f_1"] & \pi_1 C \arrow[d, "\alpha"] \\
\cdots & D_n \arrow[r, "\delta_n"'] & D_{n-1} \arrow[r] & \cdots \arrow[r] & D_2 \arrow[r, "\delta_2"'] & D_1 \arrow[r, "\psi"'] & \pi_1 D
\end{tikzcd}$$

in which ϕ, ψ are the quotient morphisms.

Let the free basis of C be denoted by X_*, where $X_0 = C_0$, and we assume X_n is a subgraph of C_n.

For $x \in X_1$ we choose $f_1(x) \in D_1$ such that $\psi f_1(x) = \alpha\phi(x)$. This is possible because ψ is surjective. Since X_1 is a free basis of C_1, this choice extends uniquely to a morphism

$$f_1 : C_1 \to D_1.$$

Since $\psi f_1 = \alpha\phi$ on the generating set X_1, it follows that

$$\psi f_1 = \alpha\phi$$

on C_1. Note also that

$$\psi f_1 \delta_2 = \alpha\phi\delta_2 = 0.$$

Since $\operatorname{Ker}\psi = \operatorname{Im}\delta_2$, it follows that $\operatorname{Im} f_1\delta_2 \subseteq \operatorname{Im}\delta_2$. For all $x \in X_2$, we choose $f_2(x) \in D_2$ so that

$$\delta_2 f_2(x) = f_1\delta_2(x).$$

Now we proceed inductively. Suppose that

$$f_{n-1} : C_{n-1} \to D_{n-1}$$

has been defined so that

$$\delta_{n-1} f_{n-1} = f_{n-2}\delta_{n-1}.$$

Then

$$\delta_{n-1} f_{n-1}\delta_n = f_{n-2}\delta_{n-1}\delta_n = 0.$$

By asphericity of D, $\operatorname{Im}(f_{n-1}\delta_n) \subseteq \operatorname{Im}\delta_n$. So for all x in the free basis X_n, there is an $f_n(x) \in D_n$ such that $\delta_n f_n(x) = f_{n-1}\delta_n(x)$. This defines a morphism

$$f_n : C_n \to D_n$$

such that $\delta_n f_n = f_{n-1}\delta_n$. □

Exercise 11.1.10 Let C_q and C_{qr} be cyclic groups of order q and qr with generators c and c_1 respectively. Consider their free crossed complex resolutions $F(C_q)$ and $F(C_{qr})$ studied in Example 11.1.4. Given the morphism $\alpha : C_q \to C_{qr}$ which sends c to c_1^r, find a morphism $F(C_q) \to F(C_{qr})$ which lifts α. □

Theorem 11.1.11 *Let C, D be crossed complexes such that C is free and D is aspherical. Let $\alpha : \pi_1 C \to \pi_1 D$ be a morphism of groupoids and $f^-, f^+ : C \to D$ morphisms of crossed complexes such that $\pi_1(f^-) = \pi_1(f^+) = \alpha$. Then there is a homotopy $h : f^- \simeq f^+$.*

Proof We proceed as before to define the homotopy (see Definition 9.3.6) starting for

$$h_0 : C_0 \to D_1.$$

Since $\pi_1(f^-) = \pi_1(f^+) = \alpha$, we have $\psi f_1^- = \alpha\phi = \psi f_1^+$. We set $h_0(c) = 1_{\alpha c} \in D_1$, for $c \in C_0$.

We have to define a map

$$h_1 : C_1 \to D_2.$$

such that for every $c \in C_1$ satisfies

$$f_1^-(c) = (h_0 s(c))(f_1^+ c)(\delta_2 h_1 c)(h_0 t(c))^{-1}$$

which because of our definition of h_0 reduces to

$$f_1^-(c) = f_1^+(c)(\delta_2 h_1 c)$$

or

$$\delta_2 h_1 c = f_1^+(c)^{-1} f_1^-(c).$$

But

$$\psi(f_1^+(c)^{-1} f_1^-(c)) = 1.$$

Hence for each $x \in X_1$ we can choose an $h_1(x)$ such that $\delta_2 h_1 x = f_1^+(x)^{-1} f_1^-(x)$. This extends to an f_1^+ derivation $h_1 : C_1 \to D_2$, as explained in Remark 9.3.4.

At the next level, for $c \in C_2$, we note that $f_1^+\delta_2 = \delta_2 f_2^+$, $f_1^-\delta_2 = \delta_2 f_2^-$ and we require h_2 such that

$$f_2^-(c) = f_2^+(c) h_1 \delta_2(c) \delta_3 h_2(c). \qquad (*)$$

But

$$\delta_2(h_1\delta_2(c)^{-1} f_2^+(c)^{-1} f_2^-(c)) = (f_1^+(\delta_2 c)^{-1} f_1^-(\delta_2 c)^{-1} \delta_2 f_2^+(c)^{-1} \delta_2 f_2^-(c) = 1.$$

So again, we can choose $h_2(x)$ for $x \in X_2$ so that (*) holds for $c = x$. This extends to an f_1^+-morphism $h_2 : C_2 \to D_3$ as required.

We now look at the situation around dimension n.

$$\begin{tikzcd}
{} \arrow[r] & C_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "f^-_{n+1}"', "f^+_{n+1}"] & C_n \arrow[r, "\delta_n"] \arrow[d, "f^-_n"', "f^+_n"] \arrow[dl, dashed, "h_n"'] & C_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, "f^-_{n-1}"', "f^+_{n-1}"] \arrow[dl, "h_{n-1}"'] & C_{n-2} \arrow[r] \arrow[d, "f^-_{n-2}"', "f^+_{n-2}"] \arrow[dl, "h_{n-2}"'] & {} \\
{} \arrow[r] & D_{n+1} \arrow[r, "\delta_{n+1}"'] & D_n \arrow[r, "\delta_n"'] & D_{n-1} \arrow[r, "\delta_{n-1}"'] & D_{n-2} \arrow[r] & {}
\end{tikzcd}$$

We suppose given the morphisms f^-, f^+ and also the h_{n-2}, h_{n-1} such that

$$f^-_{n-1} = f^+_{n-1} + h_{n-2}\delta_{n-1} + \delta_n h_{n-1}.$$

But for $c \in C_n$

$$\begin{aligned}
\delta_n(f_n^- c - f_n^+ c - h_{n-1}\delta_n c) &= f^-_{n-1}\delta_n c - f^+_{n-1}\delta_n c - \delta_n h_{n-1}\delta_n c \\
&= f^-_{n-1}\delta_n c - f^+_{n-1}\delta_n c - (f^-_{n-1}\delta_n c - f^+_{n-1}\delta_n c - h_{n-2}\delta_{n-1}\delta_n c) \\
&= 0 \qquad \text{since } \delta_{n-1}\delta_n = 0.
\end{aligned}$$

By asphericity of D, for each x in the basis X_n we can find an $h_n x$ in D_{n+1} such that

$$h_n x = f_n^- x - f_n^+ x - h_{n-1}\delta_n x.$$

This extends to an operator morphism $h_n : C_n \to D_{n+1}$ with the required properties for the next stage of the induction. □

This proof is typical of the method of constructing homotopies, and is going to be useful again in later sections.

Corollary 11.1.12 *Any two free crossed resolutions of a group* G *are homotopy equivalent.*

Remark 11.1.13 A refinement of Theorem 11.1.11 is to assume that we have two morphisms $\alpha^-, \alpha^+ : G \to H$, that the morphisms f^-, f^+ lift α^-, α^+ respectively and that η is a homotopy (or natural transformation) $\alpha^- \simeq \alpha^+$. Then we assert that under the same conditions of freeness and asphericity, η lifts to a homotopy $h : f^- \simeq f^+$. Let us assume ϕ, ψ are the identity on objects. Here η_0 assigns to each p in C_0 an element $\eta(p) \in H(\alpha^- p, \alpha^+ p)$ such that the usual naturality condition holds: if $g \in G(p, q)$ then $\alpha^-(g)\eta(q) = \eta(p)\alpha^+(g)$. For each $p \in C_0$ choose an $h_0(p) \in D_1(p, q)$ such that $\psi(h_0(p)) = \eta(p)$. Now we repeat the arguments of the proof of Theorem 11.1.11 but using the more complicated formulae for homotopies which involve h_0. We leave the details as an exercise for the reader. □

However we still want a method of constructing in a more or less algorithmic way, or at least in terms of data specifying a group or groupoid, some construction of a free crossed resolution. This is given in Section 11.2 for a group defined by a presentation by generators and relations. The next section describes some other ways of constructing resolutions according to other constructions of groups.

11.1.2 Some more complex examples: Free products with amalgamation and HNN-extensions

We will prove the following theorem in Corollary 15.8.1, using cubical methods, covering crossed complexes, and the notion of dense subcategory. This result, combined with the fact that the tensor product of free crossed complexes is free, gives one method of making new free crossed resolutions from old ones.[10]

Theorem 11.1.14 *If* C, D *are aspherical free crossed complexes, then their tensor product* $C \otimes D$ *is also aspherical.*

Example 11.1.15 Let $\mathcal{P}_G = \langle X_G \mid R_G \rangle$, $\mathcal{P}_H = \langle X_H \mid R_H \rangle$ be presentations of groups G, H respectively, and let $\mathcal{F}(\mathcal{P}_G)$, $\mathcal{F}(\mathcal{P}_H)$ be the corresponding free crossed modules, regarded as 2-truncated crossed complexes. The tensor product $T = \mathcal{F}(\mathcal{P}_G) \otimes \mathcal{F}(\mathcal{P}_H)$ is 4-truncated and is given as follows (where we now use additive notation in dimensions $3, 4$ and multiplicative notation in dimensions $1, 2$):

- T_1 is the free group on generating set $X_G \sqcup X_H$;
- T_2 is the free crossed T_1-module on $R_G \sqcup (X_G \otimes X_H) \sqcup R_H$ with the boundaries on R_G, R_H as given before and
$$\delta_2(g \otimes h) = h^{-1}g^{-1}hg \quad \text{for all } g \in X_G,\ h \in X_H\ ;$$
- T_3 is the free $(G \times H)$-module on generators $r \otimes h$, $g \otimes s$, $r \in R_G$, $s \in R_H$ with boundaries
$$\delta_3(r \otimes h) = r^{-1}r^h(\delta_2 r \otimes h), \qquad \delta_3(g \otimes s) = (g \otimes \delta_2 s)^{-1}s^{-1}s^g\ ;$$
- T_4 is the free $(G \times H)$-module on generators $r \otimes s$, with boundaries
$$\delta_4(r \otimes s) = (\delta_2 r \otimes s) + (r \otimes \delta_2 s)\ .$$

The important point is that we can if necessary calculate with these formulae, because elements such as $\delta_2 r \otimes h$ may be expanded using the rules for the tensor product. Alternatively, the forms $\delta_2 r \otimes h, g \otimes \delta_2 s$ may be left as they are since they naturally represent subdivided cylinders. □

We next illustrate the use of crossed complexes of groupoids, rather than just of groups, by the construction of a free crossed resolution of a free product with amalgamation, and a similar result for HNN-extensions, given free crossed resolutions of the individual groups. [11]

Suppose the group G is given as a free product with amalgamation

$$G \;=\; A *_C B,$$

which we can alternatively describe as a pushout of groups

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d, "i"'] & B \arrow[d, "i'"] \\ A \arrow[r, "j'"'] & G\,. \end{tikzcd}$$

We are assuming the morphisms i, j are injective so that, by standard results, i', j' are injective. Suppose we are given free crossed resolutions $\mathcal{A} = \mathcal{F}(A)$, $\mathcal{B} = \mathcal{E}(B)$, $\mathcal{C} = \mathcal{F}(C)$. The morphisms i, j may then be lifted (non uniquely) to morphisms $i'' : \mathcal{C} \to \mathcal{A}$, $j'' : \mathcal{C} \to \mathcal{B}$. However we cannot expect that the pushout of these morphisms in the category Crs gives a free crossed resolution of G.

To see this, suppose that these crossed resolutions are realised by CW-filtrations $K(Q)$ for $Q \in \{A, B, C\}$, and that i'', j'' are realised by cellular maps $K(i) : K(C) \to K(A)$, $K(j) : K(C) \to K(B)$. However, the pushout in topological spaces of cellular maps does not in general yield a CW-complex — for this it is required that one of the maps is an inclusion of a subcomplex, and there is no reason why this should be true in this case. The standard way of dealing with this problem is to form the double mapping cylinder $M(i, j)$ given by the *homotopy pushout*

$$\begin{tikzcd} K(C) \arrow[r, "K(j)"] \arrow[d, "K(i)"'] \arrow[dr, phantom, "\simeq"] & K(B) \arrow[d] \\ K(A) \arrow[r] & M(i,j) \end{tikzcd}$$

where $M(i, j)$ is obtained from $K(A) \sqcup (I \times K(C)) \sqcup K(B)$ by identifying $(0, x) \sim K(i)(x)$, $(1, x) \sim K(j)(x)$ for $x \in K(C)$. This ensures that $M(i, j)$ is a CW-complex containing $K(A), K(B)$ and $\{\frac{1}{2}\} \times K(C)$ as subcomplexes and that the composite maps $K(C) \to M(i, j)$ given by the two ways round the square are homotopic cellular maps.

It is therefore reasonable to assume that for crossed complexes the appropriate algebraic construction is also a homotopy pushout, this time in Crs, obtained by applying Π to this homotopy pushout: this yields a diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "j''"] \arrow[d, "i''"'] \arrow[dr, phantom, "\simeq"] & \mathcal{B} \arrow[d] \\ \mathcal{A} \arrow[r] & \mathcal{F}(i,j)\,. \end{tikzcd}$$

Since $M(i, j)$ is aspherical we know that $\mathcal{F}(i, j)$ is aspherical and so is a free crossed resolution. Of course $\mathcal{F}(i, j)$ has two vertices $0, 1$. Thus it is not a free crossed resolution of G but is a *free crossed* *resolution of the homotopy pushout* in the category Gpds

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d, "i"'] \arrow[dr, phantom, "\simeq"] & B \arrow[d] \\ A \arrow[r] & G(i,j) \end{tikzcd}$$

which is obtained from the disjoint union of the groupoids A, B, $\mathcal{I} \times C$ by adding the relations $(0, c) \sim i(c)$, $(1, c) \sim j(c)$ for $c \in C$. The groupoid $G(i, j)$ has two objects $0, 1$ and each of its object groups is isomorphic to the amalgamated product group G, but we need to keep its two object groups distinct. [12]

The two crossed complexes of groups $\mathcal{F}(i, j)(0)$, $\mathcal{F}(i, j)(1)$, which are the parts of $\mathcal{F}(i, j)$ lying over $0, 1$ respectively, are free crossed resolutions of the groups $G(i, j)(0)$, $G(i, j)(1)$. From the formulae for the tensor product of crossed complexes we can identify free generators for $\mathcal{F}(i, j)$: in dimension n we get

- free generators a_n at 0 where a_n runs through the free generators of A_n ;
- free generators b_n at 1 where b_n runs through the free generators of B_n ;
- free generators $\iota \otimes c_{n-1}$ at 1 where c_{n-1} runs through the free generators of C_{n-1} .

Example 11.1.16 Let A, B, C be infinite cyclic groups, written multiplicatively. The trefoil group T can be presented as a free product with amalgamation $A *_C B$ where the morphisms $C \to A$, $C \to B$ have cokernels of orders 3 and 2 respectively. The resulting homotopy pushout we call the *trefoil groupoid*. We immediately get a free crossed resolution of length 2 for the trefoil groupoid, whence we can by a retraction argument deduce the free crossed resolution $F(T)$ of the trefoil group T with presentation $\mathcal{P}_T = \langle a, b \mid a^3 b^{-2} \rangle$. By the construction in Section 11.2, there is a free crossed resolution of T of the form

$$F(T) \quad : \quad \cdots \longrightarrow 1 \longrightarrow C(r) \xrightarrow{\phi_2} F\{a, b\} \xrightarrow{\phi_1} T \qquad \text{where} \qquad \phi_2\, r = a^3 b^{-2} \ .$$

Hence a 2-cocycle on T with values in K can also be specified totally by elements $s(c, d) \in K$, $c, d \in \mathrm{Aut}(K)$ such that $\partial s(c, d) = c^3 d^{-2}$; this is a finite description. It is also easy to specify equivalence of cocycles. [13] □

Now we consider HNN-extensions. Let A, B be subgroups of a group G and let $k : A \to B$ be an isomorphism. Then we can form a pushout of groupoids

$$\begin{tikzcd} \{0,1\} \times A \arrow[r, "{(k_0, k_1)}"] \arrow[d, "i"'] & G \arrow[d, "j"] \\ \mathcal{I} \times A \arrow[r, "f"'] & *_k\ G \end{tikzcd}$$

where

$$k_0(0, a) \ = \ ka, \quad k_1(1, a) \ = \ a, \quad \text{and } i \text{ is the inclusion.}$$

In this case of course $*_k\ G$ is a group, known as the HNN-extension. It can also be described as the factor group

$$(Z * G) \,/\, \{z^{-1} a^{-1} z\,(ka) \mid a \in A\}$$

of the free product, where Z is the infinite cyclic group generated by z.

Now suppose we have chosen free crossed resolutions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{G}$ of A, B, G respectively. Then we may lift k to a crossed complex morphism $k'' : \mathcal{A} \to \mathcal{B}$ and k_0, k_1 to

$$k''_0,\, k''_1 \;:\; \{0,1\} \times \mathcal{A} \to \mathcal{G}\,.$$

Next we form the pushout in the category of crossed complexes:

$$\begin{tikzcd} \{0,1\} \otimes \mathcal{A} \arrow[r, "(k''_0,\, k''_1)"] \arrow[d, "i''"'] & \mathcal{G} \arrow[d, "j''"] \\ \mathcal{I} \otimes \mathcal{A} \arrow[r, "f''"'] & \otimes_{k''} \mathcal{G} \end{tikzcd}$$

Theorem 11.1.17 *The crossed complex $\otimes_{k''} \mathcal{G}$ is a free crossed resolution of the group $*_k\, G$.*

The proof is given in [BMPW02] as a special case of a theorem on the free crossed resolutions of the fundamental groupoid of a graph of groups . Here we show that Theorem 11.1.17 gives a means of calculation. Part of the reason for this success is that we do not need to know in detail the definition of free crossed resolution and of tensor products, we just need free generators, boundary maps, values of morphisms on free generators, and how to calculate in the tensor product with $\mathcal{I}$ using the rules given previously.

Example 11.1.18 The Klein Bottle group K has a presentation $\langle\, c, z \mid z^{-1}c^{-1}z\,c^{-1}\,\rangle$. Thus $K = *_k\, C$ where C is infinite cyclic generated by c and $kc = c^{-1}$. This yields a free crossed resolution

$$F(K) \qquad \cdots \longrightarrow 1 \longrightarrow C(r) \xrightarrow{\phi_2} F\{c, z\} \dashrightarrow^{\phi_1} K$$

where $\phi_2\, r = z^{-1}c^{-1}z\,c^{-1}$. Of course this was already known since K is a surface group, and so is aspherical, and also because it is a one relator group whose relator is not a proper power. $\square$

Example 11.1.19 Developing the previous example, let $\langle\, c, z \mid c^q,\; z^{-1}c^{-1}z\,c^{-1}\,\rangle$ be a presentation of the group L. Then $L = *_k\, C_q$ where C_q is the cyclic group of order q generated by c and $k : C_q \to C_q$ is the isomorphism $c \mapsto c^{-1}$. A small free crossed resolution of C_q is given in Subsection 11.1.1 as

$$F(C_q) \quad : \quad \cdots \longrightarrow \mathbb{Z}[C_q] \xrightarrow{\chi_n} \mathbb{Z}[C_q] \longrightarrow \cdots \longrightarrow \mathbb{Z}[C_q] \xrightarrow{\chi_2} A \dashrightarrow^{\chi_1} C_q$$

with a free generator a of A in dimension 1 ; free generators c_n in dimension $n \geqslant 2$; with $\chi_1\, a = c$; $\chi_2(c_2) = a^q$ and

$$\chi_n\, c_n \;=\; \begin{cases} c_{n-1}\,(1-c) & \text{if } n \text{ is odd,} \\ c_{n-1}\,(1 + c + c^2 + \cdots + c^{q-1}) & \text{otherwise.} \end{cases}$$

The isomorphism k lifts to a morphism $k'' : F(C_q) \to F(C_q)$ which is also inversion in each dimension. Hence L has a free crossed resolution $\otimes_{k''} C_q$ given by

$$\cdots \xrightarrow{\lambda_{n+1}} L_n \xrightarrow{\lambda_n} \cdots\cdots \xrightarrow{\lambda_3} L_2 \xrightarrow{\lambda_2} L_1 \xrightarrow{\phi} G$$

having free generators a, z in dimension 1; generators c_2, $z \otimes a$ in dimension 2; and generators c_n, $z \otimes c_{n-1}$ in dimension $n \geqslant 3$. The extra boundary rules are

$$\begin{aligned} \lambda_2(z \otimes a) &= z^{-1}a^{-1}z\,a^{-1}\,, \\ \lambda_3(z \otimes c_2) &= (z \otimes a^q)^{-1}\, c_2^{-1}\, (c_2^{-1})^z\,, \\ \lambda_{n+1}(z \otimes c_n) &= -(z \otimes \chi_n c_n) - c_n - {c_n}^z \qquad \text{for } n \geqslant 3\,. \end{aligned}$$

In particular, the identities among relations for this presentation of L are generated by

$$c_2 \qquad \text{and} \qquad \lambda_3(z \otimes c_2) \;=\; (z \otimes \chi_2 c_2)^{-1}\, {c_2}^{-1}\, ({c_2}^{-1})^z\ .$$

Similarly, relations for the module of identities are generated by

$$c_3 \qquad \text{and} \qquad \lambda_4(z \otimes c_3) \;=\; -(z \otimes c_2(1-c)) - c_3 - {c_3}^z\ .$$

Of course we can expand expressions such as $(z \otimes \chi_n c_n)$ using the rules for the cylinder given in Example 9.3.19. [14] □

11.2 Construction of free crossed resolutions of groups from a presentation

In this section we are going to address the problem of getting a resolution for a group G defined by a presentation $\langle X \mid R \rangle$. As we have seen Theorem 11.1.2 gives a theoretical solution. First we construct the free crossed module

$$C(R) \xrightarrow{\delta_2} F(X) \xrightarrow{\phi} G \to 1.$$

Now we take a free resolution of the G-module $A = \operatorname{Ker} \partial$ in the 'usual way' of constructing 'chains of syzygies'.

That means that at each step we have to get a free G-module mapping surjectively to a G-module

$$F(X_n) \to A_{n-1}$$

and get a set of generators X_n of the kernel.

How can we proceed? Theoretically the answer is easy. If in trouble, just take $X_n = A_n$. Obviously this has no practical application (as A_n could be infinite), so we would like to have a way of constructing smaller resolutions.

11.2.1 Home for a contracting homotopy: chain complexes

In order to introduce our method of 'constructing a home for a contracting homotopy', we consider first the case of chain complexes of R-modules. For these we show there is an easily described way of constructing inductively at the same time the resolution and the contracting homotopy (in chain complexes resolutions are contractible). The method uses homotopy information in dimensions $\leqslant n$ to construct C_{n+1}, δ_{n+1} and h_n from free generators of C_n

Assume that we have constructed both the resolution and the contracting homotopy up to dimension n. Thus, we have the diagram

$$\begin{tikzcd}
C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] \arrow[dl, "h_{n-1}"'] & \cdots \arrow[r] & C_3 \arrow[r, "\partial_3"] & C_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] \arrow[dl, "h_2"'] & C_1 \arrow[d, "f_1"] \arrow[dl, "h_1"'] \\
C_n \arrow[r, "\partial_n"'] & C_{n-1} \arrow[r, "\partial_{n-1}"'] & \cdots \arrow[r] & C_3 \arrow[r, "\partial_3"'] & C_2 \arrow[r, "\partial_2"'] & C_1
\end{tikzcd}$$

where the squares commute and

$$\partial_{i+1} h_i + h_{i-1} \partial_i = 1$$

for $i \leqslant n-1$ and C_i is free on say X_i.

We want a free G-module C_{n+1} and a morphism $h_n : C_n \to C_{n+1}$ satisfying $\partial_{n+1}h_n + h_{n-1}\partial_{n-1} = 1$. We construct the C_{n+1} with just enough room to define the h_n as follows. If X_n is a free generating set of C_n, we consider a set X'_{n+1} in one-to-one correspondence with X_n by $x' \mapsto x$ and define C_{n+1} to be the free R-module on the set X'_{n+1}. We define $\partial_{n+1} : C_{n+1} \to C_n$ as the unique morphism extending

$$\partial_{n+1}(x') = x - h_{n-1}\partial_n(x)$$

for all $x' \in X'_{n+1}$ and

$$h_n : C_n \to C_{n+1}$$

to be the unique morphism extending the bijection $X_n \to X'_{n+1}$.

Clearly this produces an homotopy and one checks that

$$\begin{aligned} \partial_n\partial_{n+1}x &= \partial_n(x - h_{n-1}\partial_n x) \\ &= \partial_n x - (1 - h_{n-2}\partial_{n-1})\partial_n x \qquad \text{by the inductive assumption} \\ &= 0. \end{aligned}$$

In practice one can usually find a subset X_{n+1} of X'_{n+1} such that $\partial_{n+1}X_{n+1}$ also generates $\partial_{n+1}C_{n+1}$; indeed, for many $x' \in X'_{n+1}$ we may have $\partial_{n+1}x' = 0$, so such x' may be eliminated immediately. This enables one to find a smaller candidate for the next step. We shall see this in practice in Subsection 11.2.4.

By iterating we get a free chain complex and a contracting homotopy, so the resulting chain complex is a free resolution. This method we call 'constructing a home for a contracting homotopy', in contrast to the traditional method of 'killing kernels'.

The immediate problem with repeating this process for crossed resolutions of a group G is that such resolutions are not contractible, since the fundamental group is isomorphic to G. We resolve this by passing to the 'universal covering groupoid' $p : \widetilde{G} \to G$ which we set up in the next sections, and construct a free crossed resolution of $\widetilde{G}$, by essentially the above process, taking care of the extra complications of homotopies for crossed complexes as against chain complexes. It is then easy to pass from the free crossed resolution of $\widetilde{G}$ to one for G.

We will see that there are many choices involved in this process. The process deals with Cayley graphs, a standard tool in combinatorial group theory, and we start by choosing a maximal tree in the Cayley graph. The theory well reflects the geometry of covering spaces and extends the notion of Cayley graph to include higher dimensional information.

First we develop the machinery of coverings of crossed complexes in order to prove that a cover of a free crossed complex is free. Only then can we properly give the computational procedure. This is used at the end of the Section to get some free crossed resolutions of groups that we have already stated.

11.2.2 Covering morphisms of crossed complexes

In this subsection we assume as known the notions of covering morphisms of groupoids dealt with in [Bro06, Hig71]; some details and the notation and conventions as used here are given in Subsection A.10.1 of the Appendix.

Definition 11.2.1 A morphism $p : \widetilde{C} \to C$ of crossed complexes is a *covering morphism* if

(i) the morphism $p_1 : \widetilde{C}_1 \to C_1$ is a covering morphism of groupoids;

(ii) for each $n \geqslant 2$ and $\widetilde{x} \in \widetilde{C}_0$, the morphism of groups $p_n : \widetilde{C}_n(\widetilde{x}) \to C_n(p\widetilde{x})$ is an isomorphism.

In such case we call $\widetilde{C}$ a *covering crossed complex* of C. □

This definition may also be expressed in terms of the unique covering homotopy property similar to the one given for fibrations in Section 10.5. Actually, coverings are fibrations with discrete fibre. So we can use the long exact sequence of a fibration Theorem 12.4.1, given in the next Chapter.

Proposition 11.2.2 *Let $p : \widetilde{C} \to C$ be a covering morphism of crossed complexes and let $\tilde{a} \in \mathrm{Ob}(\widetilde{C})$. Let $a = p\tilde{a}$, and let $K = p_0^{-1}(a) \subseteq \mathrm{Ob}(\widetilde{C})$. Then p induces isomorphisms $\pi_n(\widetilde{C}, \tilde{a}) \to \pi_n(C, a)$ for $n \geqslant 2$ and a sequence*

$$1 \to \pi_1(\widetilde{C}, \tilde{a}) \to \pi_1(C, a) \to K \to \pi_0(\widetilde{C}) \to \pi_0(C)$$

which is exact in the sense of the exact sequence of a fibration of groupoids.

The comment about exactness has to do with operations on the pointed sets, see Theorem 12.4.1.

The following result gives a basic homotopical example of a covering morphism of crossed complexes.

Theorem 11.2.3 *Let X_* and Y_* be filtered spaces and let*

$$f : X \to Y$$

be a covering map of spaces such that for each $n \geqslant 0$, $f_n : X_n \to Y_n$ is also a covering map with $X_n = f^{-1}(Y_n)$. Then

$$\Pi f : \Pi X_* \to \Pi Y_*$$

is a covering morphism of crossed complexes.

Proof By [Bro06, 10.2.1],

$$\pi_1 f_1 : \pi_1(X_1, X_0) \to \pi_1(Y_1, Y_0)$$

is a covering morphism of groupoids.

Now for each $n \geqslant 2$ and for each $x_0 \in X_0$, it is a standard result in homotopy theory that

$$f_* : \pi_n(X_n, X_{n-1}, x_0) \to \pi_n(Y_n, Y_{n-1}, p(x_0))$$

is an isomorphism (see for example, [Hu59]). □

Proposition 11.2.4 *Let $p : \widetilde{C} \to C$ be a covering morphism of crossed complexes. Then the induced morphism $\pi_1(p) : \pi_1\widetilde{C} \to \pi_1 C$ is a covering morphism of groupoids.*

Proof Let $\tilde{x} \in \widetilde{C}_0$. We will show that $p'_{\tilde{x}} : \mathrm{St}_{\pi_1\widetilde{C}}\, \tilde{x} \to \mathrm{St}_{\pi_1 C}\, p\tilde{x}$ is bijective. Let $[a] \in \mathrm{St}_{\pi_1 C}\, p\tilde{x}$, where $a \in \mathrm{St}_C p\tilde{x}$. Since p is a covering morphism, there exists a unique $\tilde{a}$ of $\mathrm{St}_{\widetilde{C}}\tilde{x}$ such that $p\tilde{a} = a$. So $p'_{\tilde{x}}[\tilde{a}] = [a]$ and thus $p'_{\tilde{x}}$ is surjective.

Now suppose that $p'_{\tilde{x}}[\tilde{a}] = p'_{\tilde{x}}[\tilde{b}]$. Then $(p\tilde{b})^{-1}p\tilde{a} \in \delta C_2(p\tilde{x})$ which implies that $(p\tilde{b})^{-1}(p\tilde{a}) = \delta p\tilde{c}$ for a unique $\tilde{c} \in \widetilde{C}_2(\tilde{x})$. Because p is a covering morphism, we need only show that $(\tilde{b})^{-1}\tilde{a} = \delta\tilde{c}$. This follows by star injectivity. Therefore $p'_{\tilde{x}}$ is injective and so is bijective. Hence $\pi_1(p)$ is a covering morphism of groupoids. □

Here is an important method of constructing new covering morphisms. Let C be a crossed complex. We write CrsCov/C for the full subcategory of the slice category Crs/C whose objects are the covering morphisms of C.

Proposition 11.2.5 *Suppose given a pullback diagram of crossed complexes*

$$\begin{tikzcd} \widetilde{C} \arrow[r, "\overline{f}"] \arrow[d, "\overline{q}"'] & \widetilde{E} \arrow[d, "q"] \\ C \arrow[r, "f"'] & E \end{tikzcd}$$

in which q is a covering morphism. Then $\overline{q}$ is a covering morphism.

Proof The groupoid case is [Bro06, 9.7.6]. We leave the rest of the proof to the reader. □

Our next result is the analogue for covering morphisms of crossed complexes of a classical result for covering maps of spaces (see, for example, [Bro06, 9.6.1]). It gives a complete classification of covering morphisms of crossed complexes.

Theorem 11.2.6 *If C is a crossed complex, then the functor $\pi_1 : \mathsf{Crs} \to \mathsf{Gpds}$ induces an equivalence of categories*

$$\pi_1' : \mathsf{CrsCov}/C \to \mathsf{GpdsCov}/(\pi_1 C).$$

Proof If $p : \widetilde{C} \to C$ is a covering morphism of crossed complexes, then $\pi_1 p : \pi_1\widetilde{C} \to \pi_1 C$ is a covering morphism of groupoids, by Proposition 11.2.4. Since π_1 is a functor, we also obtain the functor π_1'. To prove π_1' is an equivalence of categories, we construct a functor

$$\rho : \mathsf{GpdsCov}/(\pi_1 C) \to \mathsf{CrsCov}/C$$

and prove that there are equivalences of functors $1 \simeq \rho\pi_1'$ and $1 \simeq \pi_1'\rho$.

Let C be a crossed complex, and let $q : D \to \pi_1 C$ be a covering morphism of groupoids. We consider the crossed complex $\mathrm{sk}^1(G)$ associated to a groupoid G defined in 7.3.10. Let $\widetilde{C}$ be given by the pullback diagram in the category of crossed complexes:

$$\begin{tikzcd} \widetilde{C} \arrow[r, "\bar{\phi}"] \arrow[d, "\bar{q}"'] & \mathrm{sk}^1(D) \arrow[d, "q"] \\ C \arrow[r, "\phi"'] & \mathrm{sk}^1(\pi_1 C) \end{tikzcd}$$

By Proposition 11.2.5, $\bar{q} : \widetilde{C} \to C$ is a covering morphism of crossed complexes.

We define the functor ρ by $\rho(q) = \bar{q}$, and extend ρ in the obvious way to morphisms.

The natural transformation $\pi_1'\rho \simeq 1$ is defined on a covering morphism $q : D \to \pi_1 C$ to be the composite morphism

$$\lambda : \pi_1(\widetilde{C}) \xrightarrow{\pi_1(\bar{\phi})} \pi_1(\mathrm{sk}^1(D)) \cong D$$

where $\bar{\phi} : \widetilde{C} \to \mathrm{sk}^1(D)$ is given in the pullback diagram. The proof that λ is an isomorphism is simple and is left to the reader.

To prove that $1 \simeq \rho\pi_1'$, we show that the following diagram is a pullback:

$$\begin{tikzcd}
\widetilde{C} \arrow[r, "\widetilde{\phi}"] \arrow[d, "q"'] & \mathrm{sk}^1(\pi_1\widetilde{C}) \arrow[d, "\pi_1(q)"] \\
C \arrow[r, "\phi"'] & \mathrm{sk}^1(\pi_1 C)
\end{tikzcd}$$

This is clear in dimension 0 and in dimensions $\geqslant 2$. For the case of dimension 1, let $c : x \to y$ in C, and $[\tilde{c}] \in (\pi_1\widetilde{C})(\tilde{x}, \tilde{y})$ be such that $q[\tilde{c}] = \phi(c)$. Then there exists a unique $\tilde{c}' : \tilde{x} \to \tilde{y}$ such that $\widetilde{\phi}(\tilde{c}') = [\tilde{c}]$ and $\bar{q}(\tilde{c}') = c$. Now, $\bar{q}(\tilde{c}\delta\widetilde{C}_2(\tilde{x})) = \phi(c) = c\delta C_2(x)$. This implies that $(\bar{q}\tilde{c})\delta C_2(x) = c\delta C_2(x)$. So $\bar{q}(\tilde{c}) = c(\delta c_2)$ for some $c_2 \in C_2(x)$. Therefore there exists a unique $\tilde{c}_2 \in \widetilde{C}_2(\tilde{x})$ covering c_2, and $\bar{q}(\tilde{c}(\delta\tilde{c}_2)^{-1}) = c$. So the above diagram is a pullback and thus we have proved that $1 \simeq \rho\pi_1'$. This proves the equivalence of the two categories. □

11.2.3 Coverings of free crossed complexes

Recall that the utility of a free crossed complex is that if C is a free crossed complex on X_*, then a morphism $f : C \to D$ can be constructed inductively provided one is given the values $f_n x \in D_n, x \in X_n, n \geqslant 0$ and provided the following geometric conditions are satisfied: (i) $\delta^\alpha f_1 x = f_0 \delta^\alpha x, x \in X_1, \alpha = 0, 1$; (ii) $\beta f_n(x) = f_0(\beta x), x \in X_n, n \geqslant 2$; (iii) $\delta_n f_n(x) = f_{n-1}\delta_n(x), x \in X_n, n \geqslant 2$.

Notice that in (iii), f_{n-1} has to be defined on all of C_{n-1} before this condition can be verified.

We now show that freeness can be lifted to covering crossed complexes, using the following result of Howie ([How79, Theorem 5.1]).

Theorem 11.2.7 *Let $p : A \to B$ be a morphism of crossed complexes. Then p is a fibration if and only if the pullback functor $p^* : \mathsf{Crs}/B \to \mathsf{Crs}/A$ has a right adjoint.*

As a consequence we get the following: [15].

Corollary 11.2.8 *If $p : A \to B$ is a covering morphism of crossed complexes, then $p^* : \mathsf{Crs}/B \to \mathsf{Crs}/A$ preserves all colimits.*

We shall use this last result to prove that coverings of free crossed complexes are free.

Theorem 11.2.9 *Suppose given a pullback square of crossed complexes*

$$\begin{tikzcd}
\widetilde{A} \arrow[r, "\bar{j}"] \arrow[d, "p'"'] & \widetilde{C} \arrow[d, "p"] \\
A \arrow[r, "j"'] & C
\end{tikzcd}$$

in which p is a covering morphism and $j : A \to C$ is relatively free. Then $\bar{j} : \widetilde{A} \to \widetilde{C}$ is relatively free.

Proof We suppose given the sequence of pushout diagrams

$$\begin{tikzcd}
\coprod_{\lambda\in\Lambda_n} \mathbb{S}(n-1) \arrow[r] \arrow[d] & C^{n-1} \arrow[d] \\
\coprod_{\lambda\in\Lambda_n} \mathbb{F}(n) \arrow[r] & C^n.
\end{tikzcd}$$

defining C as relatively free. Let $\hat{C}^n = p^{-1}(C^n)$. By Corollary 11.2.8, the following diagram is also a pushout:

$$\begin{tikzcd} p^*\left(\coprod_{\lambda\in\Lambda_n}\mathbb{S}(n-1)\right) \arrow[r] \arrow[d] & \hat{C}^{n-1} \arrow[d] \\ p^*\left(\coprod_{\lambda\in\Lambda_n}\mathbb{F}(n)\right) \arrow[r] & \hat{C}^n. \end{tikzcd}$$

Since p is a covering morphism, we can write $p^*\left(\coprod_{\lambda\in\Lambda_n}\mathbb{F}(n)\right)$ as $\coprod_{\lambda\in\widetilde{\Lambda}_n}\mathbb{F}(n)$ for a suitable $\widetilde{\Lambda}_n$. This completes the proof. □

Corollary 11.2.10 *Let $p : \widetilde{C} \to C$ be a covering morphism of crossed complexes. If C is free on X_*, then $\widetilde{C}$ is free on $p^{-1}(X_*)$.*

A similar result to Corollary 11.2.10 applies in the m-truncated case.

The significance of these results is as follows. We start with an m-truncated free crossed resolution C of a group G, so that we are given $\phi : C_1 \to G$, and C is free on X_*, where X_n is defined only for $n \leqslant m$. Our extension process of Subsection 11.2.4 will start by constructing the universal cover $p : \widetilde{C} \to C$ of C; this is the covering crossed complex corresponding to the universal covering groupoid $p_0 : \widetilde{G} \to G$. By the results above, $\widetilde{C}$ is the free crossed complex on $p^{-1}(X_*)$. It also follows from Proposition 11.2.2 that the induced morphism $\widetilde{\phi} : \widetilde{C} \to \widetilde{G}$ makes $\widetilde{C}$ a free crossed resolution of the contractible groupoid $\widetilde{G}$. Hence $\widetilde{C}$ is an acyclic and hence, since it is free, also a contractible crossed complex.

11.2.4 Computing a free crossed resolution

The initial motivation for the work of this subsection was to determine in an algorithmic mode generators and relations for the G-module $\pi(\mathcal{P})$ of identities among relations for a presentation $\mathcal{P} = \langle X \mid \omega\rangle$ of a group G. Here $\omega : R \to F(X)$ is a function and we regard R as a set disjoint from $F(X)$. The advantages of this procedure and of using the function ω are (i) to allow for the possibility of repeated relations, and (ii) to distinguish between an element $r \in R$ and the corresponding element $\omega(r) \in F(X)$.

Associated to this presentation of G, we shall be constructing by induction on dimension a free crossed resolution of the universal covering $\tilde{G}$ with a contracting homotopy and projecting to a free resolution of G. If G is finite, and the presentation is finite, then this free crossed resolution will have a finite number of free generators in each dimension.

Let us start in low dimensions.

2a. Resolution of G up to dimension 2

In Chapter 3 we proved that a presentation $\mathcal{P} = \langle \omega : R \to F(X) \mid X\rangle$ gives the beginning of a crossed resolution

$$C(R) \xrightarrow{\delta_2} F(X) \xrightarrow{\phi} G \tag{11.2.1}$$

where δ_2 is the free crossed module associated to ω. Then $\pi(\mathcal{P})$ is defined to be $\operatorname{Ker}\delta_2$.

The elements of $C(R)$ are 'formal consequences'

$$c = \prod_{i=1}^{n}(r_i^{\varepsilon_i})^{u_i}$$

where $n \geqslant 0, r_i \in R, \varepsilon_i = \pm 1, u_i \in F(X)$, $\delta_2(r^\varepsilon)^u = u^{-1}(\omega r)^\varepsilon u$, subject to the crossed module rule $ab = ba^{\delta_2 b}$, $a, b \in C(R)$.

Remark 11.2.11 It follows from the Higher Homotopy van Kampen Theorem for crossed modules (as in the proof of Theorem 5.4.8) that $\pi(\mathcal{P})$ is given geometrically as the second homotopy group $\pi_2(K(\mathcal{P}))$ of the cell complex of the presentation. This result is not necessary for the work of this chapter, but it does emphasise the topological importance of our methods. [16] □

Actually equation (11.2.1) is equivalent to the more general situation

$$F_2 \to F_1 \to G$$

where F_1 is a free groupoid, F_2 is a free crossed module over F_1, G is a groupoid and ϕ induces an isomorphism $\mathrm{Cok}\,\delta_2 \cong G$; this is because the free generators of F_1 have to map to a set of generators X of G so $F_1 = F(X)$ and F_2 has to be free on some map $\omega : R \to F_1 = F(X)$.

Thus, we want to extend (11.2.1) to a crossed resolution of G. To do this we require algebraic analogues developed in the preceding Sections of methods of covering spaces. We are following the method outlined in the introduction to Section 11.2 (p. 306) for the case of chain complexes. The crucial point is that the algorithmic nature of the argument derives from the construction of homotopies; the fact that these are strong deformation retractions also simplifies the conditions on the homotopies, as shown in Proposition **??**. [17]

2b.- Resolution of the covering $\widetilde{G}$ up to dimension 2.

First we construct a covering of part of diagram (11.2.1) getting

$$\begin{tikzcd} & F(\widehat{X}) \arrow[r, "\widetilde{\phi}"] \arrow[d, "p_1"'] & \widetilde{G} \arrow[d, "p_0"'] \\ C(R) \arrow[r, "\delta_2"'] & F(X) \arrow[r, "\phi"'] & G \end{tikzcd} \tag{11.2.2}$$

where

1. The morphism $p_0 : \widetilde{G} \to G$ is the universal covering groupoid of the group G. The objects of $\widetilde{G}$ are the elements of G, and an arrow of $\widetilde{G}$ is a pair $(g, g') \in G \ltimes G$ with source $gg' = \delta^-(g, g')$ and target $g' = \delta^+(g, g')$. The projection morphism p_0 is given by $(g, g') \mapsto g$. For more details on this and the following, see Example A.10.3.

2. Here $\widehat{X}$ is the *Cayley graph* of the pair (G, X). Its objects are the elements of G and its arrows are pairs $(x, g) \in X \times G$ with source $\delta^-(x, g) = (\phi x)g$ and target $\delta^+(x, g) = g$. Notice that $\widehat{X}$ is in general not the universal covering of X.

3. The groupoid $F(\widehat{X})$ is the *free groupoid* on $\widehat{X}$. Its objects are the elements of G and its arrows can be identified with pairs $(u, g) \in F(X) \times G$ with source $(\phi u)g$ and target g. We also write $\beta(u, g) = g$. The multiplication in $F(\widehat{X})$ is given by $(v, (\phi u)g)((u, g) = (vu, g)$, $v, u \in F(X), g \in G$. The inverse of (u, g) is $(u^{-1}, (\phi u)g)$.

4. The morphism $\hat{\phi}$ is given by $(u, g) \mapsto ((\phi u)g, g)$. The morphism p_1 is given by $(u, g) \mapsto u$. It maps the object group $F(\widehat{X})(1)$ isomorphically to $N = \mathrm{Ker}\,\phi$.

As we will see in Subsection A.10.1, $\widetilde{G} \to G$ is the covering morphism corresponding to the trivial subgroup of G, and $F(\widehat{X}) \to F(X)$ is the covering morphism corresponding to the normal subgroup $N = \mathrm{Ker}\,\phi$ of $F(X)$.

The next step is to take diagram (11.2.2) one dimension higher, getting

$$\begin{tikzcd} C(\hat{R}) \arrow[r, "\hat{\delta}_2"] \arrow[d, "p_2"'] & F(\hat{X}) \arrow[r, "\hat{\phi}"] \arrow[d, "p_1"'] & \hat{G} \arrow[d, "p_0"'] \\ C(R) \arrow[r, "\delta_2"'] & F(X) \arrow[r, "\phi"'] & G \end{tikzcd} \qquad (11.2.3)$$

where $\widehat{R} = R \times G$ and $\hat{\delta}_2 : C(\widehat{R}) \to F(\widehat{X})$ is the free crossed $F(\widehat{X})$-module on $\hat{\omega} : \widehat{R} \to F(\widehat{X})$, $(r, g) \mapsto (\omega(r), g)$. This is the free crossed module of the *covering presentation* $\{\widehat{X} \mid \widehat{R}\}$ of the universal cover $\widetilde{G}$ of G.

Thus $C(\widehat{R})$ is the disjoint union of groups $C(\widehat{R})(g), g \in G$, all mapped by p_2 isomorphically to $C(R)$. Elements of $C(\widehat{R})(g)$ are pairs $(c, g) \in C(R) \times \{g\}$, with multiplication $(c, g)(c', g) = (cc', g)$. The action of $F(\widehat{X})$ is given by $(c, (\phi u)g)^{(u,g)} = (c^u, g)$. The boundary $\hat{\delta}_2$ is given by $(c, g) \mapsto (\delta_2 c, g)$. The morphism $p_2 : C(\widehat{R}) \to C(R)$ is given by $(c, g) \mapsto c$.

The elements of $C(\widehat{R})(g)$ are also all 'formal consequences'

$$(c, g) = \prod_{i=1}^{n} ((r_i, (\phi u_i)g_i)^{\varepsilon_i})^{(u_i, g_i)} = (\prod_{i=1}^{n} (r_i^{\varepsilon_i})^{u_i}, g)$$

where $n \geqslant 0$, $r_i \in R$, $\varepsilon_i = \pm 1$, $u_i \in F(X)$, $g_i \in G$, $(\phi u_i)g_i = g$, subject to the crossed module rule $ab = ba^{\hat{\delta}_2 b}$, $a, b \in C(\hat{R})$.

It is useful to think of these formulae topologically in terms of CW-complexes. The generating set X should be thought of as a set of loops giving the 1-cells of a reduced CW-complex Y, so that we identify $F(X)$ with $\pi_1(Y^1, *)$. The elements $r \in R$ can be thought of as defining the 2-cells of Y, each attached according to the formula for ωr, so that $G = \pi_1(Y, *)$. The element (r, g) for $r \in R, g \in G$ then corresponds to the covering cell of the cell r at the point g, considered as a vertex of $\widetilde{Y}$, and (r, g) is also a relator for the 'covering presentation' of $\widetilde{G}$. Let $\widehat{Y}^n$ be the n-skeleton of $\widetilde{Y}$; then $\pi_1(\widehat{Y}^1, \widehat{Y}^0)$ may be identified with the groupoid $F(\widehat{X})$. If $(u, g) : (\phi u)g \to g$ is a path in $F(\widehat{X})$, and $(r, (\phi u)g)$ is a free generator corresponding to a 2-cell of the universal cover, then this generator also contributes to the *group* $C(\widehat{R})(g)$ with the element $(c, (\phi u)g)^(u, g) = (c^u, g)$.

In effect, we are giving:
1) a presentation $\langle \hat{X} \mid \hat{\omega} \rangle$ of the groupoid $\widetilde{G}$, and
2) the free crossed module corresponding to this presentation. That this construction gives a free crossed module is thanks to Theorem 11.2.9.

2c. Contractibility of the covering up to dimension 2**.**

So we have started the construction of a crossed complex that we want to be acyclic. To prove this acyclicity we construct a contracting homotopy at the same time as we are constructing the crossed complex. So we need to construct h_0, h_1 in the following diagram:

$$\begin{tikzcd} C(\hat{R}) \arrow[r, "\hat{\delta}_2"] \arrow[d, "1"] & F(\hat{X}) \arrow[r, "\hat{\phi}"] \arrow[d, "1"] \arrow[dl, "h_1"'] & G \arrow[d, "1"] \arrow[dl, "h_0"'] \\ C(\hat{R}) \arrow[r, "\hat{\delta}_2"'] & F(\hat{X}) \arrow[r, "\hat{\phi}"'] & G \end{tikzcd} \qquad (11.2.4)$$

Remark 11.2.12 We note for the record that by Proposition **??** on the conditions for a contracting homotopy, h_1 is to be a morphism. Later we will use that for $n \geqslant 2$, h_n is to be a morphism killing the operation of the groupoid $F(\widehat{X})$. □

For h_0, choose a section $\sigma : G \to F(X)$ of ϕ such that $\sigma(1) = 1$. Then σ determines

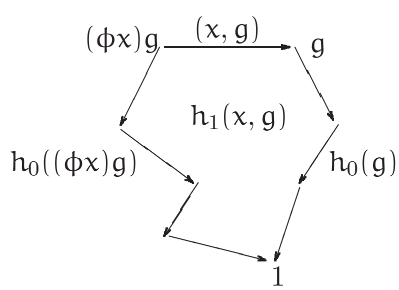
$$h_0 : G \to F(\hat{X}), \quad g \mapsto (\sigma g, 1). \tag{11.2.5}$$

Thus for $g \in G$, $h_0(g)$ is a path $g \to 1$ in the Cayley graph $\widehat{X}$, $p_1 h_0(g) = \sigma(g)$, and $h_0(1) = (1, 1)$. Remark A.10.7 is relevant here, for an expansion of $h_0(g)$ in terms of the generating set X, and also in the argument which follows.

Remark 11.2.13 Such a choice σg writing g as a word in the generators is called a 'normal form' for the element g of G; even for a finite presentation, σ cannot always be found over all of G by a finite algorithm. The usual way of finding it is by a 'rewriting' process, which may not complete in finite time. [18]

The choice of h_0 is often, but not always, made by choosing a maximal tree in the graph $\widehat{X}$ – such a choice is equivalent to a choice of what is called in group theory a Schreier transversal for the subgroup $N = \operatorname{Ker} \phi$ of the free group $F(X)$. □

In the following picture, $(x, g) : (\phi x)g \to g$ is an arrow in $F(\widehat{X})$; $h_0((\phi x)g)$ represents a path in $\widehat{X}$ from $(\phi x)g$ to 1, thought of as an element of $F(\widehat{X})$; and $h_0(g)$ represents an path in $\widehat{X}$ from g to 1, again thought of as an element of $F(\widehat{X})$:



We now construct an element $h_1(x, g) \in C(\widehat{R})$ which fills the middle, as follows.

For each arrow (x, g) of $\widehat{X}$ the element $\ell = (h_0(\phi x)g)^{-1}(x, g)h_0(g)$ is a loop at 1 in $F(\widehat{X})$; so ℓ maps to 1 in the singleton $\widetilde{G}(1, 1)$. Hence ℓ is in the image of $\hat{\delta}_2$. For each arrow (x, g) of $\widehat{X}$ choose an element $h_1(x, g) \in C(\widehat{R})(1)$ such that

$$\hat{\delta}_2(h_1(x, g)) = \ell = (h_0(\phi x)g)^{-1}(x, g)h_0(g). \tag{11.2.6}$$

Then, recalling Remark 11.2.12, and because $F(\widehat{X})$ is free on these generators, h_1 extends uniquely to a morphism

$$h_1 : F(\widehat{X}) \to C(\widehat{R})(1) \tag{11.2.7}$$

which, because it is a morphism, see again Remark A.10.7, satisfies

$$\hat{\delta}_2(h_1(u, g)) = h_0((\phi u)g)^{-1}(u, g)h_0(g) \tag{11.2.8}$$

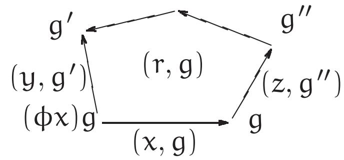
for all arrows (u, g) of $F(\widehat{X})$. In particular for $r \in R$

$$\hat{\delta}_2(h_1(\omega r, g)) = h_0(g)^{-1}(\omega r, g)h_0(g). \tag{11.2.9}$$

It follows also that $\hat{\delta}_2 h_1(h_0(g)) = (1,1)$ for all $g \in G$. Further, if h_0 is determined by a choice of maximal tree T in the Cayley graph, then for each (x, g) in T we may choose $h_1(x, g) = (1, 1)$.

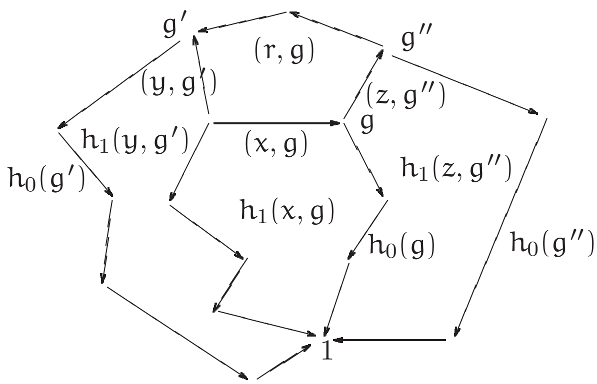
Remark 11.2.14 The specification of h_1 is equivalent to choosing for each element of $n \in N$, given as a word in the elements of X, a representation of n as a consequence of the relations R. There is no algorithm for such a choice. [19] □

Here are pictures of what we have so far. For each $g \in G, r \in R$ we have a 2-cell (r, g) in the Cayley graph with relations, where the boundary of (r, g) in $F(\widehat{X})$, illustrated in the following picture, is $(\omega r, g)$.



In this situation we have of course $g = (\phi z)g''$ and $(\phi x)g = (\phi y)g'$.

The $h_1(e)$ for all edges e of (r, g) together form a kind of cone $h_1(\omega r, g)$ on the boundary of (r, g), see equation (11.2.9); gluing this cone to (r, g) along the common boundary forms what is known as a 'separation element', giving a polygonally subdivided 2-sphere as partially shown in the following picture:



This 'separation element' defines geometrically an element of $\pi_2 = \pi(\mathcal{P})$, the module of identities among relations. We now show that these separation elements form a set of generators of $\pi(\mathcal{P})$ as a G-module; they are determined by h_0 and h_1, but the proof that they generate uses h_2.

This gives all the maps shown in diagram (11.2.3) necessary to give a contracting homotopy up to dimension 2. We now extend these to dimension 3, by constructing elements which 'fill' our separation elements.

3a. Resolution of the covering up to dimension 3.

Let I be a set in one-to-one correspondence with $R \times G$ with elements written $[r, g], r \in R, g \in G$. Let $C_3(I)$ be the free G-module on I. For any $[r, g] \in I$ we define

$$\delta_3[r, g] = p_2\left((h_1(\omega r, g))^{-1}\right)\ r^{\sigma g}.$$

This definition on the free generators extends uniquely to an operator morphism

$$\delta_3 : C_3(I) \to C(R).$$

It follows from equation (11.2.7) that $\delta_2\delta_3[r,g]=1$, and so the given values $\delta_3[r,g]$ lie in $\pi(\mathcal{P}) = \operatorname{Ker}\delta_2$, the G-module of identities among relations. Hence we have a truncated crossed complex:

$$C_3(I) \xrightarrow{\delta_3} C(R) \xrightarrow{\delta_2} F(X) \xrightarrow{\phi} G \tag{11.2.10}$$

and we now extend our previous covering truncated crossed complex by including $C_3(\widehat{I})$, defined to be the free $\widetilde{G}$-module on the projection $\widehat{I} = I \times G \to G$. This implies that $C_3(\widehat{I})$ is the disjoint union of abelian groups $C(\widehat{I})(g), g \in G$, all mapped by p_3 isomorphically to $C_3(I)$. Elements of $C_3(\widehat{I})(g)$ are pairs $(i,g) \in C_3(I)\times\{g\}$ with addition $(i,g)+(i',g)=(i+i',g)$. The action of $\widetilde{G}$ on $C_3(\widehat{I})$ is given by $(i,gg')^{(g,g')} = (i,g')$; note that this makes sense since in $\widetilde{G}$ $(g,g'): gg' \to g'$.

Let $\hat{\delta}_3 : C_3(\widehat{I}) \to C(\widehat{R})$ be the $\widetilde{G}$-morphism given by $\hat{\delta}_3(c,g) = (\delta_3 c, g),\ c \in C_3(I), g\in G$.

These definitions give the morphism of truncated crossed complexes:

$$\begin{array}{ccccccc}
C_3(\widehat{I}) & \xrightarrow{\hat{\delta}_3} & C(\widehat{R}) & \xrightarrow{\hat{\delta}_2} & F(\widehat{X}) & \xrightarrow{\hat{\phi}} & \widetilde{G} \\
{\scriptstyle p_3}\downarrow & & {\scriptstyle p_2}\downarrow & & {\scriptstyle p_1}\downarrow & & {\scriptstyle p_0}\downarrow \\
C_3(I) & \xrightarrow[\delta_3]{} & C(R) & \xrightarrow[\delta_2]{} & F(X) & \xrightarrow[\phi]{} & G
\end{array} \tag{11.2.11}$$

where the upper row is acyclic up to dimension 1.

3b. Contractibility of the covering up to dimension 2.

To construct the next part of the homotopy, and again recalling Remark 11.2.12, we define $h_2 : C(\widehat{R}) \to C_3(\widehat{I})(1)$ to be the groupoid morphism given on generators by $(r,g) \mapsto ([r,g],1), (r,g) \in R \times G$, and killing the operation of $F(\widehat{X})$, i.e. it satisfies $h_2((c,g)^{(u,g)}) = h_2(c,g)$ for all $(c,g) \in C(\widehat{R}), u \in F(X)$.

Then from the definition of $\hat{\delta}_3$ we deduce that

$$\hat{\delta}_3 h_2(c,g) = (h_1(\delta_2 c, g))^{-1}\ (c^{\sigma g}, 1)$$

for all $g \in G, c \in C(R)$ and we have got a contracting homotopy up to dimension 2:

$$\begin{tikzcd}
C_3(\widehat{I}) \arrow[r, "\hat{\delta}_3"] \arrow[d, "1"] & C(\widehat{R}) \arrow[r, "\hat{\delta}_2"] \arrow[d, "1"] \arrow[dl, "h_2"'] & F(\widehat{X}) \arrow[r, "\hat{\phi}"] \arrow[d, "1"] \arrow[dl, "h_1"'] & G \arrow[d, "1"] \arrow[dl, "h_0"'] \\
C_3(\widehat{I}) \arrow[r, "\hat{\delta}_3"'] & C(\widehat{R}) \arrow[r, "\hat{\delta}_2"'] & F(\widehat{X}) \arrow[r, "\hat{\phi}"'] & G
\end{tikzcd} \tag{11.2.12}$$

We use h_2 to prove that we have from the presentation and the choices determining h_0, h_1 constructed all identities among relations.

Theorem 11.2.15 *The module $\pi(\mathcal{P})$ of identities among relations is generated as G-module by the elements*

$$\delta_3[r,g] = (p_2 h_1(\omega r, g))^{-1}\ r^{\sigma g}$$

for all $g \in G, r \in R$.

Proof Since h_2 and h_1 give a contracting homotopy, we have $\hat{\delta}_2\hat{\delta}_3 = 0$, and so the elements $p_2(\hat{\delta}_3 h_2(c,g))$ do give identities. On the other hand, if $c \in C(R)$ and $\delta_2 c = 1$, then $(c,1) = \hat{\delta}_3 h_2(c,1)$, and so $c = \delta_3(d)$ for some d. □

Our algebraic setup is rich enough to be able to write this element in terms of the formula in Theorem 11.2.15.

4. Dimension 4 and higher.

However some of the elements of $\delta_3(I)$ may be trivial, and others may depend $\mathbb{Z}G$-linearly on a smaller subset. That is, there may be a proper subset J of I such that $\delta_3(J)$ also generates the module $\pi(\mathcal{P})$. Then for each element $i \in I \setminus J$ there is a formula expressing $\delta_3 i$ as a $\mathbb{Z}G$-linear combination of the elements of $\delta_3(J)$. These formulae determine a $\mathbb{Z}G$-retraction $\rho : C_3(I) \to C_3(J)$ such that for all $d \in C_3(I), \delta_3(\rho d) = \delta_3(d)$. So we replace I in the above diagram by J, replacing the boundaries by their restrictions. Further, and this is the crucial step, we replace h_2 by $h_2' = \rho' h_2$ where $\rho' : C_3(\widehat{I})(1) \to C_3(\widehat{J})(1)$ is mapped by p_3 to ρ.

This $h_2' : C(\widehat{R}) \to C_3(\widehat{J})(1)$ is now used to continue the above construction. We define $C_4(\overline{J})$ to be the free G-module on elements written $[d, g] \in \overline{J} = G \times J$, with

$$\delta_4[d, g] = -p_3(h_2'(\delta_3 d, g)) + d.g^{-1}.$$

These boundary elements give generators for the relations among the generators $\delta_3(J)$ of $\pi(\mathcal{P})$.

Theorem 11.2.16 *A G-module generating set of relations among these generators $\delta_3(J)$ of $\pi(\mathcal{P})$ is given by*

$$\delta_4[\gamma, g] = -k_2(\delta_3\gamma, g) + \gamma.g^{-1}$$

for all $g \in G, \gamma \in J$, where $k_2 : C(\widehat{R}) \to C_3(J)$ is a morphism from the free crossed $F(\widehat{X})$-module on $\hat{\delta}_2 : G \times R \to F(\widehat{X})$ such that k_2 kills the operation of $F(\widehat{X})$ and is determined by a choice of writing the generators $\delta_3[r, g] \in \delta_3(I)$ for $\pi(\mathcal{P})$ in terms of the elements of $\delta_3(J)$.

Proof This is a similar argument to the proof of Theorem 11.2.15, using the definition of δ_4 and setting $k_2 = p_3 h_2'$. □

From here onwards we proceed as indicated for the chain complex case in the introductory paragraphs of this Section (p. 306).

Remark 11.2.17 In the above we have defined morphisms and homotopies by their values on certain generators, and so it is important for this that the structures be free. For example, h_2' is defined by its values on the elements $(r, g) \in R \times G$. So, noting that h_2 kills the operation of $F(\widehat{X})$, we calculate for example $h_2'(r^u s^v, g) = h_2'(r, g(\phi u)^{-1}) + h_2'(s, g(\phi v)^{-1})$. In this way the formulae reflect the choices made at different parts of the Cayley graph in order to obtain a contracting homotopy. □

Remark 11.2.18 The determination of minimal subsets J of I such that $\delta_3 J$ also generates $\pi(\mathcal{P})$ is again not straightforward. Some dependencies are easy to find, and others are not. A basic result given in Corollary 7.5.24 is that the abelianisation map $C(R) \to (\mathbb{Z}G)^R$ maps $\pi(\mathcal{P})$ isomorphically to the kernel of the Whitehead-Fox derivative $(\partial r/\partial x) : (\mathbb{Z}G)^R \to (\mathbb{Z}G)^X$. Hence we can test for dependency among identities by passing to the free $\mathbb{Z}G$-module $(\mathbb{Z}G)^R$, and we use this in the next section. For bigger examples, this testing can be a formidable task by hand. [20] □

Exercise 11.2.19 Carry out the above procedure for calculating identities among relations for the presentation $\langle a, b \mid a^2, b^2, a^{-1}b^{-1}ab\rangle$ of the Klein four group $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

As an example of these techniques, we give the universal cover and contracting homotopy for an earlier example.

Example 11.2.20 Here we shall prove that the free crossed resolution of a finite cyclic group given in Example 11.1.4 is a resolution by describing its universal cover and a contracting homotopy.

We write C_∞ for the (multiplicative) infinite cyclic group with generator x, and C_q for the finite cyclic group of order q with generator c. Let $\phi : C_\infty \to C_q$ be the morphism sending x to c. We show how the inductive procedure given earlier recovers the small free crossed resolution of C_q together with a contracting homotopy of the universal cover.

Let $p_0 : \widetilde{C}_q \to C_q$ be the universal covering morphism, and let $p_1 : \widehat{C}_\infty \to C_\infty$ be the induced cover of C_∞. Then $\widehat{C}_\infty$ is the free groupoid on the Cayley graph $\widehat{X}$ pictured as follows:

$$\begin{tikzcd}
1 & c \arrow[l, "{(x,1)}"] & c^2 \arrow[l, "{(x,c)}"] & \cdots\cdots \arrow[l, "{(x,c^2)}"] & c^{q-2} \arrow[l] & c^{q-1} \arrow[l, "{(x,c^{q-2})}"] \arrow[lllll, bend right, "{(x^{q-1},1)}"']
\end{tikzcd}$$

A section

$$\sigma : C_q \to C$$

of ϕ is given by $c^i \mapsto x^i, i = 0, \ldots, q-1$, and this defines

$$h_0 : C_q \to \widehat{F}_1$$

by $c^i \mapsto (x^i, 1)$. It follows that for $i = 0, \ldots, q-1$ we have

$$h_0(c^{i+1})^{-1}(x, c^i)h_0(c^i) = \begin{cases} (1,1) & \text{if } i \neq q-1, \\ (x^q, 1) & \text{if } i = q-1. \end{cases}$$

So we take a new generator x_2 for F_2 with $\delta_2 x_2 = x^q$ and set

$$h_1(x, c^i) = \begin{cases} (1,1) & \text{if } i \neq q-1, \\ (x_2, 1) & \text{if } i = q-1. \end{cases}$$

Then for all $i = 0, \ldots, q-1$ we have

$$\widetilde{\delta}_2 h_1(x, c^i) = h_0(c^i)^{-1}(x, c^i)h_0(c^{i+1}),$$

and it follows that

$$h_1(x^q, c^i) = h_1((x, c^i)(x, c^{i+1}) \ldots (x, c^{i+q-1})) = (x_2, 1).$$

Hence

$$-h_1\widetilde{\delta}_2(x_2, c^i) + (x_2, c^i).x^{-i} = (1, -x_2) + (1, x_2 \cdot c^{-i}) = (1, x_2 \cdot (c^{q-i} - 1))$$

This gives 0 for $i = 0$, and $(1, x_2 \cdot (c-1))$ for $i = q-1$. Let $N(i) = 1 + c + \cdots + c^{i-1}$, so that $c^{q-i} - 1 = (c-1)N(q-i)$ for $i = 1, \ldots, q-1$. Hence we can take a new generator x_3 for F_3 with $\delta_3 x_3 = x_2 \cdot (c-1)$ and define

$$h_2(x_2, c^i) = \begin{cases} (1, 0) & \text{if } i = 0, \\ (1, x_3 \cdot N(q-i)) & \text{if } 0 < i \leqslant q-1. \end{cases}$$

Now we find that if we evaluate

$$-h_2\widetilde{\delta}_2(c^i, x_3) + (1, x_3 \cdot c^{-i}) = -h_2((c^{i-1}, x_2) \cdot c + (c^i, x_2)) + (1, x_3 \cdot c^{-i})$$

we obtain for $i = 0$

$$-h_2(c^{q-1}, x_2) + (1, x_3) = (1, 0),$$

for $i = 1$

$$0 + h_2(c, x_2) + (1, x_3 \cdot c^{q-1}) = (1, x_3 \cdot (N(q-1) + c^{q-1})) = (1, x_3 \cdot N(q))$$

and otherwise

$$(1, x_3(-N(q-i+1) + N(q-i) + c^{q-i})) = (1, 0).$$

Thus we take a new generator x_4 for F_4 with $\delta_4 x_4 = x_3 \cdot N(q)$ and

$$h_3(c^i, x_3) = \begin{cases} (1, x_4) & \text{if } i = 1, \\ (1, 0) & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} -h_3\widetilde{\delta}_4(c^i, x_4) + (1, x_4 \cdot c^{-i}) &= -h_3(c^i, x_3 \cdot N(q)) + (1, x_4 \cdot c^{-i}) \\ &= -h_3(1, x_3 \cdot N(q) \cdot c^{-i}) + (1, x_4 \cdot c^{-i}) \\ &= (1, x_4 \cdot (c^{q-i} - 1)). \end{aligned}$$

Thus we are now in a periodic situation and we have the theorem:

Theorem 11.2.21 *A free crossed resolution F_* of C_q may be taken to have single free generators x_n in dimension $n \geqslant 1$ with $\phi(x_1) = c$, $\delta_2(x_2) = x_1^q$ and when $n \geqslant 3$*

$$\delta_n(x_n) = \begin{cases} x_{n-1} \cdot (c - 1) & \textit{if } n \textit{ is odd}, \\ x_{n-1} \cdot (1 + c + \cdots + c^{q-1}) & \textit{if } n \textit{ is even}. \end{cases}$$

Remark 11.2.22 These methods may also be used to derive the standard free crossed resolution of a group or groupoid which we have given in Example 11.1.6. □

Remark 11.2.23 The above methods are used in [BRS99] to construct levels 1, 2 and 3 of a free crossed resolution for the symmetric group S_3. It would take us to long to give more details. □

11.3 Acyclic models

The theory of acyclic models in the traditional methods of homology, i.e. using chain complexes, is a powerful tool for comparing different representations of homology by chain complexes. It has also been useful for comparing cohomology theories of algebraic structures. The same sort of technique works for crossed complexes, but with some technical differences. The methods of proof are closely related to those of Theorem 11.1.9. . The main theorem has a proof entirely analogous to that of the traditional theorem, see for example [Dol95], but the special features of crossed complexes in dimensions $\leqslant 2$ have to be taken into account.

11.3.1 The Acyclic Model Theorem

First we recall for the reader the notion of projective module, and its relation to free modules, as the method will be essential in what follows.

Definition 11.3.1 A module P is called *projective* if for any epimorphism of modules $E \xrightarrow{\partial} A$ and any module morphism $f : P \to A$ then there is a morphism $g : P \to E$ such that $\partial g = f$. The morphism g is called a *lift* of f.

Remark 11.3.2 Note that this is not a universal property since no uniqueness of the lift is required. The situation is often shown in a diagram

$$\begin{tikzcd} & E \arrow[d, "\partial"] \\ P \arrow[ur, dashed, "g"] \arrow[r, "f"'] & A \end{tikzcd}$$

Proposition 11.3.3 *A module is projective if and only if it is a retract of a free module.*

Proof Suppose first P is a retract of a free module F, so that there are morphisms $\eta : P \to F, \mu : F \to P$ such that $\mu\eta = 1$. Consider the following diagram:

$$\begin{tikzcd} F \arrow[r, dashed, "k"] \arrow[d, shift left, "\mu"] & E \arrow[d, "\partial"] \\ P \arrow[u, shift left, "\eta"] \arrow[r, "f"'] & A \end{tikzcd}$$

Since F is free and ∂ is surjective, there is a morphism $k : F \to E$ such that $\partial k = f\mu$. Then $g = k\eta$ satisfies $\partial g = \partial k\eta = f\mu\eta = f1 = f$.

Conversely, if P is any module, we can find a free module F with a surjection $\mu : F \to P$. If P is projective, the identity $1 : P \to P$ lifts to a morphism $\eta : P \to F$ such that $\mu\eta = 1$. □

Definition 11.3.4 Let C be a category and let $F : C \to \mathsf{Crs}$ be a functor. A *base* for F is a family $B_j, n \geqslant 0, j \in J^n$ of objects of C together with sets of elements $b_j \in F(B_j)$ such that for all objects X of C and $n \geqslant 0$ we have $F(X)_n$ is "free" on the elements $F(\sigma)(b_j)$ for $j \in J^n$ and $\sigma : B_j \to X$ in C. This means that for each $n \geqslant 0$ the elements $F(\sigma)(b_j)$:

(i) if $n = 0$, are distinct and give all elements of $F(X)_0$;

(ii) if $n = 1$, freely generate the groupoid $F(X)_1$;

(iii) if $n = 2$, freely generate the crossed module $F(X)_2 = (\delta_2 : C_2 \to C_1)$ as C_1-module;

(iv) if $n \geqslant 3$, freely generate the $\pi_1 F(X)$-module $F(X)_n$.

If $B \subseteq \mathrm{Ob}(C)$ is a class of objects containing all $B_j, j \in J^n, n \geqslant 0$ then we also say F has a base in B. □

Example 11.3.5 If $C = \mathsf{Top}$, then $F = \Pi \circ S^\Delta : \mathsf{Top} \to \mathsf{Crs}$ is in dimension n freely generated by $\{\sigma(\iota_n)\}$, where ι_n is the identity map on Δ^n. If $C = \mathsf{Top} \times \mathsf{Top}$, then F given on (X, Y) by $\Pi S^\Delta(X \times Y)$ is free with base in dimension n with base (ι_n, ι_n). If $F(X, Y) = \Pi S^\Delta X \otimes \Pi S^\Delta Y$, then F has a base in dimension n in the (Δ^p, Δ^q) with $p + q = n$, namely $\{\iota_p \otimes \iota_q\}$. □

Definition 11.3.6 Let C be a category. A functor $P : C \to \mathsf{Crs}$ is said to be *projective* if there is a functor $F : C \to \mathsf{Crs}$ which has a base and such that for each $n \geqslant 1$:

(i) $\mathrm{tr}^2 P_2$ is a natural retract of $\mathrm{tr}^2 F_2$ considered as functors to XMod;

(ii) if $n \geqslant 3$ then P_n is a natural retract of F_n, considered as functors to Mod, and over the natural retraction $\pi_1 P$ of $\pi_1 F$ induced by that given by (i). □

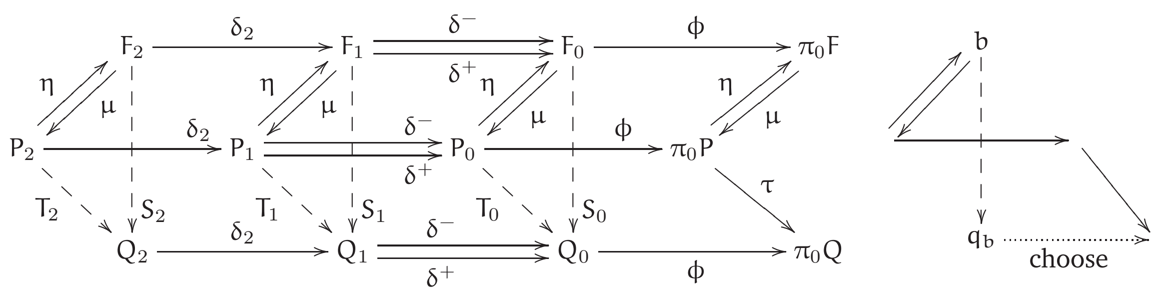
Remark 11.3.7 It is because the operations in a crossed complex are so to speak intrinsic that we have to make this careful definition of projective, which is satisfied in useful circumstances as we shall see. □

Definition 11.3.8 Let $Q : \mathsf{C} \to \mathsf{Crs}$ be a functor. We say Q is *acyclic* on the base $\mathcal{B}$ if $Q(B)$ is an acyclic crossed complex for all elements B of the base. □

Theorem 11.3.9 (Acyclic Model Theorem) *Let* $P, Q : \mathsf{C} \to \mathsf{Crs}$ *be functors such that* Q *is acyclic and suppose there is a base* $\mathcal{B}$ *and functor* $F : \mathsf{C} \to \mathsf{Crs}$ *free on* $\mathcal{B}$*, and for which* P *is projective. Then any natural transformation* $\tau : P_0 \to Q_0$ *is realised by a natural transformation* $T : P \to Q$ *and any two such realisations are naturally homotopic.*

Proof Let $\eta_n : P_n \to F_n, \mu_n : F_n \to P_n$ be the family of natural transformations supplied by the definition of projective, so that $\mu_n \eta_n = 1$. We will often drop the suffix n when it can be understood from the context.

We consider first the right hand part of the first diagram in the following:



Notice that by our assumptions, η, μ give natural morphisms of the crossed module parts $P \to F, F \to P$ respectively. Our method is to construct $S_n : F_n \to Q_n$ and then, following the pattern of Proposition 11.3.3, define $T_n = S_n \eta$, and find this is the appropriate extension.

We are trying to find a natural crossed complex morphism $T : P \to Q$ which induces τ. We first define T in dimension 0.

The points of $F(X)_0$ are of the form $F(\sigma)(b)$ for $b \in F(B)_0$, $B \in \mathcal{B}$ and all morphisms $\sigma : B \to X$. Choose a point $q_b \in Q(B)_0$ such that $\phi q_b = \tau \phi \mu b$, and define $S_0^X(F(\sigma)(b)) = Q(\sigma)(q_b)$. This defines S_0^X and we set $T_0^X = S_0^X \eta$. Then

$$\phi T_0^X = \phi S_0^X \eta = \tau \phi \mu \eta = \tau \phi.$$

We next verify naturality of S_0, and so of T_0. Let $f : X \to Y$ be a morphism in C. Then we check the naturality condition on the basis elements $F(\sigma)(b)$ of $F_0(X)$. Then

$$\begin{aligned} Q_0(f)S_0(X)(F(\sigma)(b)) &= Q_0(f)(Q_0(\sigma)S_0(B)(b)) \\ &= Q_0(f\sigma)(S_0(B)(b)) \\ &= S_0(Y)F_0(f)(S_0(B)(b)). \end{aligned}$$

Thus naturality is automatic from the construction, and we will not repeat this proof in higher dimensions.

We next define a morphism of groupoids $S_1 : F(X)_1 \to Q(X)_1$.

Let $X \in C$. We know the groupoid $F_1(X)$ has a free basis of elements $F(\sigma)(b)$ for $b \in F_1(B)$, $B \in \mathcal{B}$ and all morphisms $\sigma : B \to X$. Consider $q_b^- = T_0\delta^-\mu b$, $q_b^+ = T_0\delta^+\mu b \in Q(B)_0$. By acyclicity of $Q(B)_1$ there is an element $q_b : q_b^- \to q_b^+ \in Q(B)_1$. We therefore define $S_1^X : F_1(X) \to Q_1(X)$ to have value $Q(\sigma)(q_b)$ on the basis element $F(\sigma)(b)$. We then set $T_1 = S_1\eta$. By naturality, $\delta^\pm S_1 = S_0\delta^\pm = T_0\delta^\pm\mu$. Hence $T_0\delta^\pm = \delta^\pm T_1$.

At the next stage we consider the above diagram and a basis element $F(\sigma)(b)$ for $b \in F(B)_2$. Then from the commutativity of the diagram of solid arrows, $T_1\delta_2\mu b$ is a loop in $Q_2(B)$. By acyclicity of $Q(B)$ we can find $q_b \in Q_2(B)$ such that $\delta_2 q_b = T_1\delta_2\mu b$, and we set $S_2^X(F(\sigma)(b)) = Q_2(\sigma)(q_b)$. This defines S_2 and we set $T_2 = S_2\eta$. Since η, μ in dimensions 1 and 2 give crossed module morphisms, because $\mu\eta = 1$, and by the above argument on naturality, T_2 gives gives the required extension of T_1.

Continuing this argument, gives the natural transformation T as required.

We now have to show that any two such natural transformations, say T, U are naturally homotopic. For this, we replace F by $F' = \mathcal{I} \otimes F$ and use analogous arguments to extend the natural transformation defined by T, U on $\{0,1\} \otimes F$ to $\mathcal{I} \otimes F$ on the extra basis elements $\iota \otimes (F(\sigma)(b))$. We omit further details. □

Corollary 11.3.10 *Let* $P, Q : C \to \mathsf{Crs}$ *be functors such that* P *is projective with respect to a free functor with base* $\mathcal{B}$ *on which* Q *is acyclic, and* Q *is projective with respect to a free functor with base* $\mathcal{B}'$ *on which* P *is acyclic. Then any natural equivalence* $\pi_0 P \to \pi_0 Q$ *extends to a natural homotopy equivalence* $P \to Q$.

Exercise 11.3.11 Develop a version of the acyclic model theorem in which the notion of free is replaced by relatively free. □

11.3.2 Simplicial sets and normalisation

A *simplicial set* is a family $K = \{K_n\}_{n \geqslant 0}$ of sets together with face operations $\partial_i : K_n \to K_{n-1}$ for $n \geqslant 1$ and *degeneracy operations* $\epsilon_i : K_n \to K_{n+1}$ for $i = 0, \ldots n$ and $n \geqslant 0$, satisfying the usual simplicial relations. It is standard to consider K also as a functor $K : \boldsymbol{\Delta}^{\mathrm{op}} \to \mathsf{Sets}$, where $\boldsymbol{\Delta}$ is the category called the *simplicial site*.

We shall also need the notion of simplicial set without degeneracies: this is given by a functor $\Upsilon^{\mathrm{op}} \to \mathsf{Sets}$ where Υ is the appropriate subcategory of $\boldsymbol{\Delta}$. Clearly any simplicial set determines a simplicial set without degeneracies (often called a Δ-set) by means of the inclusion $\Upsilon \to \boldsymbol{\Delta}$.

As explained in a previous paper, [BS07], there is for each $n \geqslant 0$ a crossed complex $a\Delta^n$ which is a crossed complex model of the n-dimensional simplex; thus the boundary is given by the homotopy addition lemma. This family of crossed complexes $a\Delta^n$ can be regarded as a cosimplicial set $a\Delta : \boldsymbol{\Delta} \to \mathsf{Crs}$ so that we obtain the fundamental crossed complex of a simplicial set K as a coend

$$\Pi K = \int^{\boldsymbol{\Delta}, n} K_n \times a\Delta^n. \tag{11.3.1}$$

Also we have the unnormalised crossed complex

$$\Pi^{\Upsilon} K = \int^{\Upsilon, n} K_n \times a\Delta^n. \tag{11.3.2}$$

These crossed complexes are homotopy equivalent; this is the normalisation theorem of [BS07] for which we will give an acyclic model proof here [At the moment we do not have this written down since we do not have a direct proof of acyclicity of the unnormalised simplex! However we can get by without it for the EZ theorem, for which it is a good thing to give a proof.]: also both crossed complexes are needed for the purposes of acyclic models.

Theorem 11.3.12 *For all* $q \geqslant 0$ *the functors* $(\Pi K)_q$, $(\Pi^{\Upsilon} K)_q$ *on simplicial sets* K *have the property that the first is a natural retract of the second.*

Proof We first construct an intermediate functor $\Pi^{red}K$.

A simplicial set K contains its subsimplicial set generated by the elements of K_0: we write this as $\bar{K}_0$. It is the disjoint union of the simplicial sets generated by the elements of K_0. We form the crossed complex $\Pi^{red}K$ by the pushout in the category Crs

$$\begin{tikzcd} \Pi^{\Upsilon}\bar{K}_0 \arrow[r, "\xi"] \arrow[d] & K_0 \arrow[d] \\ \Pi^{\Upsilon}K \arrow[r, "\bar{\xi}"'] & \Pi^{red}K \end{tikzcd} \tag{11.3.3}$$

where K_0 denotes here also the trivial crossed complex on the set K_0. Because as a crossed complex K_0 is a natural retract of $\Pi^{\Upsilon}\bar{K}_0$ it follows that $\Pi^{red}K$ is also a natural retract of $\Pi^{\Upsilon}K$. In fact ξ is a homotopy equivalence of crossed complexes and so it follows from the gluing theorem for homotopy equivalences in the category Crs (see [BG89b]), that $\bar{\xi}$ is also a homotopy equivalence of crossed complexes.

We have to consider the dimensions $1, 2$ and $q \geqslant 3$.

The groupoid $C_1 = (\Pi^{red}K)_1$ is the free groupoid on the elements of K_1, but with the elements $\epsilon_0 \nu$ equated to identities for each $\nu \in K_0$. Thus $(\Pi^{red}K)_1 = (\Pi K)_1$.

In dimension 2, we note that if $x \in K_1$, then in $(\Pi^{red}K)_2$, $\delta_2(\epsilon_i x) = 1_{tx}$ for $i = 0, 1$: this is a reason for constructing $(\Pi^{red}K)$. For $i = 0, 1$, let $\Phi_i : (\Pi^{red}K)_2 \to (\Pi^{red}K)_2$ be given on the basis elements by $\Phi_i k = k(\epsilon_i \partial_{i+1} k)^{-1}$. Then

$$\Phi_i \epsilon_i x = 1, \Phi_1 \epsilon_0 x = (\epsilon_0 x)(\epsilon_0^2 t x)^{-1} = \epsilon_0 x$$

in $(\Pi^{red}K)_2$, so that $\Phi = \Phi_0\Phi_1$ vanishes on degenerate elements of K_2. Further, $\delta_2\Phi = \delta_2$. So Φ defines a morphism $(\Pi^{red}K)_2 \to (\Pi^{red}K)_2$ of crossed $(\Pi K)_1$-modules which vanishes on degenerate elements and hence defines in dimension 2 a section of the projection $\Pi^{red}K \to \Pi K$.

In dimensions $q \geqslant 3$ and for $0 \leqslant j < q$ we define $\Phi_j : (\Pi^{red}K)_q \to (\Pi^{red}K)_q$ on the free basis of elements of K_q not degeneracies of the vertices by $\Phi_j k = k - \epsilon_j \partial_{j+1} k$, and set $\Phi = \Phi_0 \dots \Phi_{q-1}$. Then $\Phi_j \epsilon_j x = 0$ and for $i < j$ we have $\Phi_j \epsilon_i x = \epsilon_i x - \epsilon_i \epsilon_{j-1} \partial_{j+1} x$. Hence Φ is trivial on degeneracies and so determines a section of $(\Pi^{red}K)_q \to (\Pi K)_q$ which is also natural for maps of K. $\Box$

11.3.3 Cubical sets and normalisation

A *cubical set* is a family $K = \{K_n\}_{n \geqslant 0}$ of sets together with face operations $\partial_i^{\pm} : K_n \to K_{n-1}$ and *degeneracy operations* $\epsilon_i : K_{n-1} \to K_n$ for $i = 1, \dots n$ and $n \geqslant 1$, satisfying the usual cubical relations. It is standard to consider K also as a functor $K : \Box^{op} \to \mathsf{Sets}$, where $\Box$ is the category called the *cubical site*.

We shall also need the notion of cubical set without degeneracies: this is given by a functor $\Xi^{op} \to \mathsf{Sets}$ where Ξ is the appropriate subcategory of $\Box$. Clearly any cubical set determines a cubical set without degeneracies by means of the inclusion $\Xi \to \Box$.

The family of crossed complexes $\mathcal{I}^n$ can be regarded as a cocubical set $\mathcal{I}^\bullet : \square \to \mathsf{Crs}$ so that we obtain the fundamental crossed complex of a cubical set K as a coend

$$\Pi K = \int^{\square,n} K_n \times \mathcal{I}^n. \tag{11.3.4}$$

Also we have the unnormalised crossed complex

$$\Pi^{\Xi} K = \int^{\Xi,n} K_n \times \mathcal{I}^n. \tag{11.3.5}$$

These crossed complexes are not homotopy equivalent, but we need both for the purposes of acyclic models.

Theorem 11.3.13 *For all* $q \geqslant 0$ *the functors* $(\Pi K)_q$, $(\Pi^{\Upsilon} K)_q$ *on cubical sets* K *have the property that the first is a natural retract of the second.*

Proof Again we work through an intermediate step. A cubical set K contains its subcubical set generated by the elements of K_0: we write this as $\bar{K}_0$. It is the disjoint union of the cubical sets generated by the elements of K_0. We form the crossed complex $\Pi^{red} K$ by the pushout in the category Crs

$$\begin{tikzcd} \Pi^{\Xi}\bar{K}_0 \arrow[r] \arrow[d] & K_0 \arrow[d] \\ \Pi^{\Xi}K \arrow[r] & \Pi^{red}K \end{tikzcd} \tag{11.3.6}$$

where K_0 denotes here also the trivial crossed complex on the set K_0. Because as a crossed complex K_0 is a natural retract of $\Pi^{\Xi}\bar{K}_0$ it follows that $\Pi^{red}K$ is also a natural retract of $\Pi^{\Xi}K$.

We have to consider the dimensions $1, 2$ and $q \geqslant 3$.

The groupoid $C_1 = (\Pi^{red}K)_1$ is the free groupoid on the elements of K_1, but with the elements $\epsilon_1 \nu$ equated to identities for each $\nu \in K_O$. Thus $(\Pi^{red}K)_1 = (\Pi K)_1$.

In dimension 2, we note that if $x \in K_1$, then in $(\Pi^{red}K)_2$, $\delta_2(\epsilon_i x) = 1_{tx}$ for $i = 1, 2$: this is a reason for constructing $(\Pi^{red}K)$. Let $\Phi_i : (\Pi^{red}K)_2 \to (\Pi^{red}K)_2$ be given on the basis elements by $\Phi_i k = k(\epsilon_i k \partial_i^+ k)^{-1}$. Then

$$\Phi_i \epsilon_i x = 1, \Phi_2 \epsilon_1 x = (\epsilon_1 x)(\epsilon_1^2 tx)^{-1} = \epsilon_1 x$$

in $(\Pi^{red}K)_2$, so that $\Phi = \Phi_2\Phi_1$ vanishes on degenerate elements of K_2. Further, $\delta_2\Phi = \delta_2$. So Φ defines a morphism of crossed $(\Pi K)_1$-modules $(\Pi^{red}K)_2 \to (\Pi^{red}K)_2$ which vanishes on degenerate elements and hence defines in dimension 2 a section of the projection $\Pi^{red}K \to \Pi K$.

In dimensions $q \geqslant 3$ we define $\Phi_i : (\Pi^{red}K)_q \to (\Pi^{red}K)_q$ on the free basis of elements of K_q which are not degeneracies of the vertices by $\Phi_i k = k - \epsilon_i \partial_i^+ k$, and set $\Phi = \Phi_1 \ldots \Phi_q$. Then Φ is trivial on degeneracies and so determines a section of $(\Pi^{red}K)_q \to (\Pi K)^q$ which is also natural for maps of K. $\square$

11.3.4 Relating simplicial and cubical by acyclic models

Consider the functors $P, Q : \mathsf{Top} \to \mathsf{Crs}$ given by

$$P(X) = \Pi|S^{\Delta}(X)|_*, \quad Q(X) = \Pi|S^{\square}(X)|_*$$

where the filtrations are the skeletal filtrations of the geometric realisations.

Proposition 11.3.14 *Each of the functors* P, Q *are natural retracts of free functors and are acyclic on a base.*

Proof In the case of P, the free functor is $\Pi||S^{\Delta}(X)||_*$ with base consisting of the geometric n-simplices $|\Delta^n|$ for $n \geqslant 0$. □

11.3.5 The crossed complex simplicial Eilenberg-Zilber theorem

11.3.6 Excision

The work for this section is a modified version of [Sch76]. His cubical methods are also relevant to the general area.

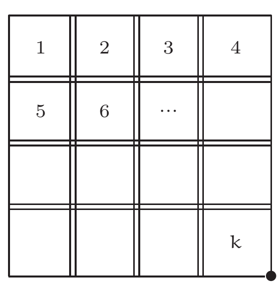
We write TopCov for the category of pairs $(X, \mathcal{U})$ where X is a topological space and $\mathcal{U}$ is a covering of X having an open refinement. A morphism $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is a map $f : X \to Y$ such that for every $U \in \mathcal{U}$ there is a $V \in \mathcal{V}$ such that $f(U) \subseteq V$. Two maps $f, g : (X, \mathcal{U}) \to (Y, \mathcal{V})$ are called *homotopic*, $f \simeq g$, if there exists a homotopy $H : I \times X \to Y$ from f to g such that for every $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ such that $H(I \times U) \subseteq V$. The definitions of a homotopy equivalence and of strong deformation retract are the obvious ones. The trivial pair $(X, \mathcal{T})$ has $\mathcal{T}$ consisting solely of X.

Definition 11.3.15 We write $S^{\Delta}(X, \mathcal{U})$ for the subsimplicial set of $S^{\Delta}(X)$ of simplices $\sigma : \Delta^n \to X$ such that $\sigma(\Delta^n) \subseteq U$ for some $U \in \mathcal{U}$.

Lemma 11.3.16 *Every object* $(I^n, \mathcal{U})$ *in* TopCov *is contractible.*

Proof We construct a finite sequence $(X_i, \mathcal{U}_i) : i = 0, \dots, k$ of objects in TopCov such that $(X_0, \mathcal{U}_0) = (I^n, \mathcal{U})$ and $(X_k, \mathcal{U}_k) = (*, \mathcal{T})$ with * one point, and $(X_i, \mathcal{U}_i)$ a strong deformation retract of $(X_{i-1}, \mathcal{U}_{i-1})$.

By the Lebesgue covering lemma, the cube I^n may be subdivided by hyperplanes parallel to its faces into a finite number say k of subcubes each of which is contained in some $U \in \mathcal{U}$. Now beginning in one corner we collapse one subcube after another into that part of its boundary which is in common with the remaining ones, shown in the diagram by double lines. The last cube is retracted onto the corner •. (An analogous argument is used in the proof of Proposition 14.2.8.)



So we define X_i as X_{i-1} with the ith cube retracted off and $\mathcal{U}_i$ as $\mathcal{U}_{i-1} \cap X_i$. Obviously $(X_i, \mathcal{U}_i)$ is a strong deformation retract of $(X_{i-1}, \mathcal{U}_{i-1})$. □

Theorem 11.3.17 *The inclusion*

$$i : S^{\Delta}(X, \mathcal{U}) \to S^{\Delta}(X) \qquad (*)$$

is a homotopy equivalence.

Proof We actually prove that the induced morphism

$$i' : \Pi S^{\Delta}(X, \mathcal{U}) \to \Pi S^{\Delta}(X) \qquad (**)$$

is a homotopy equivalence of crossed complexes.

We consider both sides of (**) as functors from TopCov to the category of crossed complexes: $P(X, \mathcal{U}) = \Pi S^{\Delta}(X, \mathcal{U})$, $Q(X, \mathcal{U}) = \Pi S^{\Delta}(X)$. As models in TopCov we choose all pairs $(\Delta^n, \mathcal{V}) : n \geqslant 0$ with Δ^n a standard simplex and $\mathcal{V}$ a covering of Δ^n having an open refinement. Both functors are acyclic on models, by Lemma 11.3.16 and the homeomorphism $\Delta^n \cong I^n$.

Let $F : \mathsf{TopCov} \to \mathsf{Crs}$ be given by $\Pi\bar{S}^{\Delta}(X, \mathcal{U})$ where this singular complex has n-simplices the maps $(\Delta^n, \mathcal{V}) \to (X, \mathcal{U})$ in TopCov. Then F has a base the identities $(\Delta^n, \mathcal{V}) \to (\Delta^n, \mathcal{V})$. The inclusion $i : Q(X, \mathcal{U}) \to F(X, \mathcal{U})$ is given by considering $\sigma : \Delta^n \to X$ as $\sigma : (\Delta^n, \sigma^{-1}\mathcal{U}) \to (X, \mathcal{U})$, and the forgetful functor $\mathsf{TopCov} \to \mathsf{Top}$ defines $r : F \to Q$ such that $ri = 1$. So Q is a retract of a free functor, while P is actually free with base in dimension n the identity $(\Delta^n, \mathcal{T}) \to (\Delta^n, \mathcal{T})$. So the Acyclic Model Theorem and its Corollary 11.3.10 applies. □

Notes

[10]p.302 Another proof using the simplicial classifying space of a crossed complex is given in Tonks' thesis, [Ton94].

[11]p. 303 These are special cases of results on graphs of groups which are given in [Moo01, BMPW02], but these cases nicely show the advantage of the method and in particular the necessary use of groupoids.

[12]p. 304 This idea of forming a fundamental groupoid is due to Higgins in the case of a graph of groups [Hig76], where it is shown that it leads to convenient normal forms for elements of this fundamental groupoid. This view is pursued in [Moo01], from which this section is largely taken.

[13]p. 304 More elaborate examples and discussion are given in [Moo01, BMPW02].

[14]306 Further examples are developed in [Moo01].

[15]p.310 For more references on this kind of argument, see [BH87, HK89]

[16]p. 312 The earlier results of Peiffer, Reidemeister and Whitehead on the relations between identities among relations and second homotopy groups of 2-complexes were given an exposition in [BH82], written in memory of Peter Stefan who died in a climbing accident in 1979. The notion of calculating using pictures explained there was developed by a number of authors, see the survey in [HAM93]. The paper [BRS99] gave the calculation method explained here, which has the advantage of being able to calculate higher syzygies (identities among identities, and so on).

[17]p. 312 This use of homotopies was inspired by work on the Homological Perturbation Lemma, [BL91], where the construction of homotopies is crucial. Such use also agrees with the general groupoid philosophy, in which an arrow $g : a \to b$ in a groupoid gives a 'reason why a and b are equivalent'.

[18]p. 314 See works on 'rewriting', and references in, say, Wikipedia.

[19]p. 315 It is shown in [HW03] how a 'logged Knuth-Bendix procedure' will give such a choice if the monoid rewrite system determined by R may be completed, and that this allows for an implementation of the determination of h_1.

[20]p. 317 An implementation of Gröbner basis procedures for finding minimal subsets which still generate is described in [HR]. Related calculations by hand for the group S_3 are given in [BRS99].

General notes

It was only in 1989 that a generalisation of Hopf's formula to all dimensions was published in [BE88]. This involved the notion of 'double' and 'n-fold presentation', which eventually was reformulated into that of n-fold extension, in work of Janelidze, Everaert, and others. This suggests that the reason for the limitations of the crossed complex methods is that crossed complexes form a linear theory, as is shown by the linear form of a crossed complex. For further work, we would seem to need double and higher crossed complexes. The natural conjecture is that the quadratic theory needs double crossed complexes, and so on! This is matter for considerable further exploration.

The free crossed resolution of finite cyclic groups was introduced by Brown and Wensley in [BW95].

The notion of free crossed resolution was crucial in the work of Huebschmann [Hue80, Hue81a, Hue81b].

The theory of acyclic models was founded in the classic paper [EM53], and was in the adjacent paper used to prove what is now called the Eilenberg-Zilber theorem in [EZ53], determining the chain complex of a product of simplicial sets as a tensor product of chain complexes. Applications were subsequently developed for example in [GM57]. The work of Barr which is given an exposition in [Bar02] gives an advanced view of the traditional chain complex theory, and it would be interesting to know if analogues can be usefully developed for crossed complexes.

The notion of resolution by chain complexes has led to an advanced view of homological algebra using the notion of triangulated category. A substantial reference is [Kün07]. Again, it is not clear how this can be usefully developed for crossed complexes, to give a nonabelian view of these ideas.

The methods of Section 11.2.4 have been developed by Ellis in [Ell04] and in subsequent GAP programs, [Ell08]. He works by constructing a universal covering CW-complex rather than the corresponding crossed complex.

Chapter 12

Nonabelian cohomology of spaces and of groups

Our intention in this chapter is to sketch an account of the cohomology of groups and of spaces based on the notion of crossed complex, and the homotopy theory of these. The advantage over the traditional chain complex approach is that it easily allows for the coefficients to be a crossed module, and hence it gives a more computational approach to the Schreier theory of nonabelian extensions of groups. It also, by the use of the theory of fibrations of crossed complexes, allows a broad form of exact sequence of a fibration of crossed complexes to be applied to extension problems and to obtain results on the homotopy classification of maps of spaces.

The key point for the homology of groups is that to a group G we can assign the standard free crossed resolution $F^{st}_*(G)$ of G; if M is a G-module, and $n \geqslant 2$, then we can form the crossed complex $\mathbb{K}(M, n; G, 1)$ which is G in dimension 1, M in dimension n, with the given action, and all boundaries trivial. We have a standard morphism $\phi : F^{st}_1(G) \to G$ and we can define $H^n(G, M)$ as the set

$$[F^{st}_*(G), \mathbb{K}(M, n; G, 1); \phi] \tag{12.0.1}$$

of homotopy classes of morphisms of crossed complexes relative to the morphism ϕ. An examination of the consequences of this shows that we recover the usual definition of cohomology in terms of n-cocycles $G^n \to M$.

One immediate advantage of this definition is that up to a bijection of the homotopy classes we can replace the standard free crossed resolution of G by any free crossed resolution $F(G)$ with its morphism $\phi : F_1(G) \to G$. Next we can generalise and replace $\mathbb{K}(M, n; G, 1)$ by any crossed complex, and in particular a crossed module. This gives a form of nonabelian cohomology of G, which we find is relevant to considering nonabelian extensions of groups and their description in terms of factor sets.

Then we can consider an analogous concept of cohomology of a space X, by replacing $F(G)$ by the free crossed complex $\Pi S(X)$ where $S(X)$ is the singular complex of X, either cubical or simplicial. Thus the natural concept is the notion of homotopy classes

$$[F, A; i] \tag{12.0.2}$$

where F is a free crossed complex, A is a crossed complex, and i is what we call a *local system*, namely a morphism $i : F_1 \to A_1$ such that $i(\delta_2 F_2) \subseteq \delta_2 A_2$.

In this way the nonabelian homological algebra of groups is seen here as a special case of nonabelian algebraic topology. This reflects the historical process, where the homology of groups arose out of considerations of algebraic topology in work of Hopf given in our Example 5.5.5 and Proposition 8.3.20, see [ML78].

Exact sequences play an important role in traditional homological algebra. Here they are also important but somewhat more elaborate than the traditional ones, but give useful results not usually related to an exact sequence. In Section 12.4 we introduce the family of long exact sequences of a fibration of crossed complexes, and in subsequent sections apply this to analyse the homotopy classes of maps from a free crossed complex to what we call n-aspherical crossed complexes. This has useful applications to the homotopy classification of maps of spaces and to a generalisation of abstract kernel theory in the cohomology of groups.

In a final section we indicate how to apply the standard free crossed resolution of groupoids to nonabelian Čech cohomology of a space.

12.1 Cohomology of a group

Recall that if G is a group, M is a G-module, and $n \geqslant 2$, then we write

$$\mathbb{K}(M, n; G, 1) := \qquad \cdots \longrightarrow 0 \longrightarrow \underset{n}{M} \longrightarrow 0 \longrightarrow \cdots\cdots \longrightarrow 0 \longrightarrow \underset{1}{G}$$

for the crossed complex which is G in dimension 1, M in dimension n, is trivial otherwise, and has zero boundary maps (this is automatic if $n > 2$).

Using this notation, we define the cohomology groups of a group G in terms of homotopy classes of maps from the standard free crossed resolution $F^{st}_*(G)$ of the group to this particular structure. The fact that any two free crossed resolutions of G are homotopy equivalent implies that we can at will replace the standard free crossed resolution by any free crossed resolution convenient for the purposes at hand.

A crossed resolution $F(G)$ of G also includes a morphism $\phi : F_1(G) \to G$ inducing an isomorphism $\phi' : \pi_1 F(G) \to G$. So this information should be taken in account in the homotopy classes. Accordingly, we introduce the notion of homotopy relative to a fixed map.

Definition 12.1.1 Let F, C be crossed complexes, and let $f : F \to C$ be a morphism. Let $i : A \to F$ be a morphism of crossed complexes. A *homotopy* $h : f \simeq g$ *rel* i (or rel A) is a homotopy which satisfies hi is a constant homotopy, so that this implies $fi = gi$. The resulting set of homotopy classes we write

$$[C, D; fi].$$

We often employ this when A is a truncation say $\mathrm{tr}^n F$ of F and i is the inclusion, so that we are dealing with a homotopy relative to levels $\leqslant n$.

This set of homotopy classes will also be usefully interpreted using the internal hom: the morphism i induces a morphism of crossed complexes $i^* : \mathsf{CRS}(F, C) \to \mathsf{CRS}(A, C)$, and $[F, C; fi]$ is $\pi_0(\mathcal{F}_f)$, where $\mathcal{F}_f$ is the fibre of i^* over fi. □

We will use such homotopy classes to define notions of cohomology of a group G (or also of a groupoid) with coefficients in a G-module, which will determine the choice of the crossed complex C. Full information on the group G is encapsulated by the choice of F as a free crossed resolution of the group G.

Definition 12.1.2 Let G be a group, and M a G-module. The *nth cohomology of G with coefficients in M* is defined to be the set of homotopy classes

$$H^n(G, A) = [F_*^{st}(G), \mathbb{K}(M, n; G, 1); \phi] \tag{12.1.1}$$

where $F_*^{st}(G)$ is the standard free crossed resolution of the group G, and $\phi : F_1^{st}(G) \to G$ is the standard morphism; recall that $F_1^{st}(G)$ is the free group on the elements of G. □

Recall from Example 11.1.6 that the standard free crossed resolution of a groupoid G is:

$$\cdots \longrightarrow F_*^{st}(G)_3 \xrightarrow{\delta_3} F_*^{st}(G)_2 \xrightarrow{\delta_2} F_*^{st}(G)_1 \xrightarrow{\phi} G \longrightarrow 1$$

in which $F_n^{st}(G)$ is free on the set $(N^{\Delta}G)_n$ of composable sequences $[g_1, g_2, \ldots, g_n], g_i \in G$ of elements of G, where the base point $t[g_1, g_2, \ldots, g_n]$ is the final point tg_n of g_n. For $n \geqslant 2$ the boundary

$$\delta_n : F_n^{st}(G) \to F_{n-1}^{st}(G)$$

is given by

$$\begin{aligned} \delta_2[g, h] &= [gh]^{-1}[g][h], \\ \delta_3[g, h, k] &= [g, h]^k[h, k]^{-1}[g, hk]^{-1}[gh, k], \end{aligned}$$

and for $n \geqslant 4$

$$\begin{aligned} &\delta_n[g_1, g_2, \ldots, g_n] \\ &\quad = [g_1, \ldots, g_{n-1}]^{g_n} + (-1)^n[g_2, \ldots, g_n] + \sum_{i=1}^{n-1}(-1)^{n-i}[g_1, g_2, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_n]. \end{aligned}$$

See also the pictures in Example 9.9.7. [21]

Example 12.1.3 Let G, M be groups, $\mathcal{M} = (\chi : M \to \mathrm{Aut}M)$ be the automorphism crossed module of M and let $\kappa : F_*^{st}(G) \to \mathrm{sk}^2\,\mathcal{M}$ be a morphism of crossed complexes. Then κ is determined by its values on the free generators of $F_*^{st}(G)$ in dimensions 1 and 2 and so is equivalent to a pair of functions

$$\kappa^1 : G \to \mathrm{Aut}M, \quad \kappa^2 : G \times G \to M$$

satisfying

$$\chi\kappa^2(g, h) = \kappa^1(gh)^{-1}\kappa^1(g)\kappa^1(h) \tag{fs1}$$

$$1 = \kappa^2(g, h)^{\kappa^1(k)}\kappa^2(h, k)^{-1}\kappa^2(g, hk)^{-1}\kappa^2(gh, k) \tag{fs2}$$

for all $g, h, k \in G$. The last two conditions are (possibly with different conventions) the conditions for what is known in the literature as a *factor set*, usually with $\mathcal{M}$ being the crossed module $M \to \mathrm{Aut}M$. Then an extension of M by G may be defined by giving a product structure on $E = G \times M$ by the rule

$$(g, m)(h, n) = (gh, \kappa^2(g, h)m^{\kappa^1 h}n), \tag{pr}$$

and defining $i : M \to E, p : E \to G$ by $i(m) = (1, m), p(g, m) = g$. The condition (fs2) is then exactly the condition for the product on E to be associative. This is not surprising because of the relation of the boundary δ_2 in $F_*^{st}(G)$ to associativity.

Conversely, given an extension $1 \to M \xrightarrow{i} E \xrightarrow{p} G \to 1$ of M by G, then choose a section $s : G \to E$ of p such that $s(1) = 1$. This defines a bijection $\alpha : E \to G \times M$ by $e \mapsto (pe, i^{-1}((sp(e))^{-1}e))$. Note that $p((sp(e))^{-1}e)) = 1$ since $ps = 1_G$. The problem is to define a multiplication on $G \times M$ so that α is a morphism (and so an isomorphism). This choice of s is also equivalent to choosing $\kappa' : F_1^{st}(G) \to E$ such that $p\kappa' = \phi$. But since M is normal in E there is a morphism $\chi_E : E \to \mathrm{Aut}M$. Let $\kappa^1 = \chi_E \kappa'$. The rule (pr) then gives the 'obstruction' to the product on E being just the semidirect product. □

Exercise 12.1.4 Verify the assertions of the last example, and relate the construction to that in Proposition 12.3.1. □

12.2 Cohomology of groups as classes of crossed extensions

Let us start by introducing the idea of crossed n-fold extensions of a group G by a G-module M.

Definition 12.2.1 A *crossed n-fold extension of M by G* is a crossed resolution E of G such that $E_{n+1} = M$ as a G-module, and $E_i = 0$ for all $i > n+1$. This may also be written as an exact sequence

$$E := \quad 0 \longrightarrow M \xrightarrow{\partial_{n+1}} E_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_3} E_2 \xrightarrow{\partial_2} E_1 \xrightarrow{\phi} G \to 1.$$

This means of course that the above sequence is exact, E_1 and the part above is a (truncated) crossed complex, and ϕ maps $\mathrm{Cok}\,\partial_2$ isomorphically to G. Thus for $n = 1$ we have exactly an *abelian extension of M by G*, since by the crossed module rule CM2), M abelian is equivalent to E_1 acts on M via G. For $n = 2$ this is also called a *crossed sequence*. □

Example 12.2.2 Let G be a group and E a crossed resolution of G. Then $\mathrm{Cosk}^n E$ is the crossed n-fold extension

$$E^{n+1} := \qquad 0 \longrightarrow \mathrm{Ker}\,\partial_n \xrightarrow{i} E_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_3} E_2 \xrightarrow{\partial_2} E_1. \qquad \square$$

Definition 12.2.3 A *morphism* $E \to E'$ of crossed n-fold extensions of M by G is a morphism of crossed resolutions which induces the identity on M and on G as shown in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & E_n \arrow[r] \arrow[d] & E_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & E_1 \arrow[r, "\phi"] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\
0 \arrow[r] & M \arrow[r] & E'_n \arrow[r] & E'_{n-1} \arrow[r] & \cdots \arrow[r] & E'_1 \arrow[r, "\phi'"] & G \arrow[r] & 1
\end{tikzcd}$$

Two crossed n-fold extensions resolutions E, E' of M by G are *similar* if there is a ziz-zag (see page A.7) of morphisms $E \to E'$. This relation is an equivalence relation, and we denote the quotient set by $\mathrm{OpExt}^n(G, M)$. □

This quotient set may be given an abelian group structure called the "Baer sum" whose definition may be found for the case $n = 1$ in [ML63]. Here we merely note that for $n > 1$ there is a class which we call 0 namely the class of the crossed n-fold extension

$$0 \longrightarrow M \xrightarrow{=} \underset{n}{M} \xrightarrow{0} \cdots \qquad \longrightarrow \underset{1}{G} \xrightarrow{\phi} G \longrightarrow 1$$

Further we shall below give a bijection

$$\mathrm{OpExt}^n(G, M) \cong H^{n+1}(G, M)$$

so that this also defines an abelian group structure on the set $\mathrm{OpExt}^n(G, M)$. The class in $H^{n+1}(G, M)$ of the crossed n-fold extension will be called its *Postnikov invariant*, or k-*invariant*.

Example 12.2.4 We recall the *dihedral crossed module* $\mu : \widetilde{D}_{2n} \to D_{2n}$ from Example 5.6.9. Here D_{2n} has a presentation $\langle x, y \mid x^n, y^2, xyxy\rangle$, $\widetilde{D}_{2n}$ has presentation $\langle u, v \mid u^n, v^2, uvuv\rangle$, and $\mu(u) = x^2, \mu(v) = y$. We show this represents the trivial cohomology class in $H^3(\mathrm{Cok}\,\mu, \mathrm{Ker}\,\mu)$.

For n odd, we know that μ is an isomorphism, so the result is trivial.

For n even, we have $\mathrm{Ker}\,\mu \cong \mathrm{Cok}\,\mu \cong C_2$ and we simply construct a morphism of crossed 2-fold extensions as in the following diagram

$$\begin{tikzcd}
0 \arrow[r] & C_2 \arrow[r, "1"] \arrow[d, "="] & C_2 \arrow[r, "0"] \arrow[d, "f_2"] & C_2 \arrow[r, "1"] \arrow[d, "f_1"] & C_2 \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & C_2 \arrow[r] & D_{2n} \arrow[r, "\mu"'] & D_{2n} \arrow[r] & C_2 \arrow[r] & 0
\end{tikzcd}$$

where if c denotes the non trivial element of C_2 then $f_1(c) = x, f_2(c) = u^{n/2}$. □

Example 12.2.5 In this example, we give a crossed 2-fold extension $\mathcal{A}$ of A by G which represents 0 in its class, but to prove this we use an intermediate crossed 2-fold extension to give maps $0 \leftarrow \mathcal{B} \to \mathcal{A}$. It is not clear how to construct a direct map between 0 and $\mathcal{A}$.

Let C_n denote the cyclic group of order n (including the case $n = \infty$), written multiplicatively, with generator t. Let $\gamma_n : C_{n^2} \to C_{n^2}$ be given by $t \mapsto t^n$. This defines a crossed module, with trivial operations. This crossed module represents the trivial cohomology class in $H^3(C_n, C_n)$, in view of the morphisms of crossed 2-fold extensions

$$\begin{tikzcd}
0 \arrow[r] & C_n \arrow[r, "1"] & C_n \arrow[r, "0"] & C_n \arrow[r, "1"] & C_n \arrow[r] & 0 \\
0 \arrow[r] & C_n \arrow[r, "{(1,0)}"] \arrow[u, "1"] \arrow[d, "1"'] & C_n \times C_\infty \arrow[r, "h"] \arrow[u, "p_1"] \arrow[d, "g"'] & C_\infty \arrow[r, "\lambda"] \arrow[u, "\lambda"'] \arrow[d, "\mu"] & C_n \arrow[r] \arrow[u, "1"'] \arrow[d, "1"] & 0 \\
0 \arrow[r] & C_n \arrow[r, "i"'] & C_{n^2} \arrow[r, "\gamma_n"'] & C_{n^2} \arrow[r] & C_n \arrow[r] & 0
\end{tikzcd}$$

where $g(t, 1) = t^n, g(1, t) = t, h(t, 1) = 1, h(1, t) = t^n, i(t) = t^n$ and λ, μ are given by $t \mapsto t$. You should check that each square of this diagram is commutative, and each row is exact. □

We now sketch the relation between crossed n-fold extensions and cohomology.

Proposition 12.2.6 *For a group* G *and* G*-module* M*, a crossed* n*-fold extension* E *of* M *by* G *determines a cohomology class* $k_E \in H^{n+1}(G, M)$*. Conversely, any such class determines a crossed* n*-fold extension of* M *by* G*.*

Proof Suppose given the crossed n-fold extension as in Definition 12.2.1. Let F be a free crossed resolution of G. Since F is free and E is aspherical, there is a morphism $f : F \to E$ over the identity on G. Then f_{n+1} determines the required cohomology class.

Conversely, suppose given a morphism of G-modules $f_{n+1} : F_{n+1} \to M$ such that $f_{n+1}\delta_{n+2} = 0$. Suppose first $n > 1$. We form a crossed n-fold extension E of M by G on setting

$$E_i = \begin{cases} F_i & \text{if } i < n \\ (F_n \times M)/D & \text{if } i = n \end{cases}$$

where D is the submodule of the product module generated by the elements $(\delta_{n+1}c, f_{n+1}c)$ for all $c \in F_{n+1}$. The morphism $i : M \to E_n$ is induced by the inclusion $m \mapsto (0, m)$ into the product, and the morphism $E_n \to F_{n-1}$ is induced by $(x, m) \mapsto \delta_n x$. To prove that i is injective, suppose that $im = (\delta_{n+1}c, f_{n+1}c)$ for come $c \in F_{n+1}$. Then $\delta_{n+1}c = 0$ and so $c = \delta_{n+2}c'$ for some $c' \in F_{n+2}$. By the condition $f_{n+1}\delta_{n+2} = 0$, we have $m = 0$.

If $n = 1$, then F_1 operates nontrivially on M via $\phi : F_1 \to G$, and instead of the product in the above formula we take the semidirect product $(F_1 \ltimes M)$.

We also want to show that similar crossed n-fold extensions E, E' give rise to cohomologous invariants. For this is enough to assume there is a morphism $g : E \to E'$. Let $f : F \to E, f' : F \to E'$ be morphisms. Then $gf : F \to E'$ and so gf, f' are homotopic. Hence their corresponding k-invariants are the same.

We omit further details. □

Exercise 12.2.7 Complete the details of the above proof. In particular, prove that the construction does give a crossed n-fold extension, i.e. verify exactness. Also show that homotopic k-invariants give rise to equivalent crossed n-fold extensions. For some extra points with regard to the case $n = 1$ we refer forward to Proposition 12.3.1. □

Now we give examples of crossed modules $M \to G$ with non-trivial Postnikov invariants. Our examples have G finite cyclic and so we use the free crossed resolution of finite cyclic groups given in Example 11.1.4. Thus the point we want to stress here is that the use of crossed techniques is amenable to calculation. These examples show that success in computing a Postnikov invariant of a crossed n-fold extension of A by G is increased by having a convenient small free crossed resolution of G. Methods for the computation of such from a presentation for a finite group are given in subsection 11.2.4. [22]

Theorem 12.2.8 *For $n \geqslant 2$, we consider the following two cyclic groups $C_n = \langle t \mid t^n \rangle$, $C_{n^2} = \langle u \mid u^{n^2} \rangle$ and let $\iota : C_n \to C_{n^2}$ denote the injection sending t to u^n. Let $\nu : \iota_* C_n \to C_{n^2}$ be the crossed module of $1 : C_n \to C_n$ induced by ι from $1 : C_n \to C_n$ and let A_n denote the C_n-module which is the kernel of ν. Then $H^3(C_n, A_n)$ is cyclic of order n and has as generator the class of this induced crossed module.*

Proof Let us describe A_n as C_n-module

By Corollary 5.6.11 the abelian group $\iota_* C_n$ is the product $V = (C_n)^n$. As a C_n-module it is cyclic, with generator v, say. Write $v_i = v^{t^i}$, $i = 0, 1, \ldots, n-1$. Then each v_i is a generator of a C_n factor of V.

The kernel A_n of $\hat{\mu} : \iota_* C_n \to C_{n^2}$ is a cyclic C_n-module on the generator $a = v_0 v_1^{-1}$. Write $a_i = a^{t^i} = v_i v_{i+1}^{-1}$. As an abelian group, A_n has generators $a_0, a_1, \ldots, a_{n-1}$ with relations $a_i^n = 1,\ a_0 a_1 \ldots a_{n-1} = 1$.

We use the free crossed resolution of C_n described in Example 11.1.4:

$$F_* = \cdots \to (C_\infty)^n \xrightarrow{N} (C_\infty)^n \xrightarrow{M} (C_\infty)^n \xrightarrow{\phi} C_\infty$$

where $\phi(w_i) = w^n$.

We define a morphism

$$\begin{tikzcd}
(C_\infty)^n \arrow[r, "N"] \arrow[d, "0"'] & (C_\infty)^n \arrow[r, "M"] \arrow[d, "f_3"'] & (C_\infty)^n \arrow[r, "\phi"] \arrow[d, "f_2"] & C_\infty \arrow[d, "f_1"] \\
0 \arrow[r] & A_n \arrow[r] & (C_n)^n \arrow[r, "\nu"'] & C_{n^2}
\end{tikzcd}$$

as follows:

1. f_1 maps w to u, inducing the identity on sC_n
2. f_2 maps the module generator w_0 of F_2 to $v = v_0$.
3. f_3 maps the module generator w_0 of F_3 to a_0.

The morphisms f_1 and the operator morphism f_2 over f_1 are defined completely by these conditions.

The abelian group of operator morphisms $g : (C_\infty)^n \to A_n$ over f_1 may be identified with A_n under $g \mapsto g(w_0)$. Under this identification, the boundaries δ_4, δ_3 are transformed respectively to 0 and to $a_i \mapsto a_i(a_i^t)^{-1}$. So the 3-dimensional cohomology group is the group A_n with a_i identified with $a_{i+1}, i = 0, \dots, n-1$. This cohomology group is therefore isomorphic to C_n, and a generator is the class of the above cocycle f_3. □

The following is another example of a determination of a non trivial cohomology class by a crossed module. The method of proof is similar to that of Theorem 12.2.8, and is left to the reader.

Example 12.2.9 Let n be even. Let C'_n denote the C_n-module which is C_n as an abelian group but in which the generator t of the group C_n acts on the generator t' of C'_n by sending it to its inverse. For $n = 2$, this gives the trivial module. Then $H^3(C_n, C'_n) \cong C_2$ and a generator of this group is represented by the crossed module $\nu_n : C_n \times C_n \to C_{n^2}$, with generators t_0, t_1, u say, and where $\nu_n t_0 = \nu_n t_1 = u^n$. Here $u \in C_{n^2}$ operates by switching t_0, t_1. However, it is not clear if this crossed module can be an induced crossed module for $n > 2$. □

Remark 12.2.10 The reason for the success of the previous determinations is that we have a convenient small free crossed resolution of the cyclic group C_n. □

Let us extract some more applications of this crossed resolution. We study the induced crossed module through the inclusion of a normal subgroup $\iota : P \lhd Q$ when the crossed module $\mu : M \to P$ is also the inclusion $M \lhd P$ of a normal subgroup such that M is also normal in Q. Then Theorem 5.8.12 shows that the induced crossed module may be described as

$$\zeta : M \times (M^{ab} \otimes I(Q/P)) \to Q$$

where for $m, n \in M,\ x \in I(Q/P)$, and $I(Q/P)$ denotes the augmentation ideal of the quotient group Q/P. The map ζ is defined by $\zeta(m, [n] \otimes x) = m \in Q$ and the action of $q \in Q$ is given by

$$(m, [n] \otimes x)^q = (m^q, [m^q] \otimes (\bar{q} - 1) + [n^q] \otimes x\overline{q})$$

where $\bar{q}$ denotes the image of q in Q/P.

Remark 12.2.11 It might be imagined from this that the Postnikov invariant of this crossed module is trivial, since one could argue that the projection

$$pr_2 : P \times P^{ab} \otimes I(Q/P) \to P^{ab} \otimes I(Q/P)$$

should give a morphism from $\iota_* P$ to the crossed module $0 : P^{ab} \otimes I(Q/P) \to Q/P$, which represents 0 in the cohomology group $H^3(Q/P, P^{ab} \otimes I(Q/P))$. However, the projection pr_2 is a P-morphism, but is not in general a Q-morphism, as the above results show. In fact, in the next Theorem we give a precise description of the Postnikov invariant of $\iota_* P$ when Q/P is cyclic of order n. This generalises the result for the case $P = C_n,\ Q = C_{n^2}$ in 12.2.8. □

Theorem 12.2.12 *Let P be a normal subgroup of Q such that P/Q is isomorphic to C_n, the cyclic group of order n. Let t be an element of Q which maps to the generator $\bar{t}$ of C_n under the quotient map. Then the first Postnikov invariant k^3 of the induced crossed module of the inclusion $BP \to BQ$ lies in a third cohomology group*

$$H^3(C_n, P^{ab} \otimes I(C_n)).$$

This group is isomorphic to

$$P^{ab} \otimes C_n,$$

and under this isomorphism the element k^3 is taken to the element

$$[t^n] \otimes \bar{t}.$$

Proof We have to determine the cohomology class represented by the crossed module

$$\xi : P \times P^{ab} \otimes I(C_n) \to Q.$$

Let $A = P^{ab} \otimes I(C_n)$. As in 12.2.8 for the case $Q = C_{n^2},\ P = C_n$, we consider the diagram

$$\begin{tikzcd}
\mathbb{Z}C_n \arrow[r, "N"] \arrow[d, "0"'] & \mathbb{Z}C_n \arrow[r, "M"] \arrow[d, "f_3"'] & \mathbb{Z}C_n \arrow[r, "\phi"] \arrow[d, "f_2"] & C_\infty \arrow[d, "f_1"] \\
0 \arrow[r] & A \arrow[r, "i"'] & P \times A \arrow[r, "\nu_n"'] & Q.
\end{tikzcd}$$

Here the top row is the beginning of a free crossed resolution of C_n. The free C_n-modules $\mathbb{Z}[C_n]$ have generators y_4, y_3, y_2 respectively, C_∞ has generator y_1 and $\phi(y_2) = y_1^n,\ M(y_3) = y_2.(\bar{t} - 1)$ (here C_∞ operates on each $\mathbb{Z}C_n$ via the morphism to C_n); $N(y_4) = y_3.(1 + \bar{t} + \bar{t}^2 + \cdots + \bar{t}^{n-1})$. Further, we define $f_1(y_1) = t,\ f_2(y_2) = (t^n, 0),\ f_3(y_3) = [t^n] \otimes (\bar{t} - 1)$, and $i(a) = (1, a),\ a \in A$. Thus the diagram gives a morphism of crossed complexes, and the cohomology class of the cocycle f_3 is the Postnikov invariant of the crossed module.

As in 12.2.8 since $\mathbb{Z}[C_n]$ is a free C_n-module on one generator, the cohomology group $H^3(C_n, A)$ is isomorphic to the homology group of the sequence

$$A \xleftarrow{M} A \xleftarrow{N} A$$

where N is multiplication by $1 + \bar{t} + \bar{t}^2 + \cdots + \bar{t}^{n-1}$ and M is multiplication by $\bar{t} - 1$. It follows that $N = 0$, and it is easy to check that $I(C_n)/I(C_n)(\bar{t} - 1)$ is a cyclic group of order n generated by $\bar{t} - 1$. The cocycle f_3 determines the element $f_3(y_3) = [t^n] \otimes (\bar{t} - 1)$ of A, and the result follows. □

12.3 Dimension 2 cohomology with coefficients in a crossed module and extension theory

This section gives an account of the theory of nonabelian extensions of a group M by a group G, that is the aim is to classify extensions $1 \to M \to E \to G \to 1$. An immediate difference between this and the abelian case is that we do not get an action of G on M from such an extension. We have already mentioned in Example 12.1.3 the 'factor sets' which arise. It turns out that it is convenient to be more specific about how the actions arise by using extensions of the type of a crossed module.[23]

We first give a formulation as a kind of pushout of the construction of nonabelian extensions of groups. This formulation is convenient for the development of the theory and the proof of theorems. Our

Proposition 12.3.1 *Suppose given a crossed sequence*

$$0 \to \pi \xrightarrow{i} F_2 \xrightarrow{\delta} F_1 \xrightarrow{\phi} G \to 1$$

and a morphism of groups $k^2 : F_2 \to M$*, together with an action of* F_1 *on* M *such that*

AP1) *if* $m \in M$, $r \in F_2$ *then* $(k^2 r)^{-1} m (k^2 r) = m^{\delta r}$;

AP2) *if* $r \in F_2,\ x \in F_1$ *then* $k^2(r^x) = (k^2 r)^x$.

Then there is a commutative square

$$\begin{tikzcd} F_2 \arrow[r, "\delta"] \arrow[d, "k^2"'] & F_1 \arrow[d, "k^1"] \\ M \arrow[r, "i_k"'] & E_k \end{tikzcd}$$

such that

CP1) $i_k : M \to E_k$ *is a crossed module;*

CP2) *if* $m \in M,\ x \in F_1$ *then* $m^x = m^{k^1 x}$;

CP3) *the square is universal for properties CP1), CP2);*

CP4) *there is an exact sequence*

$$M \xrightarrow{i_k} E_k \xrightarrow{\psi} G \to 1;$$

CP5) *the morphism* i_k *is injective if and only if* $k^2(i\pi) = 1$.

Proof Let C_k be the semidirect product group $F_1 \ltimes M$ formed with the given action of F_1 on M. Let C_k act on F_2 via the projection to F_1 and the action of F_1 on F_2.

We first prove that the function $\xi : F_2 \to C_k,\ r \mapsto (\delta r, (k^2 r)^{-1})$ is a morphism. Let $r, s \in F_2$. Then by the semidirect product rule

$$\begin{aligned}\xi(r)\xi(s) &= (\delta r\, \delta s, k^2((r^{-1})^{\delta s}\, s^{-1}) \\ &= (\delta r\, \delta s, k^2(s^{-1} r^{-1})) \\ &= \xi(rs).\end{aligned}$$

Next we prove ξ preserves the action. Let $r \in F_2,\ (x, m) \in C_k$. Then

$$\begin{aligned}(x, m)^{-1}\xi(r)(x, m) &= (x^{-1}, (m^{-1})^{x^{-1}})(\delta r, (k^2 r)^{-1})(x, m) \\ &= (x^{-1}(\delta r)x, (m^{-1})^{x^{-1}(\delta r)x}(k^2 r^{-1})^x m) \\ &= (x^{-1}(\delta r)x, (k^2 r^x)^{-1} m^{-1}(k^2 r^x)(k^2(r^{-1}))^x m) \\ &= \xi(r^x).\end{aligned}$$

Finally we prove easily the second crossed module rule:

$$\begin{aligned} r^{-1}sr &= s^{\delta r} && \text{by the crossed module rule for } \delta \\ &= s^{(\delta r, k^2 r^{-1})} && \text{by definition of the action of } C_k \\ &= s^{\xi r} && \text{as required.} \end{aligned}$$

Let $E_k = \operatorname{Cok}\xi$, and let $[x, m]$ denote the image in E_k of $(x, m) \in C_k$. Let $i_k : M \to E_k$ be given by $m \mapsto [1, m]$. Then the formula $m^{[x,n]} = n^{-1}m^x n$ gives, by AP1), a well defined action of E_k on M which is easily shown to make i_k a crossed module.

Let $\psi : E_k \to G$ be given by $[x, m] \mapsto \phi(x)$. Then ψ is well defined and is the cokernel of i_k.

Suppose given a morphism of crossed modules

$$\begin{tikzcd} F_2 \arrow[r, "\delta"] \arrow[d, "k^2"'] & F_1 \arrow[d, "l"] \\ M \arrow[r, "\alpha"'] & Q \end{tikzcd}$$

such that $m^x = m^{lx}, x \in F_1, m \in M$. Suppose $\omega : E_k \to Q$ determines a morphism of crossed modules such that $\omega l = k^1,\ \omega i_k = \alpha$. Since $[x, m] = [x, 1][1, m]$, we easily check that $\omega[x, m] = (lx)(\alpha m)$. So such an ω is unique. On the other hand, we easily check this does define a morphism as required.

The morphism ψ of CP4) is defined by $\psi[x, m] = \phi x$. This gives the exact sequence.

Finally $i_k m = 1$ for all $m \in M$ is equivalent to $(1, m) = (\delta r, k^2 r^{-1})$ for some $r \in F_2$; this easily proves CP5). $\square$

Definition 12.3.2 Let $\mathcal{M}$ denote the crossed module $\mu : M \to P$. An *extension* (i, p, σ) *of type* $\mathcal{M}$ of the group M by the group G is:

(i) an exact sequence of groups

$$1 \longrightarrow M \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

so that E operates on M by conjugation, and $i : M \to E$ is hence a crossed module; and

(ii) a morphism of crossed modules

$$\begin{tikzcd} 1 \arrow[r] & M \arrow[r, "i"] \arrow[d, equal] & E \arrow[d, "\sigma"] \\ & M \arrow[r, "\mu"] & P \end{tikzcd}$$

i.e. $\sigma i = \mu$ and $m^e = m^{\sigma e}$, for all $m \in M$, $e \in E$. Thus the action of E on M is also via σ.

We shall write such an extension as

$$1 \longrightarrow M \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1, \qquad E \xrightarrow{\sigma} P.$$

Two such extensions of type $\mathcal{M}$

$$1 \longrightarrow M \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1, \qquad E \xrightarrow{\sigma} P.$$

$$1 \longrightarrow M \xrightarrow{i'} E' \xrightarrow{p'} G \longrightarrow 1, \qquad E' \xrightarrow{\sigma'} P.$$

are said to be *equivalent* if there is a morphism of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & M \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\phi"'] & G \arrow[r] \arrow[d, equal] & 1, & E \arrow[r, "\sigma"] \arrow[d, "\phi"'] & P \arrow[d, equal] \\
1 \arrow[r] & M \arrow[r] & E' \arrow[r] & G \arrow[r] & 1, & E' \arrow[r, "\sigma'"] & P.
\end{tikzcd}$$

such that the right hand square also commutes. Of course in this case ϕ is an isomorphism, by the 5-lemma, and hence equivalence of extensions is an equivalence relation.

We denote by

$$\mathrm{OpExt}_{\mathcal{M}}(G, M)$$

the set of equivalence classes of all extensions of type $\mathcal{M}$ of M by G. □

The usual theory of extensions of a group M by a group G considers extensions of the type of the crossed module $\chi_M : M \to \mathrm{Aut}M$. The advantages of replacing this by a general crossed module are first that the group $\mathrm{Aut}M$ is not a functor of M, so that the relevant cohomology theory in terms of χ_M appears to have no morphisms of coefficients, and second, that the more general case occurs geometrically. [24]

The heart of the proof of the following theorem is in Proposition 12.3.1.

Theorem 12.3.3 *Suppose given a crossed sequence*

$$0 \to \pi \xrightarrow{i} F_2 \xrightarrow{\delta} F_1 \xrightarrow{\phi} G \to 1$$

and a crossed module $\mathcal{M} = (\mu : M \to P)$. Let $\mathcal{F}$ denote the crossed module $\delta : F_2 \to F_1$. Let $[\mathcal{F}, \mathcal{M}]^0$ denote the set of homotopy classes of morphisms $\mathbf{k} = (k^2, k^1) : \mathcal{F} \to \mathcal{M}$ of crossed modules, such that $k^2(i\pi) = 1$. Then there is a natural injection

$$\mathbf{E} : [\mathcal{F}, \mathcal{M}]^0 \to \mathrm{OpExt}_{\mathcal{M}}(G, M)$$

sending the class of a morphism $\mathbf{k}$ to the extension

$$1 \to M \to E(\mathbf{k}) \to G \to 1$$

where $E(\mathbf{k})$ is the quotient of the semidirect product group $F_1 \ltimes M$, in which F_1 acts on M via P. The function $\mathbf{E}$ is surjective if F_1 is a free group.

Exercise 12.3.4 Prove the last theorem, including the analysis of the equivalence of extensions.

Example 12.3.5 In the applications of Theorem 12.3.3 we would take a free crossed resolution $F(G)$ of a group G and let $\delta : F_2 \to F_1$ be the crossed module $\delta_2 : F_2(G) \to F_1(G)$ with $\phi : F_1(G) \to G$ given by the resolution. Then π is the module of identities among relations for the presentation $\langle X \mid R\rangle$ of G determined by the free bases X, R of $F_1(G), F_2(G)$ respectively. □

Example 12.3.6 Extensions by a cyclic group Let C_n denote the cyclic group of order n, written multiplicatively, and generated by an element t, so that the infinite cyclic group is C_∞. The presentation $\langle t : t^n\rangle$ for C_n gives rise to the free crossed module $\delta : (C_\infty)^n \to C_\infty$ where $(C_\infty)^n$ is generated as a group by $t_0, t_1, \ldots, t_{n-1}$ and as a crossed C_∞-module by t_0; here $t \in C_\infty$ operates on $(C_\infty)^n$ by

$(t_i)^t = t_{i+1}, i = 0, \ldots, n-1 \pmod n$; and for all i, $\delta(t_i) = t^n$. This description of this free crossed module is given in [**?**]. A morphism

$$\begin{tikzcd} (C_\infty)^n \arrow[r, "\delta"] \arrow[d, "k^2"'] & C_\infty \arrow[d, "k^1"] \\ A \arrow[r, "\alpha"'] & Q \end{tikzcd}$$

of crossed modules is thus specified by elements $q = k^1(t) \in Q,\ a = k^2(t_0) \in A$ such that $\alpha a = q^n$. Further, Ker δ, the module of identities for the presentation, is the submodule generated by the element $t_0 t_1^{-1}$. Hence the condition $k^2(\operatorname{Ker}\delta) = 0$ is equivalent to $k^2(t_0 t_1^{-1}) = a(a^q)^{-1} = 1$, that is $a = a^q$. An equivalence $(a' : q') \simeq (a : q)$ of such data is given by a derivation $h : C_\infty \to A$, and so by an element $b = h(t) \in A$, such that $q' = q(\alpha b)$ and

$$a' = ah(t^n) = ab^{q^{n-1}} b^{q^{n-2}} \ldots b^2 b.$$

This result is given in [**?**] ChIII, section 7. The extension group E determined by the data $(a : q)$ is the quotient of $C_\infty \ltimes A$ by the element (t^n, a^{-1}). □

Example 12.3.7 The trefoil group Let G be the trefoil group with presentation $\langle x, y : x^2 = y^3 \rangle$. This is a 1-relator presentation whose relator is not a proper power, and so there are no identities among the relations, as proved by Lyndon in [**?**] (see [**?**] for more information). Therefore the extension data of A by G of type $\alpha : A \to Q$ is given by elements $q_x, q_y \in Q, a_r \in A$, such that $\alpha a_r = (q_x)^2 (q_y)^{-3}$. An equivalence $(a'_r : q'_x, q'_y) \simeq (a_r : q_x, q_y)$ of such data is given by elements $b, c \in A$ such that $q'_x = q_x(\alpha b), q'_y = (q_y)(\alpha c)$ and $a'_r = a_r h(x^2 y^{-3})$ where h is the derivation $F\{x, y\} \to A$ given by $hx = b, hy = c$. Thus

$$\begin{aligned} h(x^2y^{-3}) &= h(x^2)^{q_y^{-3}} h(y^{-3}) \\ &= (b^{q_x} b)^{q_y^{-3}} (c^{-1})^{q_y^{-3}} (c^{-1})^{q_y^{-2}} (c^{-1})^{q_y^{-1}}. \end{aligned}$$

The group E determined by the extension data $(a_r : q_y, q_y)$ is the quotient of the semidirect product $F\{x, y\} \ltimes A$ by the element $(x^2 y^{-3}, a_r^{-1})$. Here $F\{x, y\}$ acts on A by $a^x = a^{q_x}, a^y = a^{q_y}, a \in A$. □

Example 12.3.8 Extensions by a product The tensor product of crossed complexes as defined in [**?**] may be used to describe extensions by a product $G \times H$ of groups. Let $F_*(G),\ F_*(H)$ be free crossed resolutions of groups G, H respectively. The tensor product $F_*(G) \otimes F_*(H)$ is then a free crossed resolution of $G \times H$. A proof of asphericity will be given in Corollary 15.8.1. It is proved in Theorem 9.6.3 that the tensor product of free crossed complexes is free on the tensor product of the free generators, so that in particular $F_*(G) \otimes F_*(H)$ is freely generated as a crossed complex by $a_i \otimes b_j$, where the a_i, b_j run over sets of free generators of $F_*(G),\ F_*(H)$ respectively. Thus it is easy to specify morphisms from $F_*(G) \otimes F_*(H)$ to a crossed module or crossed complex. Further, generators for the module of identities for a presentation of the product $G \times H$ are the images under δ_3 of free generators of $(F_*(G) \otimes F_*(H))_3$, by asphericity. Such free generators are of the form $a_3 \otimes *, * \otimes b_3, a_2 \otimes b_1, a_1 \otimes b_2$ where a_i, b_j run over free generators of $F_i(G), F_j(H)$ respectively.

This implies the following. Let $\langle X; R \rangle, \langle Y, S \rangle$ be presentations of G, H respectively, and let I, J be generating sets for the modules of identities for these presentations. Then a free crossed resolution $F_*(G)$ corresponding to X, R, I is in dimensions $\leqslant 3$ of the form

$$C_3(I) \xrightarrow{\delta_3} F_C(R) \xrightarrow{\delta_2} F(X)$$

where $C_3(I)$ is the free G-module on I, and similarly for $F_*(H)$. Thus in dimensions $\leqslant 3$, $F_*(G) \otimes F_*(H)$ has generators as follows, where for Z any set, $\bar{Z}$ denotes a set of formal generators $\bar{z}, z \in Z$:

- *dimension* 1: X, Y,
- *dimension* 2: $\bar{R}, \bar{S}, \{x \otimes y : x \in X, y \in Y\}$,
- *dimension* 3: $\bar{I}, \bar{J}, \{x \otimes \bar{s}, \bar{r} \otimes y : x \in X, y \in Y, r \in R, s \in S\}$.

The boundaries are given by:

$$\delta_2\bar{r} = r,\ \delta_2\bar{s} = s,\ \delta_2(x \otimes y) = y^{-1}x^{-1}yx,$$

$$\delta_3\bar{i} = i, \delta_3\bar{j} = j, \delta_3(x \otimes \bar{s}) = \bar{s}^{-1}\bar{s}^{x}(x \otimes s)^{-1},$$

$$\delta_3(\bar{r} \otimes y) = (r \otimes y)\bar{r}^{-1}\bar{r}^{y}.$$

Now the elements $x \otimes s, r \otimes y$ have to be expressed in terms of the free generators in dimension 2. This is done by using the biderivation rules

$$x \otimes uv = (x \otimes u)^{v}\,(x \otimes v),$$

$$\omega z \otimes y = (z \otimes y)(\omega \otimes y)^{z},$$

which are part of the crossed complex structure of the tensor product.

Note that in this example, we obtain nice generators of the module of identities for the product, by applying the boundary to free generators in dimension 3 of a crossed resolution. [25] □

12.4 The exact sequences of a fibration of crossed complexes

A fibration $p : E \to D$ of crossed complexes yields a family of exact sequences involving the H_n, π_1 and π_0, as follows. Let $x \in E_0$ and let $\mathcal{F}_x = p^{-1}(px)$ be the sub crossed complex of E of all elements of E_0 which map by p to x and otherwise all elements of some E_n which map down by p to the identity at px.

Theorem 12.4.1 *There is an exact sequence*

$$\cdots \to H_n(\mathcal{F}_x, x) \xrightarrow{i_n} H_n(E, x) \xrightarrow{p_n} H_n(B, px) \xrightarrow{\partial_n} \cdots$$
$$\cdots \to \pi_1(\mathcal{F}_x, x) \xrightarrow{i_1} \pi_1(E, x) \xrightarrow{p_1} \pi_1(B, px) \xrightarrow{\partial_1} \pi_0(\mathcal{F}_x) \xrightarrow{i_*} \pi_0(E) \xrightarrow{p_*} \pi_0(B).$$

Here the terms of the sequence are all groups, except the last three which are sets with base points the classes $x_{\mathcal{F}}, x_E, x_B$ of $x,\ x,\ px$ respectively.

(i) *There is an operation of the group $\pi_1(E, x)$ on the group $\pi_1(\mathcal{F}_x, x)$ making the morphism*

$$i_1 : \pi_1(\mathcal{F}_x, x) \to \pi_1(E, x)$$

into a crossed module.

(ii) *There is an operation of the group $\pi_1(B, px)$ on the set $\pi_0(\mathcal{F}_x)$ such that the boundary*

$$\pi_1(B, px) \xrightarrow{\partial_1} \pi_0(\mathcal{F}_x)$$

is given by $\partial_1(\alpha) = \alpha \cdot x_F$.

Further we have additional exactness at the bottom end as follows:

(a) $\partial_1\alpha = \partial_1\beta$ *if and only if there is a* $\gamma \in E(x)$ *such that* $p_1\gamma = -\beta + \alpha$*;*

(b) *if* $\bar{u}$ *denotes the component in* $\mathcal{F}_x$ *of an object* u *of* $\mathcal{F}_x$*, then* $i_*\bar{u} = i_*\bar{v}$ *if and only if there is an* $\alpha \in B(y)$ *such that* $\alpha_\#\bar{u} = \bar{v}$*;*

(c) *if* $\hat{y}$ *denotes the component of* y *in* B *then*

$$i_*(\pi_0\mathcal{F}_x) = p_*^{-1}(\hat{y}).$$

Proof The proof of this theorem is a development of the part of the theorem which deals with fibrations of groupoids and which is given for example in [Bro06, 7.2.9]. See also [Bro70]. We leave the details as an exercise. □

Corollary 12.4.2 *Under the conditions of the theorem, the set* $\pi_0(E)$ *is the disjoint union of the sets* $p_*^{-1}(\hat{y})$ *for each component ..... (to be completed).*

Remark 12.4.3 The power of this result comes when we apply it in the next section to fibrations of internal homs $\mathsf{CRS}(F, E) \to \mathsf{CRS}(F, B)$ using Proposition 10.5.6. [26]

12.5 Homotopy classification of morphisms

In this section we analyse the set $[F, C]_*$ of pointed homotopy classes of morphisms from a reduced crossed complex F to a reduced crossed complex C for particular examples of C. Usually F will be free. We stick to the reduced and pointed case as in this case it is easier to relate the results to classical theorems, but the other case can be treated by the same methods.

The cases we are thinking of are:

- $F = \Pi X_*$ for X_* the skeletal filtration of a CW-complex Z; and
- $F = F^{st}(G)$, the standard free crossed resolution of a group or groupoid G.

In the first case we think of $[F, C]_*$ as a kind of 'nonabelian cohomology set of the space X with coefficients in the crossed complex C'. In the second case it is a kind of nonabelian cohomology of the group or groupoid G. In either case, our analysis of this set works by using fibrations of the 'coefficients', i.e. fibrations of crossed complexes. It turns out that this leads to some nice formulations of or generalisations of classical results.

We shall write $\mathbb{K}(Q, 1)$ for the crossed complex which is the group or groupoid Q in dimension 1 and is otherwise trivial. Later we will write $\mathbb{K}(M, n; Q, 1)$ for the crossed complex which is Q in dimension 1, the Q-module M (with the given action of Q) in dimension $n \geqslant 2$, and is otherwise trivial; in particular, if $n = 2$ then the boundary $M \to Q$ is assumed trivial.

We start with the simplest case.

Proposition 12.5.1 *For any* $n \geqslant 2$*, reduced crossed complex* F *with fundamental group* Φ*, and any group* Q*, there is a natural bijection*

$$[F, \mathbb{K}(Q, 1)]_* \cong \mathrm{Hom}(\Phi, Q).$$

Proof This follows easily from the definitions. □

Now we take a more complicated case. First a definition.

Definition 12.5.2 Let F, C be crossed complexes, let A be a subcomplex of F, with inclusion $i : A \to F$ and let $f : A \to C$ be a morphism. We write $[F, C; f]$ for the set of homotopy classes rel A of morphisms $F \to C$ which extend f. We write similarly $[F, C; f]_*$ for the pointed homotopy classes in the case A, F, C, i are pointed.

Theorem 12.5.3 *Let F be a reduced free crossed complex and let $\Phi = \pi_1 F$. Then $[F, \mathbb{K}(M, n; Q, 1)]_*$ is the disjoint union of sets $[F, \mathbb{K}(M, n; Q, 1) : \theta\phi]_*$ one for each morphism $\theta : \Phi \to Q$, namely those homotopy classes inducing θ. Further, the morphisms $F \to \mathbb{K}(M, n; Q, 1)$ inducing $\theta : \Phi \to Q$ may be given the structure of abelian group which is inherited by homotopy classes.*

Proof The morphism $q : \mathbb{K}(M, n; Q, 1) \to \mathbb{K}(Q, 1)$ which is the identity in dimension 1 and 0 elsewhere is a fibration inducing a fibration $q_* : \mathsf{CRS}_*(F, \mathbb{K}(M, n; Q, 1)) \to \mathsf{CRS}_*(F, \mathbb{K}(Q, 1))$. The induced map on π_0 is surjective since every morphism $f : F \to \mathbb{K}(Q, 1)$ may be lifted by 0 to a morphism $F \to \mathbb{K}(M, n; Q, 1)$. By Proposition **??**, $\pi_0 \mathsf{CRS}_*(F, \mathbb{K}(Q, 1)) \cong \mathrm{Hom}(\Phi, Q)$. So we can write, using the exact sequence of Theorem 12.4.1,

$$\pi_0 \mathsf{CRS}_*(F, \mathbb{K}(M, n; Q, 1)) \cong \bigsqcup_{\theta : \Phi \to Q} [F, \mathbb{K}(M, n; Q, 1); \theta\phi]_*.$$

The abelian group structure induced by addition of values in dimension n on the set of morphisms $F \to \mathbb{K}(M, n; Q, 1)$ which extend $\theta\phi$ is clear from the diagram

$$\begin{tikzcd}
\cdots & F_{n+1} \arrow[r] \arrow[d] & F_n \arrow[r] \arrow[d] & F_{n-1} \arrow[r] \arrow[d] & \cdots & {} \arrow[r] & F_2 \arrow[r] \arrow[d] & F_1 \arrow[r, "\phi"] \arrow[d] & \Phi \arrow[d] \arrow[dl, "\theta"'] \\
\cdots & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots & {} \arrow[r] & 0 \arrow[r] & Q \arrow[r, equal] & Q
\end{tikzcd} \tag{12.5.1}$$

as is also the fact that this addition passes to homotopy classes rel $\theta\phi$. □

Remark 12.5.4 Thus the situation for crossed complexes is not quite like that for chain complexes with a group or groupoid of operators. In that category, two morphisms $C \to D$ over the same operator morphism $Q \to H$ do indeed have a sum by addition of values. □

Definition 12.5.5 We write $H^n_{\theta\phi}(F, M)$ for $[F, \mathbb{K}(M, n; Q, 1); \theta\phi]$, and call this abelian group the n*th cohomology over* $\theta\phi$ of F with coefficients in M. Thus $[F, \mathbb{K}(M, n; Q, 1)]_*$ is the disjoint union of the abelian groups $H^n_{\theta\phi}(F, M)$ for all morphisms $\theta : \Phi \to Q$. When convenient and clear, we abbreviate $\theta\phi$ to θ. □

Remark 12.5.6 In the case $F = \Pi X_*$ for a CW-filtration X_*, then we recover the cellular cohomology o X, while in the case $F = F^{st}(G)$ for a group or groupoid G, then we recover the usual notions of cohomology of G. □

A generalisation of the previous result is as follows.

Example 12.5.7 Let C be a reduced crossed complex such that $C_1 = Q$, and $\delta_2 = 0 : C_2 \to C_1$. Let F be a free crossed complex. Then $\mathsf{Crs}_*(F, C)$ and $[F, C]_{\theta\phi}$ may be given the structure of abelian group by addition of values. □

12.6 Generalisation of abstract kernel theory

We now use the previous results to analyse $[F, C]_*$ in another interesting case, namely when C has no homology between 1 and n, and is trivial above n.

Theorem 12.6.1 *Let $n \geqslant 2$ and let F, C be reduced crossed complexes such that F is free, C is n-aspherical, and $C_i = 0$ for $i > n$. Let $\Phi = \pi_1(F), Q = \pi_1 C, M = \operatorname{Ker} \delta_n : C_n \to C_{n-1}$. Let $\theta : \Phi \to Q$ be a morphism of groups. Then there is defined an element $k_\theta \in H^{n+1}_{\theta\phi}(F, M)$, called the* obstruction class *of θ, such that the vanishing of k_θ is necessary and sufficient for θ to be realised by a morphism $F \to C$.*

If $k_\theta = 0$, then the set $[F, C; \theta\phi]$ of homotopy classes of morphisms $F \to C$ realising $\theta\phi$ is bijective with $H^n_{\theta\phi}(F, M)$.

Proof Consider the morphisms of crossed complexes $C \xrightarrow{j} \xi C \xrightarrow{p} \zeta C$ as shown in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & C_n \arrow[r, "\delta_n"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_2 \arrow[r] \arrow[d] & C_1 \arrow[d] & & C \arrow[d, "j"] \\
\cdots \arrow[r] & M \arrow[r] \arrow[d] & C_n \arrow[r, "\delta_n"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & C_2 \arrow[r] \arrow[d] & C_1 \arrow[d] & & \xi C \arrow[d, "p"] \\
\cdots \arrow[r] & M \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & Q & & \zeta C
\end{tikzcd}$$

Then ξC is aspherical, $\zeta C = \mathbb{K}(M, n+1; Q, 1)$, and $p : \xi C \to \zeta C$ is a fibration of crossed complexes.

Since F is a free reduced crossed complex, we have an induced fibration of crossed complexes

$$p_* : \mathsf{CRS}_*(F, \xi C) \to \mathsf{CRS}_*(F, \zeta C). \tag{12.6.1}$$

On applying π_0 to this we get, considering previous identifications, a map of sets

$$p_* : \operatorname{Hom}(\Phi, Q) \to \bigsqcup_{\theta \in \operatorname{Hom}(\Phi, Q)} H^{n+1}_{\theta\phi}(F, M). \tag{12.6.2}$$

Lemma 12.6.2 *A morphism $\theta : \Phi \to Q$ maps to 0 in $H^{n+1}_{\theta\phi}(F, M)$ if and only if θ is induced by a morphism $F \to C$.*

Proof Suppose θ is induced by a morphism $f : F \to C$. Then f factors through pj and is therefore 0 in $H^{n+1}_{\theta\phi}(F, M)$.

Suppose conversely that θ determines 0 in $H^{n+1}_{\theta\phi}(F, M)$. We know that θ is induced by a morphism $f' : F \to \xi C$. Then pf is homotopic to 0 and so by the fibration condition f' is homotopic to f'' such that $pf'' = 0$. Hence f'' determines $f : F \to C$ such that $jf = f''$. Then f also induces θ. This proves the lemma. □

Let $\mathcal{F}(f)$ denote the fibre of p_* over pf. Then we have an exact sequence

$$\to \pi_1(\mathsf{CRS}_*(F, \xi C), f) \to \pi_1(\mathsf{CRS}_*(F, \zeta C), pf) \to \pi_0 \mathcal{F}(f) \to \pi_0 \mathsf{CRS}_*(F, \xi C) \to \pi_0 \mathsf{CRS}_*(F, \zeta C).$$

By Proposition **??**, $\pi_1(\mathsf{CRS}_*(F, \xi C), f) = 0$, so the above sequence translates to

$$0 \to H^n_{\theta\phi}(F, M) \to [F, C; \theta\phi] \to \operatorname{Hom}(\Phi, Q).$$

Further we have a free action of the abelian group $H^n_{\theta\phi}(F, A)$ on the set $[F, C]_{\theta\phi}$. This completes the proof of the theorem. □

This result generalises the classical theory of extensions of groups and Q-kernels. To apply the theory to that case, the crossed complex F is taken to be a free crossed resolution of the group G. If F is the standard free crossed resolution of G, then the relation with factor systems is shown in [BP96]. The advantage of this approach is that it is clear that the standard free crossed resolution may be replaced by any free crossed resolution of G, and in many cases it is possible to construct small such resolutions. The we may get a finite description of the classes of extensions.

12.7 Homotopy classification of maps of spaces

In order to apply these results we need to know useful circumstances when a given space Y is of the homotopy type of, or can be replaced by, BC , for some crossed complex C. An important result for this is the following:

Theorem 12.7.1 *Let Y be a CW-complex with skeletal filtration Y_*. Then there is a map $q : Y \to B\Pi Y_*$ with 2-connected homotopy fibre. Further the fibre is n-connected if Y is connected and $\pi_i Y = 0$ for $1 < i < n$.*

Corollary 12.7.2 *Let X be a connected CW-complex of dimension $\leqslant n$, and let Y be a connected CW-complex such that $\pi_i Y = 0$ for $1 < i < n$. Then the map $q : Y \to B\Pi Y_*$ induces a bijection*

$$[X, Y]_* \cong [X, B\Pi Y_*]_*.$$

Corollary 12.7.3 (Hopf classification theorem) *Let X be a connected CW-complex of dimension $\leqslant n$. Then there is a bijection*

$$[X, S^n]_* \cong H^n(X, \mathbb{Z}).$$

We can also interpret the more general homotopy classification theorem involving an obstruction.

Theorem 12.7.4 *Let X be a connected CW-complex of dimension $\leqslant n$, and let Y be a connected CW-complex such that $\pi_i Y = 0$ for $1 < i < n$. Let $\Phi = \pi_1 X$, $G = \pi_1 Y$, and let $M = \pi_n Y$ considered as a G-module. Then a morphism $\theta : \Phi \to G$ determines an element $k_\theta \in H^n_{\theta\phi}(X, M)$ whose vanishing is necessary and sufficient for θ to be realised by a map $f : X \to Y$. If θ is realisable, then the homotopy classes of maps realising θ are bijective with $H^{n-1}_{\theta\phi}(X, M)$.*

12.8 The cohomology of a cover of a topological space

In this section a way of assigning a free crossed complex to a cover $\mathcal{U}$ of a topological space X, and so leading to a notion of non abelian cohomology of the cover.

Let $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ be family of subsets of the topological space X. We define the projection

$$p : E\mathcal{U} = \bigsqcup_\lambda U_\lambda \to X \tag{12.8.1}$$

to send $(x, \lambda) \mapsto x, x \in U_\lambda$. This projection defines an equivalence relation $\mathrm{Equ}\,\mathcal{U}$ on $E\mathcal{U}$, which is of course a special kind of groupoid. The objects of $\mathrm{Equ}\,\mathcal{U}$ are pairs (x, λ) such that $x \in U_\lambda$. There is a unique arrow $(x, \lambda) \to (x, \mu)$ if and only if $x \in U_\lambda \cap U_\mu$. Hence we can form

$$F_*(\mathcal{U}) = F^{st}_*(\mathrm{Equ}\,\mathcal{U}), \tag{12.8.2}$$

which we call the *standard crossed resolution of the cover* $\mathcal{U}$. If C is a crossed complex, then we can form

$$H^0(\mathcal{U}, C) = [F_*(\mathcal{U}), C]. \tag{12.8.3}$$

Example 12.8.1 A free basis element $[g_1, \ldots, g_n]$ of $F_n(\mathcal{U})$ is equivalent to a sequence $[x, \lambda_0, \lambda_1, \ldots, \lambda_n]$ such that $x \in U_{\lambda_0} \cap \cdots \cap U_{\lambda_n}$. Then we have the boundary formulae[27] in $F_*(\mathcal{U})$:

$$\begin{aligned} \delta_2[x, \lambda, \mu, \nu] &= [x, \lambda, \nu]^{-1}[x, \lambda, \mu][x, \mu, \nu] \\ \delta_3[x, \lambda, \mu, \nu, \xi] &= [x, \lambda, \mu, \nu, \xi]^{[x,\nu,\xi]}\ [x, \mu, \nu, \xi]^{-1}[x, \lambda, \mu, \xi]^{-1}[x, \lambda, \nu, \xi]. \end{aligned}$$

So we can analyse this definition in the particular case when C is a crossed module of groups $\partial : M \to P$, and say that a *cocycle* $f = (f_1, f_2)$ of $\mathcal{U}$ with values in this crossed module consists of functions with values $f_1[x, \lambda, \mu,] \in P$, $f_2[x, \lambda, \mu, \nu] \in M$ and satisfying

$$\begin{aligned} \partial f_2[x, \lambda, \mu, \nu] &= f_1([x, \lambda, \nu]^{-1}[x, \lambda, \mu][x, \mu, \nu]) \\ f_2\delta_3[x, \lambda, \mu, \nu, \xi] &= 1. \end{aligned}$$

Now let the cover $\mathcal{V} = \{V_\mu\}_{\alpha \in A}$ of X be a refinement of the cover $\mathcal{U}$. This means there is function $\phi : A \to \Lambda$ such that for each $\alpha \in A$ we have $V_\alpha \subseteq U_{\phi(\alpha)}$. Such a *refinement map* defines a groupoid morphism $\phi_* : \mathrm{Equ}(\mathcal{V}) \to \mathrm{Equ}(\mathcal{U})$ by $(x, \alpha, \beta) \mapsto (x, \phi(\alpha), \phi(\beta))$. One easily checks that if $\psi : A \to \Lambda$ is another refinement map, then the two groupoid morphisms $\phi_*, \psi_* : \mathrm{Equ}(\mathcal{V}) \to \mathrm{Equ}(\mathcal{U})$ are homotopic by the homotopy h which assigns to (x, α) the arrow $(x, \phi\alpha) \to (x, \psi\alpha)$ of $\mathrm{Equ}(\mathcal{U})$.

The importance of this is that ϕ, ψ then induce homotopic morphisms $\phi_*, \psi_* : F(\mathcal{V}) \to F(\mathcal{U})$ of free crossed resolutions; hence ϕ_*, ψ_* are homotopic and so induce the same function

$$[F(\mathcal{U}), C] \to [F(\mathcal{V}), C].$$

This is the start of defining the Čech cohomology of the space X with coefficients in the crossed complex C using refinements of open covers and taking inverse limits of the corresponding sets of homotopy classes. □

Notes

[21]p. 331 The formulae for the differential given on this page are different in detail from those given in [Hue80, BH82, Ton94]. This reflects the different conventions we have used. The formulae are forced on us by choices made in Chapter 13 for the equivalence of categories which is central to the work of this book, and which determine the tensor product formulae.

[22]p. 334These methods have been implemented in GAP4, see [HW03]. Related methods, using universal covering cell complexes, also implemented in GAP4, but without the crossed information, are in [Ell04].

[23]p. 337 This idea was introduced by Dedecker in [Ded64]; see also Taylor [Tay53].

[24]p. 339 Such examples are in [Tay54, BM94], the first to do with bundles and the second to do with covering maps of non-connected topological groups.

[25]p. 341 The above description explains the determination of extensions by a product of cyclic groups given in [**?**]. Different conventions for the tensor product have been adopted by Baues in [**?**].

[26]p. 342 We refer to Exercises 1-4 in [Bro06, Section 10.7] for applications of the lower part of the exact sequence to problems in group theory. Notice also that there should be a useful Mayer-Vietoris type sequence for a pullback of a fibration of crossed complexes generalising that of [Bro06, 10.7.6]. We also refer to the Exercises 1-5 in that section for applications and extensions of those ideas.

[27]p. 346 Formulae of this type go back to Dedecker, [Ded60]. Compare also (2.6.5) of [Bre94].

Part III

ω-groupoids

Introduction to Part III

In Part II we have explored the techniques of crossed complexes, and hope we have convinced the reader they are a powerful tool in algebraic topology. In this part, we have to give the proofs of the main theorems on which those tools depend.

To this end, we introduce the algebra of ω-groupoids, or in full, *cubical ω-groupoids with connections*. It was the way in which this algebra could be developed to model the geometry of cubes which suggested the possibility of the theory and calculations described in this book.

As intimated in Chapter 6 of Part I, the crucial advantages of cubical methods are the capacity to encode conveniently:

A) subdivision;

B) multiple composition as an algebraic inverse to subdivision;

C) commutative cubes, and their composition.

These properties allows us to verify a universal property by using the first two properties to give a candidate for a morphism, and using the third to verify that this morphism is well defined.

The techniques which enable an analogous argument in all dimensions are more elaborate. The main achievements are as follows:

- In order to *define* the notion of commutative cube, we have to relate the cubical theory of ω-groupoids to that of crossed complexes. This purely algebraic equivalence is established in Chapter 13.
- The proof that the natural definition of the fundamental ω-groupoid ρX_* of a filtered space actually is an ω-groupoid requires the techniques, of *collapsing* for subcomplexes of a cube which were given in Chapter 14. These techniques are also used to prove the equivalence of the two functors ρ and Π under the equivalence of algebraic categories proved in Chapter 13.
- The proof of the HHvKT for the functor ρ is also given in Chapter 14.
- The final Chapter 15 constructs the monoidal closed structure on the category of ω-groupoids, and deduces the precise formulae for the equivalent structure on crossed complexes used in Part II. Also proved is the Eilenberg-Zilber type natural transformation $\rho(X_*) \otimes \rho(Y_*) \to \rho(X_* \otimes Y_*)$, for filtered spaces X_*, Y_*.

Chapter 13

The category ω-Gpds of ω-groupoids

As stated in the Introduction to Part III, this chapter contains the generalisation to all dimensions of the algebraic part of Chapter 6. There we proved the equivalence between the category XMod of crossed modules over groupoids and the category DGpds of double groupoids with connections. To obtain this equivalence we defined in Section 6.2 a functor

$$\gamma : \mathsf{DCatG} \to \mathsf{XMod}$$

and in Section 6.6 another functor

$$\lambda : \mathsf{XMod} \to \mathsf{DGpds}.$$

We proved also in Section 6.6 that these functors give an equivalence of categories. The composition $\gamma\lambda$ is clearly naturally isomorphic to the identity. Nevertheless, we had to work hard to prove that $\lambda\gamma$ is also isomorphic to the identity. We shall come back to this point later.

In this Chapter we are going to follow analogous steps. Thus the first point is to get the appropriate generalisation of both categories.

The generalisation of XMod has already been studied: it is the central algebraic category for most of Part II, namely the category Crs of crossed complexes. In the reduced case, this category had been studied in the literature, because of its connections with relative homotopy groups, and with group cohomology.

It was not so hard to write down a definition of ω-Gpds the category of multiple groupoids with connections or ω-groupoids as a reasonable generalisation to all dimensions of DGpds, the category of double groupoids with connections. The Definition and general properties of this category are given in Section 13.2 but earlier, in Section 13.1, we extend the notion of cubical sets given in Section 10.1 to include the structures of connections and compositions.

Once these two categories of ω-groupoids and of crossed complexes are fixed, it is easy to define the functor

$$\gamma : \omega\text{-}\mathsf{Gpds} \to \mathsf{Crs}.$$

As in Section 6.2, to associate a crossed complex to an ω-groupoid we take the elements of γG_n to be cubes with all faces but one (∂_1^- in our convention) concentrated at a point and the boundary maps are given by the restriction to the non trivial face. All this is developed in Section 13.3.

As in the 2-dimensional case it is considerably more difficult to define a functor back

$$\lambda : \mathsf{Crs} \to \omega\text{-}\mathsf{Gpds}.$$

The still harder part is to give the natural equivalence $\lambda\gamma \simeq 1$, by showing that an ω-groupoid G may be rebuilt from the crossed complex γG it contains.

This equivalence, which is completed in Section 13.6, is a purely algebraic equivalence between two algebraically defined categories. So we have to use only the algebraic definition, however much we rely on geometry for the structure of the proof. Each axiom for the two categories is used at least once, proving that all of them are needed.

Let us recall that in Chapter 6 to define the functor λ on a crossed module $\mathcal{M} = (\mu : M ato P)$ we used as 2-dimensional elements of $\lambda\mathcal{M}$ the 'squares of arrows in P commuting up to an element of M' as explained at the beginning of Section 6.6.

The clear generalisation of squares are the 'n-shells'. They are families of n-cubes that fit together as do the faces of an $(n+1)$-cube, that is they satisfy the appropriate face relations. These n-shells are studied in Section 13.5 where they are used to give the construction of right and left adjoints for the truncation functor.

It is more difficult to define a 'commutative n-shell'. But even in dimension 2 we found the 'commutative cube' rather an inconvenient idea and in Section 6.6 we worked instead with the 'folding map'. We explore this avenue in Section 13.4 We define first the 'folding' Φ_i in direction i and then the 'folding map'

$$\Phi = \Phi_1\Phi_2 \cdots \Phi_{n-1} : G_n \to \gamma G_n$$

as the composite of the foldings in decreasing order. The effect of Φ is to 'fold' all faces of a cube into one face, which in our convention is taken to be the $(-, 1)$ face. This folding map allows us to say that an n-shell is commutative if and only if it folds to the trivial n-shell. We define the foldings in Section 13.4 and explore their behaviour with respect to all operators: faces, degeneracies, connections, compositions.

A main result is that every element $x \in G_n$ is determined by its total boundary ∂x and the folding Φx; this is a consequence of Proposition 13.5.11. In essence, this says that the folding process can be inverted and suggests how to construct λC_n inductively using pairs $(\mathbf{x}, \xi)$ where $\mathbf{x}$ is a 'shell' (generalisation of the total boundary) and $\xi \in C_n$ 'fills' the folding of the shell ($\delta\xi = \delta\Phi\mathbf{x}$). We work inductively using the coskeleton functor of Section 13.5; the construction of λ is done in Section 13.6.

In Chapter 6 we saw that connections and the folding map give a characterisation of commutative cubes. In the general case this may be taken as the definition of commutative n-cube, i.e. of the thin cubes. The *basic* thin n-cubes are images of degeneracies and connections: the general thin n-cubes are formed from the basic ones using negatives and compositions (see Definition 13.4.17). In Proposition 13.4.18 we prove that the thin n-cubes are exactly the elements that fold to the trivial one: i.e. they represent the 'commutative n-cubes', or, more precisely, the cubes with commutative boundary, or shell. Hence we obtain that *any composite of commutative cubes is commutative*. This is a key result for the proof of the Higher Homotopy van Kampen Theorem in Chapter 14.

The last Section (13.7) contains the algebraic Homotopy Addition Lemma (HAL) 13.7.1 and some of its consequences, which will be used in Chapter 14. The HAL gives an expression for the only non-trivial face of the folding of an n-shell ($\Sigma\mathbf{x} = \delta\Phi\mathbf{x}$). Thus a commutative shell is one having $\Sigma\mathbf{x} = 0$ and by Proposition 13.5.11 any commutative n-shell has a unique thin filler. The main consequence is that the thin cubes satisfy Dakin's axioms for T-complexes ([Dak83]):

- degenerate cubes are thin;
- any box has a unique thin filler (so ω-groupoids are Kan cubical sets in a strong way);
- if a thin cube has all faces but one thin, then this last face is also thin.

This chapter involves a substantial amount of work, and checking of detail. The advantage of this is that we can often apply the main result, the equivalence of categories, without using the details, and even if the application seems simple, this may be deceptive, since powering it is a well crafted machine. Sufficient detail is given that all proofs should be checkable by a graduate student.

13.1 Connections and compositions in cubical complexes

To generalise the category of double groupoids, it is important to notice that every double groupoid has an underlying 2-truncated cubical set. Moreover they have some extra 'degeneracies' that we have called connections. In this Section we explore some definitions generalising these concepts to every dimension. We are adding extra structure to the cubical sets studied in Section 10.1.

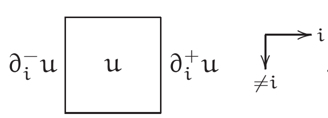
A key example of a cubical set is the singular cubical set of a space KX (see Definition **??**). But we are interested in the filtered spaces whose definition and main properties were studied in Subsection 7.1.1, partly as these are a tool for studying spaces. There is a natural generalisation of the singular cubical set of a space to the filtered case, which we call the *filtered singular cubical set*.

Definition 13.1.1 For any filtered space we denote by $R_n X_*$ the set of filtered maps $I^n_* \to X_*$ where I^n represents the standard n-cube with its standard cell structure as a product of n copies of $I = [0, 1]$.

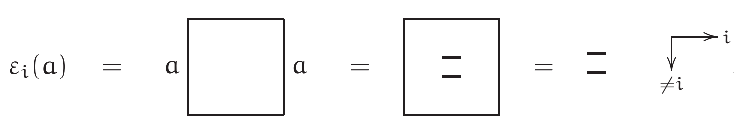
The sets $R_n X_*$ for $n \geqslant 0$, together with the face and degeneracy maps defined for the singular cubical set of a space, form a cubical set called the *filtered singular cubical complex* of the filtered space X_*, which we write RX_*. □

There is every reason to have a pictorial image for n-cubes very similar to the one we used for squares in Chapter 6 since it also useful here to state the laws of connections and compositions and to prove some results.

Remark 13.1.2 There is just a small difference with the conventions we used in Chapter 6. Since now we cannot picture all n dimensions, we have got to state which directions we are representing in any 2-dimensional picture, e.g. sometimes it is useful to show just one direction condensing all the orthogonal directions, as in:



The degeneracies can be represented



□

Singular cubical sets have a lot of extra structure arising from geometric maps on cubes, and which we used in Part I for squares. We are going to give generalisations to all dimensions of connections and compositions.

Let us first generalise the connections studied in Section 6.5; these connections should be thought of as giving more forms in which an n-cube can be 'degenerate'.

Definition 13.1.3 We say that a cubical set K has *connections* if it has additional structure maps

$$\Gamma_i : K_{n-1} \to K_n \ (i = 1, 2, \ldots, n-1)$$

(called *connections*) satisfying the relations:

$$\partial_i^\alpha \Gamma_j = \begin{cases} \Gamma_{j-1}\partial_i^\alpha & (i < j) \\ \Gamma_j \partial_{i-1}^\alpha & (i > j+1), \end{cases} \tag{1}$$

$$\partial_j^- \Gamma_j = \partial_{j+1}^- \Gamma_j = \mathrm{id},$$

$$\partial_j^+ \Gamma_j = \partial_{j+1}^+ \Gamma_j = \varepsilon_j \partial_j^+.$$

$$\Gamma_i \varepsilon_j = \begin{cases} \varepsilon_{j-1}\Gamma_i & (i < j) \\ \varepsilon_j \Gamma_{i-1} & (i > j) \end{cases} \tag{2}$$

$$\Gamma_j \varepsilon_j = \varepsilon_j^2 = \varepsilon_{j+1}\varepsilon_j,$$

$$\Gamma_i \Gamma_j = \Gamma_{j+1}\Gamma_i \qquad (i \leqslant j) \tag{3}$$

□

Remark 13.1.4 This definition generalises axioms CON1-2 of Definition 6.5.1 □

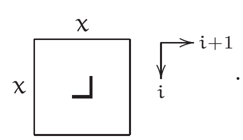
Example 13.1.5 1.- The singular cubical set KX of a space X is a cubical set with connections. The connection $\Gamma_i : K_{n-1} \to K_n$ is induced by the map $\gamma_i : I^n \to I^{n-1}$ defined by

$$\gamma_i(t_1, t_2, \ldots, t_n) = (t_1, t_2, \ldots, t_{i-1}, \max(t_i, t_{i+1}), t_{i+2}, \ldots, t_n).$$

2.- The connections of the previous example also give a structure of cubical set with connections to the filtered singular cubical set RX_* of a filtered space X_*. □

Remark 13.1.6 The connections are to be thought of as extra 'degeneracies'. (A degenerate cube of type $\varepsilon_j x$ has a pair of opposite faces equal and all other faces degenerate. A cube of type $\Gamma_i x$ has a pair of adjacent faces equal and all other faces of type $\Gamma_j y$ or $\varepsilon_j y$).

We can get a 2-dimensional picture of the connection Γ_i representing only the two dimensions i and $i+1$

$\Gamma_i(x) =$ 

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□

The singular cubical set KX of a space has another extra piece of structure which we will exploit in a substantial way: the possibility of "adding together" cubes in a direction if the appropriate faces in this direction coincide. The multiple forms of this composition give a method of 'algebraic inverse to subdivision'. The precise definition of the basic compositions is as follows.

Definition 13.1.7 A *cubical set with connections and compositions* is a cubical set K with connections in which each K_n has n partial compositions $+_i$ and n unary operations $-_i$ $i = 1, 2, \ldots, n)$ satisfying the following axioms.

If $a, b \in K_n$, then $a +_i b$ is defined if and only if $\partial_i^+ a = \partial_i^- b$, and then for $\alpha = \pm$:

$$\begin{cases} \partial_i^-(a +_i b) = \partial_i^- a \\ \partial_i^+(a +_i b) = \partial_i^+ b \end{cases} \qquad \partial_i^\alpha(a +_j b) = \begin{cases} \partial_i^\alpha a +_{j-1} \partial_i^\alpha b & (i < j) \\ \partial_i^\alpha a +_j \partial_i^\alpha b & (i > j), \end{cases} \tag{1.i}$$

If $a \in K_n$, then $-_i a$ is defined and

$$\begin{cases} \partial_i^-(-_i a) = \partial_i^+ a \\ \partial_i^+(-_i a) = \partial_i^- a \end{cases} \qquad \partial_i^\alpha(-_j a) = \begin{cases} -_{j-1}\partial_i^\alpha a & (i < j) \\ -_j \partial_i^\alpha a & (i > j) \end{cases} \tag{1.ii}$$

$$\varepsilon_i(a +_j b) = \begin{cases} \varepsilon_i a +_{j+1} \varepsilon_i b & (i \leqslant j) \\ \varepsilon_i a +_j \varepsilon_i b & (i > j) \end{cases} \tag{2.i}$$

$$\varepsilon_i(-_j b) = \begin{cases} -_{j+1}\varepsilon_i a & (i \leqslant j) \\ -_j \varepsilon_i a & (i > j) \end{cases} \tag{2.ii}$$

$$\Gamma_i(a +_j b) = \begin{cases} \Gamma_i a +_{j+1} \Gamma_i b & (i < j) \\ \Gamma_i a +_j \Gamma_i b & (i > j) \end{cases} \tag{3.i}$$

$$\Gamma_j(a +_j b) = (\Gamma_j a +_{j+1} \varepsilon_j b) +_j (\varepsilon_{j+1} b +_{j+1} \Gamma_j b)$$

(This last equation is called the *transport law*.)

$$\Gamma_i(-_j a) = \begin{cases} -_{j+1}\Gamma_i a & (i < j) \\ \quad -_j \Gamma_i a & (i > j) \end{cases} \tag{3.ii}$$

We have for $i \neq j$ and whenever both sides are defined:

$$(a +_i b) +_j (c +_i d) = (a +_j c) +_i (b +_j d) \tag{4.i}$$

These relations are called the *interchange laws*. Further:

$$\begin{aligned} &-_i(a +_j b) = (-_i a) +_j (-_i b) \text{ and } -_i (-_j a) = -_j(-_i a) \text{ if } i \neq j \\ &-_j(a +_j b) = (-_j b) +_j (-_j a) \text{ and } -_j (-_j a) = a. \end{aligned} \tag{4.ii}$$

□

Example 13.1.8 1.- It is easily verified that the singular cubical set KX of a space X satisfies these axioms if $+_j, -_j$ are defined by

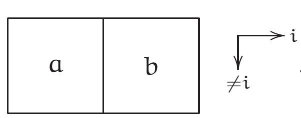
$$(a +_j b)(t_1, t_2, \ldots, t_n) = \begin{cases} a(t_1, \ldots, t_{j-1}, 2t_j, t_{j+1}, \ldots, t_n) & (t_j \leqslant \frac{1}{2}) \\ b(t_1, \ldots, t_{j-1}, 2t_j - 1, t_{j+1}, \ldots, t_n) & (t_j \geqslant \frac{1}{2}) \end{cases}$$

whenever $\partial_j^+ a = \partial_j^- b$; and

$$(-_j a)(t_1, t_2, \ldots, t_n) = a(t_1, \ldots, t_{j-1}, 1 - t_j, t_{j+1}, \ldots, t_n).$$

2.- The faces and degeneracies of the previous example also give a structure of cubical set with connections and compositions to the filtered singular cubical set RX_* of a filtered space X_*. □

Remark 13.1.9 We have a 2-dimensional pictorial image of the composition $+_i$ given by



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Also the interchange law can be stated in a matrix form. The diagram

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \; \begin{array}{l} \lrcorner\!\!\longrightarrow j \\ \downarrow \\ i \end{array}$$

will be used to indicate that both sides of the equation are defined and also to denote the unique composite of the four elements. With this notation, the transport law can be stated

$$\Gamma_j(a +_j b) = \begin{bmatrix} \Gamma_j a & \varepsilon_j b \\ \varepsilon_{j+1} b & \Gamma_j b \end{bmatrix} \; \begin{array}{l} \longrightarrow j+1 \\ \downarrow \\ j \end{array} .$$

□

Remark 13.1.10 The interchange laws in Definition 13.1.7 and the associativity laws (when they hold) have as consequence that we can define the composition of some complicated arrays of elements in any cubical set G with associative compositions.

A rectangular *array* of n-cubes is a family of n-cubes $x_{pq} \in G_n$ $(1 \leqslant p \leqslant P, 1 \leqslant q \leqslant Q)$ satisfying for some $i \neq j$ the relations

$$\begin{aligned} \partial_i^+ x_{pq} &= \partial_i^- x_{p+1,q} \; (1 \leqslant p < P, 1 \leqslant q \leqslant Q) \\ \partial_j^+ x_{pq} &= \partial_j^- x_{p,q+1} \; (1 \leqslant p \leqslant P, 1 \leqslant q < Q) \end{aligned}$$

It is written $(x_{pq})_{\{1 \leqslant p \leqslant P, 1 \leqslant q \leqslant Q\}}$ or

$$(x_{pq}) \;=\; \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1Q} \\ x_{21} & x_{22} & \cdots & x_{2Q} \\ \cdots & \cdots & \cdots & \cdots \\ x_{P1} & x_{P2} & \cdots & x_{PQ} \end{pmatrix} \; \begin{array}{l} \longrightarrow j \\ \downarrow \\ i \end{array}$$

An array (x_{pq}) has a unique *composite* $x = [x_{pq}] \in G_n$ obtained by applying the operations $+_i, +_j$ in any well-formed fashion; for example

$$x = (x_{11} +_i x_{21} +_i \cdots +_i x_{P1}) +_j \cdots +_j (x_{1Q} +_i x_{2Q} +_i \cdots +_i x_{PQ}).$$

We write

$$[x_{pq}] \;=\; \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1Q} \\ x_{21} & x_{22} & \cdots & x_{2Q} \\ \cdots & \cdots & \cdots & \cdots \\ x_{P1} & x_{P2} & \cdots & x_{PQ} \end{bmatrix} \; \begin{array}{l} \longrightarrow j \\ \downarrow \\ i \end{array}$$

The same is true for multi-dimensional arrays, and the most general situation can be described as follows. Let $(m) = (m_1, m_2, \ldots, m_n)$ be a sequence of positive integers. A *composable array* in G_n of type (m) is a family of cubes $x_{(p)} \in G_n$, where $(p) = (p_1, p_2, \ldots, p_n), 1 \leqslant p_i \leqslant m_i$, satisfying the relations

$$\partial_i^+ x_{(p)} = \partial_i^- x_{(p)'_i} \text{ for all } i$$

where $(p)'_i = (p_1, p_2, \ldots, p_{i-1}, p_i + 1, p_{i+1}, \ldots, p_n)$. We denote the unique composite in G_n of such an array by $[x_{(p)}]$. The previous case is obtained by taking $m_k = 1$ for $k \neq i, j$. We shall also sometimes write $[x_1, x_2, \ldots, x_r]_j$ for the linear composite $x_1 +_j x_2 +_j \cdots +_j x_r$, and an unlabeled $-x$ in such a composite will always mean $-_j x$. □

We introduce some notation for multiple compositions in the singular cubical sets KX and $R_n X_*$.

Remark 13.1.11 Let $(m) = (m_1, \ldots, m_n)$ be an n-tuple of positive integers and

$$\phi_{(m)} : I^n \to [0, m_1] \times \cdots \times [0, m_n]$$

be the map $(x_1, \ldots, x_n) \mapsto (m_1 x_1, \ldots, m_n x_n)$. Then a *subdivision of type* (m) of a map $\alpha : I^n \to X$ is a factorisation $\alpha = \alpha' \circ \phi_{(m)}$; its *parts* are the cubes $\alpha_{(r)}$ where $(r) = (r_1, \ldots, r_n)$ is an n-tuple of integers with $1 \leqslant r_i \leqslant m_i$, $i = 1, \ldots, n$, and where $\alpha_{(r)} : I^n \to X$ is given by

$$(x_1, \ldots, x_n) \mapsto \alpha'(x_1 + r_1 - 1, \ldots, x_n + r_n - 1).$$

We then say that α is the *composite* of the cubes $\alpha_{(r)}$ and write $\alpha = [\alpha_{(r)}]$. The *domain* of $\alpha_{(r)}$ is then the set $\{(x_1, \ldots, x_n) \in I^n : r_i - 1 \leqslant x_i \leqslant r_i, 1 \leqslant i \leqslant n\}$.

The composite is *in direction* j if m_j is the only $m_i > 1$, and we then write $\alpha = [\alpha_1, \ldots, \alpha_{mj}]_j$; the composite is *in the directions* j, k $(j \neq k)$ if m_j, m_k are the only $m_i > 1$, and we then write

$$\alpha = [\alpha_{rs}]_{j,k}$$

for $r = 1, \cdots, m_j$ and $s = 1, \cdots, m_k$.

These definitions and notations are one of the keys to our use of cubical methods in the proof of the Higher Homotopy van Kampen Theorem, since they allow for

'algebraic inverses to subdivision'.

13.2 ω-groupoids

In this section we restrict to cubical sets with connections and compositions such that each composition gives a structure of groupoid. These objects give the category ω-Gpds of ω-groupoids which generalises the category DGpds of double groupoids studied in Chapter 6.

Definition 13.2.1 An *ω-groupoid* $G = \{G_n\}_{n \geqslant 0}$ is a cubical set with connections and compositions in which each $+_j$ gives a groupoid structure on G_n such that for $x \in G_n$ the identity elements are

$$\eta_j^\alpha x = \varepsilon_j \partial_j^\alpha x$$

(the left identity when $\alpha = -$ and the right one when $\alpha = +$) and the inverse is $-_j x$.

A *morphism* of ω-groupoids is a morphism of cubical sets preserving all the connections and all the groupoid operations. We denote the resulting category of ω-groupoids by ω-Gpds. □

[This category is complete and cocomplete, as follows from general theorems of Freyd [Fre72], Bastiani-Ehresmann [BE72] and Coates [Coa74]. Is it worth saying this? This remark is relevant once one begins to look at limits and colimits, and consider free objects, and presentations. we should put it there.]

Remark 13.2.2 Of course the compositions of the cubical singular set KX of a space X are not groupoid compositions, for the same reason as the usual composition of paths in a space do not form a category. In dimension 1 it is easy to define the fundamental groupoid $\pi_1 X$ by taking homotopy classes rel end points.

For higher dimensions, there is a solution in the filtered case. A major result in Chapter 14 is the definition of the fundamental ω-groupoid ρX_* of the filtered space X_*. The applications of this construction are a major theme of this book.

Let us point out that in defining ω-groupoids some of the laws in Definition 13.1.7 are redundant.

Proposition 13.2.3 *If one assumes that each $+_j$ is a groupoid structure on G_n with identities $\eta_j^\alpha x$ for all $x \in G_n$ and inverse $-_j$, then one may omit parts* (1.ii)*,* (2.ii)*,* (3.ii) *and* (4.ii) *of all the laws in Definition 13.1.7 since they follow from the first parts and the groupoid laws. One may also rewrite the transport law* (3.i) *of the same definition in the form*

$$\Gamma_j(a +_j b) = (\Gamma_j a +_{j+1} \varepsilon_j b) +_j \Gamma_j b = (\Gamma_j a +_j \varepsilon_{j+1} b) +_{j+1} \Gamma_j b \tag{3.i*}$$

and deduce that

$$\Gamma_j(-_j a) = (-_j \Gamma_j a) -_{j+1} \varepsilon_j a = (-_{j+1} \Gamma_j a) -_j \varepsilon_{j+1} a. \tag{3.ii*}$$

Definition 13.2.4 An ω-*subgroupoid* of G is a sub cubical set closed under all the connections and all the operations $+_j, -_j$. Any set S of elements of G generates an ω-subgroupoid, namely, the intersection of all ω-subgroupoids containing S. This ω-subgroupoid can be built from S by repeated applications of all the structure maps and operations: first, it can be verified that the elements of the form $\varepsilon \dots \varepsilon \Gamma \dots \Gamma \partial \dots \partial x$ $(x \in S)$ make up the subcomplex-with-connections K generated by S; (here ∂ stands for various ∂_i^α, etc.) the ω-subgroupoid generated by S then consists, as again can be verified, of all composites of arrays of cubes of the form $-_i -_j \dots -_l\, y$ $(y \in K)$.

We also use finite-dimensional versions of the above structures and categories.

Definition 13.2.5 An n-*tuple groupoid* is an n-truncated cubical set $G = (G_n, G_{n-1}, \dots, G_0)$ with connections, having m groupoid structures in dimension m $(m \leqslant n)$, and satisfying all the laws for an ω-groupoid in so far as they make sense. We denote by ω-Gpds_n the category of n-tuple groupoids. (The category ω-Gpds_2 is another name for the category DGpds of double groupoids, the prototype for ω-Gpds, which was introduced in [BS76a] and has been studied in Chapter 6). □

13.3 The crossed complex associated to an ω-groupoid.

Analogously to Chapter 6, we consider for an ω-groupoid G the elements of G having all faces trivial but one. A main result is that these elements may be given the structure studied extensively in Part II, namely that of crossed complex:

$$\gamma G : \ \cdots \longrightarrow \gamma G_n \xrightarrow{\delta_n} \gamma G_{n-1} \xrightarrow{\delta_{n-1}} \cdots\cdots \xrightarrow{\delta_3} \gamma G_2 \xrightarrow{\delta_2} G_1 \ ,$$

where $\delta_n = \partial_1^-$. We shall prove in the next few Sections that crossed complexes are equivalent to ω-groupoids. Moreover, this associated crossed complex is obtained in such a way that the crossed complex $\gamma\rho X_*$ associated to the fundamental ω-groupoid ρX_* of the filtered space X_* is naturally isomorphic to ΠX_*, the fundamental crossed complex of a filtered space described in Subsection 7.1.3. The proof of this result is again delayed to the next Chapter (see Theorem 14.4.1).

Let us start by defining γG_n as a set. The definition is motivated by the standard definition of relative homotopy groups.

Definition 13.3.1 For any ω-groupoid G and for $n \geqslant 2$ and $p \in G_0$, we define the set of n-cubes x all of whose faces except $\partial_1^- x$ are *concentrated* at p to be

$$\gamma G_n(p) = \{x \in G_n : \partial_i^\alpha x = (\varepsilon_1)^{n-1} p \text{ for all } (\alpha, i) \neq (-,1)\}.$$

We observe that for any $p \in G_0$, such a concentrated r-cube $(\varepsilon_1)^r p$ is an identity for all compositions $+_k$ of G since $(\varepsilon_1)^r p = \varepsilon_k(\varepsilon_1)^{r-1} p$ for $1 \leqslant k \leqslant r$; accordingly, we will write 0 (sometimes 0_p) for such a cube $(\varepsilon_1)^r p\ (p \in G_0)$. With this convention, we have the rules $\partial_i^\alpha 0 = 0, \varepsilon_i 0 = 0, \Gamma_i 0 = 0$. □

Remark 13.3.2 An element of γG_n can be represented as

$$\begin{array}{ccc} 0 & \boxed{\quad u \quad} & 0 \\ & 0 & \end{array} \qquad \begin{array}{l} \longrightarrow \neq 1 \\ \downarrow 1 \end{array}.$$

□

Now we define the operations on $\gamma G_n(p)$ which them a family of groups (abelian for $n \geqslant 3$).

Proposition 13.3.3 *Let $n \geqslant 2$ and $p \in G_0$. Then each composition $+_j$ of G_n, for $2 \leqslant j \leqslant n$, induces a group structure on $\gamma G_n(p)$. For $n \geqslant 3$ this group structure is independent of j and is Abelian.*

Proof The first part is easy to verify, while the last part is proved by applying the interchange law to the composites

$$\begin{bmatrix} x & 0_p \\ 0_p & y \end{bmatrix} \qquad \begin{bmatrix} 0_p & x \\ y & 0_p \end{bmatrix} \qquad \begin{array}{l} \longrightarrow k \\ \downarrow j \end{array}$$

for $x, y \in \gamma G_n(p)$ and $2 \leqslant j < k \leqslant n$. □

Definition 13.3.4 We write $x + y$ for $x +_j y$ if $x, y \in \gamma G_n(p)$ and $2 \leqslant j \leqslant n$, and the zero element for this addition is 0_p. If $n = 1$ we also write $+$ for the groupoid operation $+_1$ on $\gamma G_1 = G_1$.

The face map $\partial_1^- : G_n \to G_{n-1}$ restricts to

$$\delta_n : \gamma G_n(p) \to \gamma G_{n-1}(p).$$

Let $n \geqslant 2, p, q \in G_0$. We define *the action* of $a \in G_1(p, q)$ on $x \in \gamma G_n(p)$ by

$$x^a = [-\varepsilon_1^{n-1} a,\ x,\ \varepsilon_1^{n-1} a]_n = \begin{array}{ccc} -a & \partial_1^- x & a \\ \boxed{\ |\,|\ } & \boxed{\ x\ } & \boxed{\ |\,|\ } \\ -a & 0 & a \end{array} \qquad \begin{array}{l} \longrightarrow n \\ \downarrow \neq n \end{array}.$$

Also, if $x \in G_1(p)$, we define $x^a = -1 + x + a$. □

We now check that these definitions imply γG is a crossed complex. We have seen that $\gamma G_n(p)$ is a group (abelian for $n \geqslant 3$) where $G_1(p) = G_1(p, p)$. It is also immediate that:

Proposition 13.3.5 *The maps δ_n are group homomorphisms and satisfy $\delta^2 = 0$.*

We now verify the main properties of the action.

Proposition 13.3.6 *Let $n \geqslant 2, p, q \in G_0$. For any $x \in \gamma G_n(p)$ and $a \in G_1(p, q)$ the element x^a defined above lies in $\gamma G_n(q)$, and the rule $(x, a) \mapsto x^a$ defines an action of the groupoid G_1 on the groupoid γG_n. This action is preserved by the map $\delta : \gamma G_n(p) \to \gamma G_{n-1}(p)$ for $n \geqslant 2$.*

Proof We follow in essence the proof of Proposition 6.2.3, but use arrays rather than pictures.

First, note that, for $1 \leqslant i < n$, $\partial_i^\alpha(x^a) = [-\varepsilon_1^{n-2}a,\ \partial_i^\alpha x,\ \varepsilon_1^{n-2}a]_{n-1}$, while $\partial_n^\alpha(x^a) = \varepsilon^{n-1}\partial_1^+ a = 0_q$. From this it follows that $x^a \in \gamma G_n(q)$ and $\delta(x^a) = (\delta x)^a$.

That $x^{a+b} = (x^a)^b$ follows from the equation

$$\varepsilon_1^{n-1}(a+b) = [\varepsilon_1^{n-1}a,\ \varepsilon_1^{n-1}b]_n.$$

□

We next show that the action by elements of $\delta_2(\gamma G_2)$ satisfies the crossed complex conditions.

Proposition 13.3.7 *Let $y \in \gamma G_2(p)$ and $a = \delta y$. If $x \in \gamma G_n(p)$, then $x^a = x$ for $n \geqslant 3$ and $x^a = -y + x + y$ for $n = 2$.*

Proof If $a = \delta y$ and $n \geqslant 2$, the two ways of composing

$$\begin{bmatrix} -_n\varepsilon_1^{n-1}a & x & \varepsilon_1^{n-1}a \\ -_n\varepsilon_1^{n-2}y & 0_p & \varepsilon_1^{n-2}y \end{bmatrix} \quad \begin{array}{l} \xrightarrow{\ n\ } \\ \downarrow\, n-1 \end{array}$$

give $x^a = [-_n\varepsilon_1^{n-2}y,\ x,\ \varepsilon_1^{n-2}y]_n$, which is the result we require when $n = 2$. For $n \geqslant 3$ we may also compose

$$\begin{bmatrix} -_n -_{n-1}\varepsilon_1^{n-2}y & 0_p & -_{n-1}\varepsilon_1^{n-2}y \\ -_n\varepsilon_1^{n-2}y & x & \varepsilon_1^{n-2}y \end{bmatrix} \quad \begin{array}{l} \xrightarrow{\ n\ } \\ \downarrow\, n-1 \end{array}$$

in two ways to obtain, by what we have just proved, $x^a = x$. □

Putting together the above properties we obtain:

Theorem 13.3.8 *If G is an ω–groupoid then γG is a crossed complex, and this defines a functor*

$$\gamma : \omega\text{-}\mathsf{Gpds} \to \mathsf{Crs}.$$

By restriction, we also have a functor $\gamma : \omega\text{-}\mathsf{Gpds}_m \to \mathsf{Crs}_m$.

We shall show in Section 13.6 that the ω-groupoid G can be reconstructed from its associated crossed complex γG and hence that $\gamma : \omega\text{-}\mathsf{Gpds} \to \mathsf{Crs}$ is an equivalence of categories.

On our way to prove this result we are going to use another description of the action of G_1 on γG_n, as in the next Proposition. The proof gives the first time in this section that we use the connections.

Proposition 13.3.9 *The action of G_1 on γG_n defined in Lemma 13.3.6 is also given by*

$$x^a = [-\varepsilon_1^{j-1}\varepsilon_2^{n-j}a,\ x,\ \varepsilon_1^{j-1}\varepsilon_2^{n-j}a]_j$$

for $x \in \gamma G_n(p)$, $a \in G_1(p,q)$ and any j with $2 \leqslant j \leqslant n$.

Proof Let $2 \leqslant j \leqslant n$, and write $b_j = \varepsilon_1^{j-1}\varepsilon_2^{n-j}a = \varepsilon_n\varepsilon_{n-1}\dots\widehat{\jmath}\dots\varepsilon_1 a \in G_n$. Then b_j is an identity for all the compositions of G_n except $+_j$. Also $\partial_j^+(-_jb_j) = \partial_j^-(b_j) = 0$ and

$$\partial_{j+1}^\alpha(b_j) = \partial_j^\alpha(b_{j+1}) = \varepsilon_{n-1}\varepsilon_{n-2}\dots\widehat{\jmath}\dots\varepsilon_1 a = c,$$

say. Thus, if $j \geqslant 2$, we may form the composite

$$y = \begin{bmatrix} -_j -_{j+1}\Gamma_j c & -_jb_j & -_j\Gamma_j c \\ -_{j+1}b_{j+1} & x & b_{j+1} \\ -_{j+1}\Gamma_j c & b_j & \Gamma_j c \end{bmatrix} \quad \begin{array}{l} \xrightarrow{\ j+1\ } \\ \downarrow\, j \end{array}$$

Since b_{j+1} is an identity for $+_j$, the composite of the last column is $\varepsilon_j \partial_j^+ \Gamma_j c = 0_p$, and similarly the composites of the first column and of the first and last rows are 0_p. Hence, computing y by rows and by columns, we have

$$[-b_{j+1},\ x,\ b_{j+1}]_{j+1} = [-b_j,\ x,\ b_j]_j\ (j \geqslant 2).$$

It follows that, for $j \geqslant 2$, $[-b_j,\ x,\ b_j]_j = [-b_n,\ x,\ b_n]_n$, which is the definition of x^a. □

13.4 Folding operations

As explained in the Introduction to this Chapter we have to take a detour to define the notion of a 'commutative n-cube'. Instead of trying to make sense of all possible compositions of the $(n-1)$-faces, we just fold all faces into one.

First we introduce a 'folding in the i-th direction' which is analogous to the 2-dimensional case with $i = 2$. The composition of the foldings in all directions gives an operation Φ on cubes in an ω-groupoid G (or in an n-tuple groupoid) which has the effect of folding all faces of $x \in G_n$ onto the face $\partial_1^- \Phi x$. The resulting face can be seen as the 'ordered sum of the faces of x'. This operation Φ transforms x into an element of the associated crossed complex γG.

Later in this Section we study the behaviour of the foldings with respect to the operators of an ω-groupoid, namely faces, degeneracies, connections and composition.

We end the Section by proving that the thin elements (i.e. the composites of an array of degeneracies and connections) are just those folding to the trivial cube, i.e. those having 'commuting boundary'.

We emphasise again that these results and techniques, though with a geometric motivation, are purely algebraic, that is we use only the operations and laws that we have given. This is essential for the theory and the geometric applications.

Definition 13.4.1 In any n-tuple groupoid G, we define operations

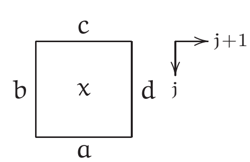
$$\Phi_j : G_m \to G_m,$$

for any $1 \leqslant j < m \leqslant n$, by the formula

$$\Phi_j x = [\varepsilon_j \partial_j^+ x,\ -\Gamma_j \partial_{j+1}^- x,\ x,\ \Gamma_j \partial_{j+1}^+ x]_{j+1}.$$

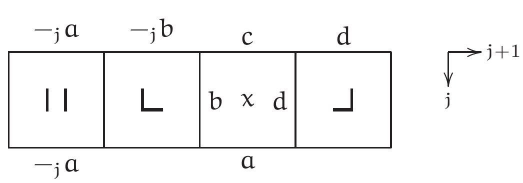
The map Φ_j is called the *folding in the* j*-th direction.* □

It is easy to check that the composite $\Phi_j x$ is defined. Writing a, b, c, d for the relevant faces of x,



the effect of Φ_j can be seen from the diagram

$$\Phi_j x =$$



in which unlabeled faces are appropriate degenerate cubes.

Next we study the various relations for the compositions of operations Φ_j with the operators of an n-tuple groupoid (i.e. faces, degeneracies, connections, compositions and inverses). Recall that to simplify the notation we have written $\eta_j^\alpha x$ for $\varepsilon_j \partial_j^\alpha x$, the left $(\alpha = 0)$ or right $(\alpha = 1)$ identity for x with respect to $+_j$.

We begin by the compositions of foldings and faces.

Proposition 13.4.2 *The faces of the folding in the j-th direction are given by:*

$$\partial_i^\alpha \Phi_j = \begin{cases} \Phi_{j-1}\partial_i^\alpha & (i < j), \\ \Phi_j \partial_i^\alpha & (i > j+1), \end{cases} \tag{i}$$

$$\partial_j^- \Phi_j x = [-\partial_j^+ x,\ -\partial_{j+1}^- x,\ \partial_j^- x,\ \partial_{j+1}^+ x]_j. \tag{ii}$$

$$\partial_{j+1}^\alpha \Phi_j = \partial_j^+ \Phi_j = \eta_j^+ \partial_j^+ = \eta_j^+ \partial_{j+1}^+. \tag{iii}$$

Proof These are proved by using the laws for faces of degeneracies, connections and compositions contained in the Remark 10.1.5 and Definitions 13.1.3 and 13.1.7. We shall prove them using the array form.

(i) If $i < j$ then

$$\begin{aligned} \partial_i^\alpha \Phi_j x &= [-\partial_i^\alpha \eta_j^+ x,\ -\partial_i^\alpha \Gamma_j \partial_{j+1}^- x,\ \partial_i^\alpha x,\ \partial_i^\alpha \Gamma_j \partial_{j+1}^+ x]_j \\ &= [-\eta_{j-1}^+ \partial_i^\alpha x,\ -\Gamma_{j-1}\partial_j^- \partial_i^\alpha x,\ \partial_i^\alpha x,\ \Gamma_{j-1}\partial_j^+ \partial_i^\alpha x]_j \\ &= \Phi_{j-1}\partial_i^\alpha x. \end{aligned}$$

The case $i > j+1$ is similar.

(ii) This is proved by a routine argument of the same kind and we will omit all such routine proofs from now on.

(iii) As before,

$$\begin{aligned} \partial_j^+ \Phi_j x &= [-\partial_j^+ \eta_j^+ x,\ -\partial_j^+ \Gamma_j \partial_{j+1}^- x,\ \partial_j^+ x,\ \partial_j^+ \Gamma_j \partial_{j+1}^+ x]_j \\ &= [-\partial_j^+ x,\ \eta_j^+ \partial_{j+1}^- x,\ \partial_j^+ x,\ \eta_j^+ \partial_{j+1}^+ x]_j. \end{aligned}$$

But $\eta_j^+ \partial_{j+1}^- x$ and $\eta_j^+ \partial_{j+1}^+ x$ are identities for $+_j$, so

$$\partial_j^+ \Phi_j x = [-\partial_j^+ x,\ \partial_j^+ x]_j = \eta_j^+ \partial_j^+ x.$$

The other cases are easily verified. □

From this proposition we deduce immediately a formula which we are going to use later in this Section.

Corollary 13.4.3 *With the notation of the above proposition*

$$\partial_{j+1}^\alpha \Phi_j \Phi_{j+1} \cdots \Phi_{n-1} = \partial_j^+ \Phi_j \Phi_{j+1} \cdots \Phi_{n-1} = \eta_j^+ \eta_{j+1}^+ \cdots \eta_{n-1}^+ \partial_n^+$$

Proof This follows from (iii). □

Now we give the relation with degeneracies.

Proposition 13.4.4 *The foldings in the* j*-th direction behave on degeneracy operators as follows:*

$$\begin{cases} \Phi_j \varepsilon_i = \varepsilon_i \Phi_{j-1}, \ \Phi_j \eta_i^\alpha = \eta_i^\alpha \Phi_j & \textit{if } i < j; \\ \Phi_j \varepsilon_i = \varepsilon_i \Phi_j, \ \Phi_j \eta_i^\alpha = \eta_i^\alpha \Phi_j & \textit{if } i > j+1. \end{cases} \tag{i}$$

$$\Phi_j \varepsilon_j = \eta_{j+1}^+ \varepsilon_j = \eta_j^+ \varepsilon_{j+1}, \ \Phi_j \eta_j^\alpha = \eta_{j+1}^+ \eta_j^\alpha. \tag{ii}$$

$$\Phi_j \varepsilon_{j+1} = \eta_{j+1}^+ \varepsilon_j = \eta_j^+ \varepsilon_{j+1}, \ \Phi_j \eta_{j+1}^\alpha = \eta_j^+ \eta_{j+1}^\alpha. \tag{iii}$$

Proof (i) and (ii) are routine; the parts about $\Phi_j \eta_j^\alpha$ involve also the use of the previous Proposition.

(iii)

$$\begin{aligned} \Phi_j \varepsilon_{j+1} x &= [-\eta_j^+ \varepsilon_{j+1} x, \ -\Gamma_j x, \varepsilon_{j+1} x, \ \Gamma_j x]_{j+1} \\ &= [-\eta_j^+ \varepsilon_{j+1} x]_{j+1} = [-\eta_{j+1}^+ \varepsilon_j x]_{j+1} \\ &= \eta_{j+1}^+ \varepsilon_j x. \end{aligned}$$

The other equations follow easily. □

From this proposition we deduce immediately another formula that we use later in this Section.

Corollary 13.4.5 *With the notation of the above proposition*

$$\Phi_1 \Phi_2 \cdots \Phi_{j-2} \eta_{j-1}^+ = \eta_1^+ \eta_2^+ \cdots \eta_{j-1}^+, \ \Phi_1 \Phi_2 \cdots \Phi_{j-1} \varepsilon_j = \eta_1^+ \eta_2^+ \cdots \eta_{j-1}^+ \varepsilon_j.$$

Proof This follows from (iii) in the preceding proposition. □

Now we give the relations with connections

Proposition 13.4.6 *The foldings in the* j*-th direction behave on connection operators as follows:*

$$\Phi_j \Gamma_i = \begin{cases} \Gamma_i \Phi_{j-1} & (i < j), \\ \Gamma_i \Phi_j & (i > j+1). \end{cases} \tag{i}$$

$$\Phi_j \Gamma_j = \varepsilon_j \eta_j^+ = \varepsilon_{j+1} \eta_j^+. \tag{ii}$$

$$\Phi_j \Gamma_{j+1} x = [-\Gamma_{j+1} \eta_j^+ x, \ -\Gamma_j x, \ \Gamma_{j+1} x, \ \Gamma_j \eta_{j+1}^+ x]_{j+1}. \tag{iii}$$

Proof (i) and (iii) are routine. For (ii),

$$\begin{aligned} \Phi_j \Gamma_j x &= [-\eta_j^+ \Gamma_j x, \ -\Gamma_j \partial_{j+1}^- \Gamma_j x, \ \Gamma_j x, \nu \Gamma_j \partial_{j+1}^+ \Gamma_j x]_{j+1} \\ &= [-\varepsilon_j \eta_j^+ x, \ -\Gamma_j x, \ \Gamma_j x, \ \Gamma_j \eta_j^+ x]_{j+1} && \text{by 13.1.3} \\ &= [-\varepsilon_{j+1} \eta_j^+ x, \ \varepsilon_{j+1} \eta_j^+ x]_{j+1} && \text{by 10.1.5 and 13.1.3} \\ &= \varepsilon_{j+1} \eta_j^+ x = \varepsilon_j \eta_j^+ x. \end{aligned}$$

□

We now define for $n \geqslant 2$ the *folding operation*

$$\Phi : G_n \to \gamma G_n$$

by folding in each direction in decreasing order.

Definition 13.4.7 On G_0 and G_1 we define Φ as the identity map. We now define for $n \geqslant 2$

$$\Phi x = \Phi_1 \Phi_2 \cdots \Phi_{n-1} x$$

for any $x \in G_n$.

Let us see that the folding has the image we want, i.e. that Φx has all faces but one trivial. To do this, we introduce some notation.

Definition 13.4.8 For $x \in G_n$, we call $(\partial_1^+)^n x$ the *base-point* of x and denote it by βx.

Proposition 13.4.9 *If* $(\alpha, j) \neq (0,1)$ *then* $\partial_j^\alpha \Phi = \varepsilon_1^{n-1}\beta$. *Hence, for any* $x \in G$, Φx *lies in the associated crossed complex* γG_n.

Proof If $2 \leqslant j \leqslant n$ then

$$\begin{aligned}
\partial_j^\alpha \Phi &= \Phi_1\Phi_2\cdots\Phi_{j-2}\partial_j^\alpha\Phi_{j-1}\cdots\Phi_{n-1} && \text{by 13.4.2 (i)}\\
&= \Phi_1\Phi_2\cdots\Phi_{j-2}\eta_{j-1}^+\cdots\eta_{n-1}^+\partial_n^+ && \text{by 13.4.3}\\
&= \eta_1^+\eta_2^+\cdots\eta_{n-1}^+\partial_n^+ && \text{by 13.4.5}\\
&= \varepsilon_1^{n-1}(\partial_1^+)^n && \text{by 10.1.5.}
\end{aligned}$$

If $j = 1$ and $n \geqslant 2$, then $\alpha = 1$ and the equation follows from Proposition 13.4.2 (iv) and Remark 10.1.5. The case $n = 1$ is trivial. Thus, for $x \in G_n$, we have $\partial_j^\alpha \Phi x = 0_p$ for $(\alpha, j) \neq (0,1)$, where $p = \beta x$. This shows that $\Phi x \in \gamma G_n(p)$. □

This gives the following important characterisation of the elements in γG as those invariant under the folding.

Corollary 13.4.10 *If* $x \in G$, *then* x *is in* γG *if and only if* $\Phi x = x$. *In particular* $\Phi^2 y = \Phi y$ *for all* y *in* G.

Proof It is clear that if $x \in C_n(p) = (\gamma G_n)(p)$, then Definition 13.4.1 implies $\Phi_j x = x$. This implies $\Phi x = x$. □

To end the study of the behaviour of the folding map with respect to the operators of a cubical set with connections, let us record the effect the folding map has on degeneracies and connections.

Proposition 13.4.11 *If* $n \geqslant 2$, *then on* G_{n-1},

$$\Phi\varepsilon_j = \varepsilon_1^n\beta \quad \textit{and} \quad \Phi\Gamma_j = \varepsilon_1^n\beta.$$

Proof Making computations

$$\begin{aligned}
\Phi_1\Phi_2\cdots\Phi_{n-1}\varepsilon_j &= \Phi_1\Phi_2\cdots\Phi_j\varepsilon_j\Phi_j\Phi_{j+1}\cdots\Phi_{n-2} && \text{by 13.4.4(i)}\\
&= \Phi_1\Phi_2\cdots\Phi_{j-1}\eta_{j+1}^+\varepsilon_j\Phi_j\cdots\Phi_{n-2} && \text{by 13.4.4(ii)}\\
&= \Phi_1\Phi_2\cdots\Phi_{j-1}\varepsilon_j\varepsilon_j\partial_j^+\Phi_j\cdots\Phi_{n-2} && \text{by 10.1.5}\\
&= \eta_1^+\eta_2^+\cdots\eta_{j-1}^+\varepsilon_j\varepsilon_j\eta_j^+\cdots\eta_{n-2}^+\partial_{n-1}^+ && \text{by 13.4.3 and 13.4.5}\\
&= \varepsilon_1^n\beta && \text{by 10.1.5.}
\end{aligned}$$

and

$$\begin{aligned}
\Phi_1\Phi_2\cdots\Phi_{n-1}\Gamma_j &= \Phi_1\Phi_2\cdots\Phi_j\Gamma_j\Phi_j\Phi_{j+1}\cdots\Phi_{n-2} && \text{by 13.4.6(i)}\\
&= \Phi_1\Phi_2\cdots\Phi_{j-1}\varepsilon_j\eta_j^+\Phi_j\cdots\Phi_{n-2} && \text{by 13.4.6(ii)}\\
&= \varepsilon_1^n\beta && \text{as above.}
\end{aligned}$$

□

We now study the behaviour of the folding map Φ with respect to composition and inverses. The rules are easy to state (see Proposition 13.4.14) but their proof involves more complicated rules for the partial foldings Φ_j.

Proposition 13.4.12 *We have the following relations of Φ_j with the compositions and inverses:*

$$\left.\begin{array}{ll} \Phi_j(x +_i y) & = \Phi_j x +_i \Phi_j y \\ \Phi_j(-_i x) & = -_i \Phi_j x \end{array}\right\} \quad \textit{if } i \neq j, j+1. \tag{i}$$

$$\Phi_j(x +_j y) = [\Phi_j y, -\varepsilon_j \partial^+_{j+1} y, \Phi_j x, \varepsilon_j \partial^+_{j+1} y]_{j+1}. \tag{ii}$$

$$\Phi_j(x +_{j+1} y) = [-\eta^+_j y, \Phi_j x, \eta^+_j y, \Phi_j y]_{j+1}. \tag{iii}$$

Proof (i) This is routine, using the interchange law for the directions i and $j+1$.

(ii) Let the relevant faces of x and y be given by

$$\begin{array}{ccc} & u & \\ a & \boxed{\;x\;} & b \\ & v & \end{array} \qquad \begin{array}{ccc} & v & \\ c & \boxed{\;y\;} & d \\ & w & \end{array} \qquad \begin{array}{l} \longrightarrow j+1 \\ \downarrow j \end{array}$$

Then

$$\Phi_j(x +_j y) = [-\varepsilon_j w,\ -\Gamma_j(a +_j c),\ (x +_j y),\ \Gamma_j(b +_j d)]_{j+1}.$$

Using the transport law, this can be written as the composite

$$A = \begin{bmatrix} -\varepsilon_j w & -\varepsilon_j c & -\Gamma_j a & x & \Gamma_j b & \varepsilon_j d \\ -\varepsilon_j w & -\Gamma_j c & -\varepsilon_{j+1} c & y & \varepsilon_{j+1} d & \Gamma_j d \end{bmatrix} \quad \begin{array}{l} \longrightarrow j+1 \\ \downarrow j \end{array}$$

where $-$ stands for $-_{j+1}$. Consider the composite

$$B = \begin{bmatrix} -\varepsilon_j w & -\varepsilon_j c & \varepsilon_j v & \varepsilon_j d & -\varepsilon_j d & -\varepsilon_j v & -\Gamma_j a & x & \Gamma_j b & \varepsilon_j d \\ -\varepsilon_j w & -\Gamma_j c & y & \Gamma_j d & -\varepsilon_j d & -\varepsilon_j v & -\varepsilon_j \eta^+_j a & \varepsilon_j v & \varepsilon_j \eta^+_j b & \varepsilon_j d \end{bmatrix} \quad \begin{array}{l} \longrightarrow j+1 \\ \downarrow j \end{array}$$

By composing the columns first, we see that B is equal to the right hand side of (ii). However, the composites of the rows of B are the same as the composites of the rows of A, since $\varepsilon_j \eta^+_j b = \varepsilon_{j+1} \eta^+_j b$ is an identity of the horizontal composition as well as the vertical one. Hence $A = B$.

(iii) This is routine. $\Box$

To state the behaviour of the folding map Φ with respect to compositions and inverses, we need some extra notation.

Definition 13.4.13 For $x \in G_n$, the *edges of x terminating at the base point* $\beta x = (\partial^+_1)^n x$ will have special importance and we denote them by

$$u_i x = \partial^+_1 \partial^+_2 \cdots \hat{\imath} \cdots \partial^+_n x$$

for all $1 \leqslant i \leqslant n$.

Proposition 13.4.14 *Let $n \geqslant 2$ and $x, y, z \in G_n$ with $\partial^+_i x = \partial^-_i y$. Then, in γG_n :*

$$\Phi(x +_i y) = \begin{cases} \Phi y + (\Phi x)^{u_1 y} & \textit{if } n = 2 \textit{ and } i = 1, \\ (\Phi x)^{u_i y} + \Phi y & \textit{otherwise;} \end{cases} \tag{i}$$

$$\Phi(-_i z) = -(\Phi z)^{-u_i z}. \tag{ii}$$

Proof (i) First consider the case $i = n \geqslant 2$. We have, by Proposition 13.4.12,

$$\begin{aligned}\Phi(x +_n y) &= \Phi_1\Phi_2 \cdots \Phi_{n-2}[-\eta^+_{n-1}y,\ \Phi_{n-1}x,\ \eta^+_{n-1}y,\ \Phi_{n-1}y]_n \\ &= [-u,\ \Phi x,\ u,\ \Phi y]_n\end{aligned}$$

where

$$\begin{aligned}u &= \Phi_1\Phi_2 \cdots \Phi_{n-2}\eta^+_{n-1}y \\ &= \eta^+_1\eta^+_2 \cdots \eta^+_{n-1}y && \text{by 13.4.5} \\ &= \varepsilon_1^{n-1}u_n y && \text{by 10.1.5.}\end{aligned}$$

Hence $\Phi(x +_n y) = (\Phi x)^{u_n y} + \Phi y$ in this case.

In the remaining cases we have $1 \leqslant i < n$, so we may put

$$\begin{aligned}X &= \Phi_{i+1}\Phi_{i+2} \cdots \Phi_{n-1}x, \\ Y &= \Phi_{i+1}\Phi_{i+2} \cdots \Phi_{n-1}y,\end{aligned}$$

and then

$$\begin{aligned}\Phi(x +_i y) &= \Phi_1\Phi_2 \cdots \Phi_i(X +_i Y) && \text{by 13.4.12 (i)} \\ &= \Phi_1 \cdots \Phi_{i-1}[\Phi_i Y,\ -\varepsilon_i\partial^+_{i+1}Y,\ \Phi_i X,\ \varepsilon_i\partial^+_{i+1}Y]_{i+1} && \text{by 13.4.12 (ii)} \\ &= [\Phi y,\ -V,\ \Phi x,\ V]_{i+1} && \text{by 13.4.12 (i) ,}\end{aligned}$$

where

$$\begin{aligned}V &= \Phi_1 \cdots \Phi_{i-1}\varepsilon_i\partial^+_{i+1}\Phi_{i+1} \cdots \Phi_{n-1}y \\ &= \eta^+_1\eta^+_2 \cdots \eta^+_{i-1}\varepsilon_i\eta^+_{i+1} \cdots \eta^+_{n-1}\partial^+_n y && \text{by 13.4.3 and 13.4.5} \\ &= (\varepsilon_1)^i(\varepsilon_2)^{n-i-1}u_i y && \text{by 10.1.5.}\end{aligned}$$

Hence, by Lemma 13.3.9, $\Phi(x +_i y) = \Phi y + (\Phi x)^{u_i y}$ in this case. (Note that $i + 1 \geqslant 2$, so addition in direction $i + 1$ is addition in γG_n). If $n = 2$ and $i = 1$, this is the required formula. Otherwise, we have $n \geqslant 3$, so γG_n is commutative and the formula can be rewritten in the required form.

(ii) Put $x = -_i x, y = z$ in (i) and note that, by 13.4.11, $\Phi((-_i z) +_i z) = \Phi\varepsilon_i\partial^+_i z = \varepsilon^n_1\beta z = 0$ in γG_n. □

The folding map is an involution. More precisely

Proposition 13.4.15 *For any* $1 \leqslant j \leqslant n - 1$, *we have*

$$\Phi\Phi_j = \Phi : G_n \to G_n.$$

Proof By definition, for $x \in G_n$,

$$\begin{aligned}\Phi_j x &= [-\varepsilon_j\partial^+_j x,\ -\Gamma_j\partial^-_{j+1}x,\ x,\ \Gamma_j\partial^+_{j+1}x]_{j+1} \\ &= [a,\ b,\ x,\ c]_{j+1},\ \text{say.}\end{aligned}$$

By Proposition 13.4.11 and 13.4.14(ii), Φa, Φb and Φc are all zero in γG_n, so Proposition 13.4.14 gives

$$\Phi\Phi_j x = (\Phi x)^u,$$

where

$$\begin{aligned} u &= u_{j+1}c \\ &= \partial_1^+ \cdots \partial_j^+ \partial_{j+2}^+ \cdots \partial_n^+ \Gamma_j \partial_{j+1}^+ x && \text{by definition of } u_{j+1} \\ &= \varepsilon_1 \partial_1^+ \partial_2^+ \cdots \partial_n^+ x && \text{by 10.1.5 and 13.1.3.} \end{aligned}$$

Thus $\Phi\Phi_j x = (\Phi x)^{\varepsilon_1 \beta x} = \Phi x$. □

Corollary 13.4.16 *The folding operation Φ is idempotent, i.e. for any n, we have*

$$\Phi\Phi = \Phi : G_n \to G_n.$$

We end this Section with the definition of the thin n-cubes and their characterisation as those n-cubes that fold to the trivial cube; thus, in particular, a thin cube has commutative boundary.

Definition 13.4.17 An element $x \in G_n$, for $n \geqslant 1$, is *thin* if it can be written as a composite of an array $x = [x_{(r)}]$, where each entry is either of the form $\varepsilon_i y$ or of the form $-_i -_j \cdots -_l \Gamma_m y$. □

The collection of all thin elements of G is clearly closed under all the ω-groupoid operations except possibly the face operations. It is useful to think of the thin elements as the most general kind of 'degenerate' cubes. They are important in the topological applications and we establish their main properties in Section 13.7. For the present we prove only the following characterisation.

Proposition 13.4.18 *An element $x \in G_n$, for $n \geqslant 1$, is thin if and only if $\Phi x = 0$.*

Proof We have shown that $\Phi\varepsilon_j y = 0, \Phi\Gamma_j y = 0$ for all $y \in G_{n-1}$ (see Proposition 13.4.11). It follows from Proposition 13.4.14 that $\Phi x = 0$ whenever x is thin. To see the converse, we recall the definition

$$\Phi_j x = [-\varepsilon_j \partial_j^+ x,\ -\Gamma_j \partial_{j+1}^- x,\ x,\ \Gamma_j \partial_{j+1}^+ x]_{j+1}$$

which can be rewritten as

$$x = [\Gamma_j \partial_{j+1}^- x,\ \varepsilon_j \partial_j^+ x,\ \Phi_j x,\ -\Gamma_j \partial_{j+1}^+ x]_{j+1}.$$

These two equations show that $\Phi_j x$ is thin if and only if x is thin. Hence Φx is thin and only if x is thin. In particular, if $\Phi x = 0$ (i.e. $\Phi x = \varepsilon_1^n \beta x$) then Φx is thin, so x is also thin. □

13.5 n-shells: coskeleton and skeleton

To work inductively on an ω-groupoid, we have at each step n to restrict our attention to dimensions $\leqslant n$ and the minimal part accompanying it. To this end, it is useful to introduce the n-skeleton of an ω-groupoid as the ω-subgroupoid generated by the part of dimensions $\leqslant n$. Let us make the construction a bit more categorically.

Definition 13.5.1 If we ignore the elements of dimension higher than n in an ω-groupoid we obtain an n-tuple groupoid. This gives the n-*truncation* functor

$$\mathrm{tr}_n : \omega\text{-}\mathsf{Gpds} \to \omega\text{-}\mathsf{Gpds}_n.$$

□

We shall show that tr_n has both a right adjoint $\mathrm{cosk}^n : \omega\text{-Gpds}_n \to \omega\text{-Gpds}$, the n-*coskeleton functor* (Definition 13.5.6) and a left adjoint $\mathrm{sk}^n : \omega\text{-Gpds}_n \to \omega\text{-Gpds}$, the n-*skeleton functor* (Definition 13.5.15).

We will see that both can be described in terms of 'shells', i.e. families of r-cubes that fit together as the faces of an $(r+1)$-cube do. A trivial example is the total boundary of an n-cube.

For any n-tuple groupoid $G = (G_n, G_{n-1}, \cdots, G_0)$ we will construct an ω-groupoid $\mathrm{cosk}^n G$ adding 'shells' in all dimensions $\geqslant n$. To check that $\mathrm{cosk}^n G$ is an ω-groupoid we need to explain how to apply faces, degeneracies and connections to these shells. As consequence, we describe the result of applying the folding operations Φ_i and Φ to these shells. In particular, we prove that Φ commutes with the total boundary.

All these results may be used to prove the existence and uniqueness of fillers for n-shells. Associated to any n-cube $x \in G_n$ we have its total boundary $\boldsymbol{\partial} x$ and its folding Φx satisfying $\boldsymbol{\partial}\Phi x = \Phi \mathbf{x}$. Conversely, for any $\mathbf{x} \in \square\, G_{n-1}$ and $\xi \in \gamma G_n(\beta \mathbf{x})$ and $n \geqslant 2$ such that $\delta\xi = \delta\Phi\mathbf{x}$ exists $x \in G_n$ with $\boldsymbol{\partial} x = \mathbf{x}$ and $\Phi x = \xi$ is. This x is unique and it is denoted $x = \langle \mathbf{x}, \xi \rangle$. This property and notation allows the reconstruction of G from γG.

We finish the Section by constructing sk^n the n-skeleton functor as an ω-subgroupoid of cosk^n, and proving that it is the left adjoint of tr_n.

Definition 13.5.2 In any cubical set K, an n-*shell* is a family $\mathbf{x} = (x_i^\alpha)$ of n-cubes $(i = 1, 2, \cdots, n+1; \alpha = \pm)$ satisfying

$$\partial_j^\beta x_i^\alpha = \partial_{i-1}^\alpha x_j^\beta \qquad \text{for} \qquad 1 \leqslant j < i \leqslant n+1 \qquad \text{and} \qquad \alpha, \beta = \pm.$$

We denote by $\square K_n$ the set of all n-shells of K. □

Example 13.5.3 Notice that the faces $\{\partial_j^\alpha y\}$ for any $(n+1)$-cube y form an n-shell $\boldsymbol{\partial} y$ that we call its *total boundary*. It could be said that an n-shell is just a collection of n-cubes that is a candidate to be the total boundary of an $(n+1)$-cube. If this $(n+1)$-cube exists it is a called 'filler' of the n-shell. □

Remark 13.5.4 We shall usually write shells in boldface.

Now, to any n-truncated cubical set we associate an $(n+1)$-truncated cubical set by adding the n-shells.

Definition 13.5.5 Let $K = (K_n, K_{n-1}, \ldots, K_0)$ be an n-truncated cubical set.

To give to $K' = (\square K_n, K_n, K_{n-1}, \ldots, K_0)$ the structure of $(n+1)$-truncated cubical set we need only to define faces and degeneracies involving the top dimension.

Thus the faces

$$\partial_i^\alpha : \square K_n \to K_n$$

are given by $\partial_i^\alpha \mathbf{x} = x_i^\alpha$ for any $\mathbf{x} \in \square K_n$, and, the degeneracies

$$\boldsymbol{\varepsilon}_j : K_n \to \square K_n$$

are given by $\boldsymbol{\varepsilon}_j y = \mathbf{z}$, for any $y \in K_n$, where

$$z_i^\alpha = \begin{cases} \varepsilon_{j-1}\partial_i^\alpha y & (i < j), \\ \varepsilon_j \partial_{i-1}^\alpha y & (i > j), \\ y & (i = j). \end{cases} \tag{i}$$

Clearly the cubical rules of 10.1.5 are satisfied.

If K has also connections, we can define connections on K' by:

$$\Gamma_j : K_n \to \Box K_n$$

given by $\Gamma_j y = \mathbf{w}$, where

$$w_i^\alpha = \begin{cases} \Gamma_{j-1}\partial_i^\alpha y & (i < j), \qquad w_j^- = w_{j+1}^- = y, \\ \Gamma_j \partial_{i-1}^\alpha y & (i > j+1); \quad w_j^+ = w_{j+1}^+ = \eta_j^+ y. \end{cases} \tag{ii}$$

Again this is the Definition needed for the connections to satisfy the relations in Definition 13.1.3. In this way K' becomes an $(n+1)$-truncated cubical set with connections.

If K has compositions, we can also define compositions in $\Box K_n$ as follows. Let $\mathbf{x}, \mathbf{y} \in \Box K_n$ with $y_j^- = x_j^+$. Define $\mathbf{x} +_j \mathbf{y} = \mathbf{t}$ and $-_j \mathbf{x} = \mathbf{s}$, where (cf. 13.1.7)

$$\begin{cases} t_j^- = x_j^-, \\ t_j^+ = y_j^+, \end{cases} \quad t_i^\alpha = \begin{cases} x_i^\alpha +_{j-1} y_i^\alpha & (i < j), \\ x_i^\alpha +_j y_i^\alpha & (i > j), \end{cases}$$
$$\begin{cases} s_j^- = x_j^+, \\ s_j^+ = x_j^-, \end{cases} \quad s_i^\alpha = \begin{cases} -_{j-1} x_i^\alpha & (i < j), \\ -_j x_i^\alpha & (i > j). \end{cases} \tag{iii}$$

Then K' becomes an $(n+1)$-truncated cubical set with connections and compositions.

Moreover, if K is an n-tuple groupoid, then K' is an $(n+1)$-tuple groupoid. The verification of these facts is a tedious but entirely routine computation. □

The coskeleton functor can now be obtained by iteration of this construction.

Definition 13.5.6 For any n-tuple groupoid $G = (G_n, G_{n-1}, \cdots, G_0)$ we define its n-*coskeleton* by

$$(\operatorname{cosk}^n G)_m = \begin{cases} G_m & \text{for} \quad m \leqslant n, \\ \Box^{m-n} G_n & \text{for} \quad m > n \end{cases}$$

with operations defined as above. □

Proposition 13.5.7 *If* $G = (G_n, G_{n-1}, \cdots, G_0)$ *is an n-tuple groupoid, then* $\operatorname{cosk}^n G$ *is an ω-groupoid. This construction gives a functor*

$$\operatorname{cosk}^n : \omega\text{-}\mathsf{Gpds}_n \to \omega\text{-}\mathsf{Gpds}$$

which is right adjoint to tr_n.

Proof By definition, it is clear that $\operatorname{cosk}^n G$ is an ω-groupoid.

If H is any ω-groupoid and $\theta_k : H_k \to G_k$ are defined for $k \leqslant n$ so as to form a morphism of n-tuple groupoids from $\operatorname{tr}_n H$ to G, then there is a unique extension to a morphism of ω-groupoids $\theta : H \to \operatorname{cosk}^n G$ defined inductively by

$$\theta_m y = \mathbf{z}, \quad \text{where} \quad z_i^\alpha = \theta_{m-1}\partial_i^\alpha y \qquad (m > n).$$

This shows that cosk^n is right adjoint to tr_n. □

Proposition 13.5.8 *If* $G = (G_n, G_{n-1}, \cdots, G_0)$ *is an* n*-tuple groupoid, then all elements of* $\mathrm{cosk}^n G$ *in dimension* $n + 2$ *and higher are thin.*

Proof To prove the result it is enough to show that, for any ω-groupoid G, elements of $\Box^2 G_r$ are always thin, or equivalently, by Proposition 13.4.18, that their foldings are trivial.

Let $\mathbf{z} \in \Box^2 G_r$ and $\mathbf{w} = \Phi\mathbf{z}$. Then $\mathbf{w} \in \Box^2 G_r$ and all its $(r+1)$-dimensional faces $\partial_i^\alpha \mathbf{w}$ are 0_p, where $p = \beta\mathbf{z}$, except possibly $\partial_1^- \mathbf{w}$. Let us check that this one is also 0_p.

The condition that all $(r+1)$-faces but one are 0_p implies that *all* the r-dimensional faces of $\mathbf{w}$ are 0_p. Hence $\partial_1^- \mathbf{w}$ is an r-shell all of whose faces are 0_p. By definition, therefore $\partial_1^- \mathbf{w} = 0_p$.

Hence $\mathbf{w}$ itself is an $(r+1)$-shell all of whose faces are 0_p and therefore $\mathbf{w} = 0_p$. By Proposition 13.4.18, $\mathbf{z}$ is thin. $\Box$

We next see that the total boundary commutes with the folding.

Proposition 13.5.9 *For any element* x *of dimension at least two in any* m*-tuple groupoid*

$$\Phi \boldsymbol{\partial} x = \boldsymbol{\partial} \Phi x$$

Proof Given an n-shell $\mathbf{y} = (y_i^\alpha) \in \Box G_n$, we obtain n-shells $\Phi_i \mathbf{y}$ and

$$\Phi\mathbf{y} = \Phi_1 \Phi_2 \ldots \Phi_{n-1} \mathbf{y}.$$

By Proposition 13.4.9, all faces of $\Phi\mathbf{y}$ except $\partial_1^- \Phi\mathbf{y}$ are 0_p, where $p = \beta\mathbf{y} = (\partial_1^+)^n y_1^+$.

If H is a given ω-groupoid, then adjointness gives a canonical morphism $\theta : H \to \mathrm{Cosk}^n H = \mathrm{cosk}^n(\mathrm{tr}_n H)$, with $\theta_{n+1} x = \boldsymbol{\partial} x$ for $x \in H_{n+1}$. Since θ preserves the folding operations we have the result. $\Box$

Remark 13.5.10 Note that by Proposition 13.4.2 the faces of $\Phi_j x$ depend only on the faces of x, and this gives a recipe for $\Phi_j \boldsymbol{\partial} x$.

We can now prove that an n-shell $\mathbf{x} \in \Box G_{n-1}$ has a unique filler $x \in G_n$ for each element $\xi \in \gamma G_n(p)$ having the same boundary as the folding $\Phi\mathbf{x}$. This is the key to the inductive reconstruction of an ω-groupoid G from its associated crossed complex γG. It essentially arises from the fact that the folding operations are invertible, given complete information on the needed boundary.

Proposition 13.5.11 *Let* G *be an* ω*-groupoid, and let* γG *be its associated crossed complex. Let* $\mathbf{x} \in \Box G_{n-1}$ *and* $\xi \in \gamma G_n(p)$*, where* $p = \beta\mathbf{x}$ *and* $n \geqslant 2$*. Then a necessary and sufficient condition for the existence of* $x \in G_n$ *such that* $\boldsymbol{\partial} x = \mathbf{x}$ *and* $\Phi x = \xi$ *is that* $\delta\xi = \delta\Phi\mathbf{x}$*. Furthermore, if* x *exists, it is unique and it is denoted* $x = \langle \mathbf{x}, \xi \rangle$*.*

Proof Clearly the condition is necessary, since if $\boldsymbol{\partial} x = \mathbf{x}$ and $\Phi x = \xi$, then $\boldsymbol{\partial}\Phi x = \Phi\boldsymbol{\partial} x = \Phi\mathbf{x}$, by the previous Proposition, so $\delta\Phi\mathbf{x} = (\Phi\mathbf{x})_1^- = \partial_1^- \Phi x = \delta\xi$.

Suppose, conversely, that we are given $\mathbf{x}$ and ξ with $\delta\xi = \delta\Phi\mathbf{x}$, i.e. $\partial_1^- \xi = (\Phi\mathbf{x})_1^-$. Since all other faces of ξ and $\Phi\mathbf{x}$ are concentrated at p, this condition is equivalent to $\boldsymbol{\partial}\xi = \Phi\mathbf{x}$, an equation in $\Box G_{n-1}$. We have to show that there is a unique $x \in G_n$ such that $\boldsymbol{\partial} x = \mathbf{x}$ and $\Phi x = \xi$.

Since $\Phi\mathbf{x} = \Phi_1 \Phi_2 \ldots \Phi_{n-1} \mathbf{x}$, by induction, it is enough to show that if $y \in G_n$ and $\boldsymbol{\partial} y = \Phi_i \mathbf{z}$ for some $1 \leqslant i \leqslant n-1$ and $\mathbf{z} \in \Box G_{n-1}$, then there is a unique $z \in G_n$ with $\boldsymbol{\partial} z = \mathbf{z}$ and $\Phi_i z = y$. But this is clear since the equation

$$[-\varepsilon_i \partial_i^+ z,\ -\Gamma_i \partial_{i+1}^- z,\ z,\ \Gamma_i \partial_{i+1}^+ z]_{i+1} = y$$

becomes

$$[-\varepsilon_i z_i^+,\ -\Gamma_i z_{i+1}^-,\ z,\ \Gamma_i z_{i+1}^+]_{i+1} = y$$

under the stated conditions, and therefore has a unique solution for z in terms of y and $\mathbf{z}$. It is easy to check that this z has boundary $\mathbf{z}$. □

An easy consequence of this and Proposition 13.4.18 is a characterisation of when an n-shell has a thin filler, plus the fact that this filler is unique.

Corollary 13.5.12 *A thin element of an ω-groupoid is determined by its faces. Given a shell $\mathbf{x}$, there is a thin element t with $\partial t = \mathbf{x}$ if and only if $\delta\Phi\mathbf{x} = 0$.*

Proof Put $\xi = 0$ in Proposition 13.5.11 and use that t is thin if and only if $\Phi t = 0$ (Proposition 13.4.18). □

Definition 13.5.13 A shell $\mathbf{x}$ will be called a *commuting shell* if its folding is trivial, i.e. if $\delta\Phi\mathbf{x} = 0$. This can be interpreted as 'the sum of its folded faces is 0'. By the previous corollary, a commuting shell has a thin filler and that filler is unique.

Another consequence of Corollary 13.5.12 is that any ω-groupoid G can be recovered from its associated crossed complex γG.

Proposition 13.5.14 *Let G be an ω-groupoid. Then γG generates G as ω-groupoid.*

Proof Let H be any ω-subgroupoid of G containing γG. Then $\gamma H = \gamma G$ by definition. We show inductively that $H_n = G_n$.

This is true for $n = 0, 1$ since $\gamma G_0 = G_0, \gamma G_1 = G_1$.

Suppose $x \in G_n (n \geqslant 2)$. Then $\Phi x \in \gamma G_n$ and, by induction hypothesis, $\partial x \in \square H_{n-1}$. By Proposition 13.5.11, there is a unique $y \in H_n$ with $\partial y = \partial x$ and $\Phi y = \Phi x$. But x is the unique element of G_n with this property, so $H_n = G_n$. □

We shall finish the section by constructing sk^n the n-skeleton functor as a substructure of cosk^n and proving that it is the left adjoint of tr_n.

Definition 13.5.15 Given an n-tuple groupoid $G = (G_n, G_{n-1}, \dots, G_0)$, the n-*skeleton* $\mathrm{sk}^n G$ of G is the ω-subgroupoid of $\mathrm{cosk}^n G$ generated by G. □

There is a characterisation of sk^n in terms of commuting shells.

Proposition 13.5.16 *Given an n-tuple groupoid $G = (G_n, G_{n-1}, \dots, G_0)$, the n-skeleton*

$$\mathrm{sk}^n G = S$$

where S is defined by

$$S_m = \begin{cases} G_m & \text{if} \quad m \leqslant n, \\ \{\mathbf{x} \in \square S_{m-1} \mid \delta\Phi\mathbf{x} = 0\} & \text{if} \quad m > n. \end{cases}$$

i.e. for $m > n$, $\mathrm{sk}^n G_m$ consists entirely of thin elements, namely, the commuting shells. Moreover, for $m \geqslant n+2$, $\mathrm{cosk}^n G_m = \mathrm{sk}^n G_m$, i.e. all shells in $\square S_{m-1}$ are commuting shells.

Proof It is clear that $S \subseteq \mathrm{cosk}^n G$. By Proposition 13.5.8 all elements of S_m are thin for $m > n$.

Clearly, S is closed under face maps, degeneracy maps and connections (since $\varepsilon_j y$ and $\Gamma_j y$ are always thin).

Also, by induction on m, S_m is closed under $+_i, -_i (1 \leqslant i \leqslant m)$; for if $\mathbf{x}, \mathbf{y} \in S_m (m > n)$ and $\mathbf{x} +_i \mathbf{y}$ is defined, then $\mathbf{x} +_i \mathbf{y}$ has faces in S_{m-1} (by induction hypothesis) and $\delta\Phi(\mathbf{x} +_i \mathbf{y}) = 0$ because composites of thin elements in $\operatorname{cosk}^n G$ are thin. Thus $\mathbf{x} +_i \mathbf{y} \in S_m$, and similarly $-_i \mathbf{x} \in S_m$. Hence S is an ω-subgroupoid of $\operatorname{cosk}^n G$.

By Corollary 13.5.12, any ω-subgroupoid of $\operatorname{cosk}^n G$ containing S_{m-1} (for $m > n$) must contain S_m, so S is generated by G and $S = \operatorname{sk}^n G$.

To prove the last statement, if $m \geqslant n+2$, all shells in $\operatorname{cosk}^n G_m = \square^{m-n} G_k$ are thin by Proposition 13.5.8 and therefore satisfy $\delta\Phi\mathbf{x} = 0$ by Corollary 13.5.12 $\square$

Proposition 13.5.17 *The functor* $\operatorname{sk}^n : \omega\text{-}\mathsf{Gpds}_n \to \omega\text{-}\mathsf{Gpds}$ *is left adjoint to* tr_n.

Proof If H is any ω-groupoid and $\psi : G \to \operatorname{tr}_n H$ is a morphism of n-tuple groupoids, then ψ extends uniquely to a morphism of ω-groupoids $\psi : \operatorname{sk}^n G \to H$ inductively.

For $m > n$, consider a commuting shell $\mathbf{x} \in \square \operatorname{sk}^n G_{m-1}$. Since the elements $\psi_{m-1} x_i^\alpha$ form a commuting shell in H, by Corollary 13.5.12 exists $t \in H_m$ thin such that $\partial_i^\alpha t = \psi_{m-1} x_i^\alpha$ for $1 \leqslant i \leqslant m$ and $\alpha = 0, 1$. Then, we define $\psi_m \mathbf{x} = t$. $\square$

Given an ω-groupoid G, we define $\operatorname{Sk}^n G = \operatorname{sk}^n(\operatorname{tr}_n G)$ and call this, by abuse of language, the n-Skeleton of G. There is a unique morphism $\sigma : \operatorname{Sk}^n G \to G$ of ω-groupoids (the adjunction) which is the identity in dimensions $0, 1, 2, \ldots, n$. Let us prove that the image is what we would call intuitively the n-skeleton of G, i.e. the ω-groupoid of G generated by G_n.

Proposition 13.5.18 *The adjunction* $\sigma : \operatorname{Sk}^n G \to G$ *is an injection and identifies* $\operatorname{Sk}^n G$ *with the* ω*-subgroupoid of* G *generated by* G_n.

Proof For $m = 0, 1, 2, \ldots, n$, $\sigma_m : G_m \to G_m$ is the identity map.

Then, for $m > n$, $(\operatorname{Sk}^n G)_m$ is the set of commuting shells in $\square_{m-1}(\operatorname{Sk}^n G)$, by Proposition 13.5.16. Suppose that, for some $m > n$, $\sigma_{m-1} : (\operatorname{Sk}^n G)_{m-1} \to G_{m-1}$ is an injection. For any $\mathbf{x} \in (\operatorname{Sk}^n G)_m$, the elements $\sigma_{m-1} x_i^\alpha$ form a commuting shell $\mathbf{y}$ in $\square G_{m-1}$ and $\sigma_m \mathbf{x}$ is the unique thin element t of G_m with $\boldsymbol{\partial} t = \mathbf{y}$. Thus $x_i^\alpha = \sigma_{m-1}^{-1} y_i^\alpha = \sigma_{m-1}^{-1} \partial_i^\alpha t$ is uniquely determined by t for all (i, α) and therefore σ_m is an injection. This shows, inductively, that σ is an injection.

Now G_n generates $\operatorname{tr}_n G$ as n-tuple groupoid (even as n-truncated cubical set) and therefore generates $\operatorname{Sk}^n G$ as ω-groupoid, by Proposition 13.5.16. It follows that G_n generates the image of $\operatorname{Sk}^n G$ in G. $\square$

13.6 The equivalence of ω-Gpds and Crs

In this Section we construct a functor

$$\lambda : \mathsf{Crs} \to \omega\text{-}\mathsf{Gpds}$$

which together with γ gives an equivalence of categories.

The key idea for constructing λ in such a way that there is an equivalence $\lambda\gamma \simeq 1_{\omega\text{-}\mathsf{Gpds}}$ comes from Proposition 13.5.11, which show that an elements of G_n is determined by its total boundary and its folding.

We have proved that given $\mathbf{x} \in \square G_{n-1}$, $\xi \in \gamma G_n$ with $\delta\xi = \delta\Phi\mathbf{x}$ there is a unique element $x \in G_n$ such that $\boldsymbol{\partial} x = \mathbf{x}$ and $\Phi x = \xi$. We write $\langle \mathbf{x}, \xi \rangle = x$.

To define λG we use these elements $\langle \mathbf{x}, \xi \rangle$. It is clear how to express its faces, degeneracies and connections of G following Definition 13.5.5. Our next proposition shows how to define the compositions.

Proposition 13.6.1 *If $x = \langle \mathbf{x}, \xi \rangle, y = \langle \mathbf{y}, \eta \rangle$ in G_n, and $x_i^+ = y_i^-$, then*

$$x +_i y = \begin{cases} \langle \mathbf{x} +_1 \mathbf{y}, \eta + \xi^{u_1 y} \rangle & \text{if } n = 2 \text{ and } i = 1, \\ \langle \mathbf{x} +_i \mathbf{y}, \xi^{u_1 y} + \eta \rangle & \text{otherwise,} \end{cases}$$

and

$$-_i x = \langle -_i \mathbf{x}, -\xi^{-u_i x} \rangle.$$

Proof This follows immediately from Proposition 13.4.14 and the rule $\boldsymbol{\partial}(x +_i y) = \boldsymbol{\partial} x +_i \boldsymbol{\partial} y$. □

These results show how to construct from any crossed complex C an ω-groupoid $G = \lambda C$ with $\gamma G \cong C$.

Theorem 13.6.2 *There is a functor λ from the category* Crs *of crossed complexes to the category* ω-Gpds *of ω-groupoids such that $\lambda : \mathsf{Crs} \to \omega\text{-}\mathsf{Gpds}$ and $\gamma : \omega\text{-}\mathsf{Gpds} \to \mathsf{Crs}$ are inverse equivalences.*

Proof Let C be any crossed complex. We construct an ω-groupoid $G = \lambda C$ and an isomorphism of crossed complexes $\sigma : C \to \gamma G$ by induction on dimension.

We start with $G_0 = C_0,\ G_1 = C_1$, so that (G_1, G_0) is a groupoid. We write γG_n (in any cubical complex) for the set of n-cubes x with all faces except $\partial_1^- x$ concentrated at a point. Then $\gamma G_0 = C_0, \gamma G_1 = C_1$, and we take $\sigma_0 : C_0 \to \gamma G_0$ and $\sigma_1 : C_1 \to \gamma C_1$ to be the identity maps.

Suppose, inductively, that we have defined G_r and $\sigma_r : C_r \to \gamma G_r$ for $0 \leqslant r < n$ (where $n \geqslant 2$) so that $(G_{n-1}, G_{n-2}, \cdots, G_0)$ is an $(n-1)$-tuple groupoid and $(\sigma_{n-1}, \sigma_{n-2}, \cdots, \sigma_0)$ is an isomorphism of $(n-1)$-truncated crossed complexes. Then $(\square G_{n-1}, G_{n-1}, \ldots, G_0)$ is an n-tuple groupoid and we define

$$G_n = \{(\mathbf{x}, \xi) \mid \mathbf{x} \in \square\, G_{n-1}, \xi \in C_n, \delta \Phi \mathbf{x} = \sigma_{n-1} \delta \xi\}.$$

For $y \in G_{n-1}$, let $\varepsilon_j y = (\boldsymbol{\varepsilon}_j y, 0)$, where $\boldsymbol{\varepsilon}_j$ is defined in Definition 13.5.5(i). Then $\varepsilon_j y \in G_n$, since $\Phi \boldsymbol{\varepsilon}_j y = 0$ by Proposition 13.4.11. The maps $\varepsilon_j : G_{n-1} \to G_n$, together with the obvious face maps $\partial_i^\alpha : G_n \to G_{n-1}$ defined by $\partial_i^\alpha(\mathbf{x}, \xi) = x_i^\alpha$, give $(G_n, G_{n-1}, \cdots, G_0)$ the structure of an n-truncated cubical set.

Similarly one can define connections $\Gamma_j : G_{n-1} \to G_n$ by $\Gamma_j y = (\boldsymbol{\Gamma}_j y, 0)$, where $\boldsymbol{\Gamma}_j$ is defined in Definition 13.5.5(ii), and the laws in Definition 13.1.3 are clearly satisfied, since they are satisfied by $\boldsymbol{\Gamma}_j$.

Recalling Proposition 13.6.1, we define operations $+_i, -_i$. For $(\mathbf{x}, \xi), (\mathbf{y}, \eta) \in G_n$ with $x_i^+ = y_i^-$, let

$$(\mathbf{x}, \xi) +_i (\mathbf{y}, \eta) = \begin{cases} (\mathbf{x} +_1 \mathbf{y}, \eta + \xi^{u_1 y}) & \text{if } n = 2 \text{ and } i = 1, \\ (\mathbf{x} +_i \mathbf{y}, \xi^{u_1 y} + \eta) & \text{otherwise,} \end{cases}$$

and

$$-_i(\mathbf{x}, \xi) = (-_i \mathbf{x}, -\xi^{u_i x}).$$

By Proposition 13.4.14, for all $(n, i) \neq (2, 1)$,

$$\begin{aligned}\delta\Phi(\mathbf{x} +_i \mathbf{y}) &= \delta((\Phi\mathbf{x})^{u_i y} + \Phi\mathbf{y}) \\ &= (\sigma_{n-1}\delta\xi)^{u_i y} + \sigma_{n-1}\delta\eta \\ &= \sigma_{n-1}\delta(\xi^{u_i y} + \eta),\end{aligned}$$

so G_n is closed under $+_i$. The case $n = 2, i = 1$ is similar. Also $\delta\Phi(-_i\mathbf{x}) = \delta(-\Phi\mathbf{x})^{-u_i x} = \sigma_{n-1}\delta(-\xi^{u_i x})$, and therefore $-_i\mathbf{x} \in G_n$.

We claim that $(G_n, G_{n-1}, \dots, G_0)$ is now an n-tuple groupoid. Firstly, it is clear that, for $t \in G_{n-1}$, $\varepsilon_i t$ acts as an identity for $+_i$, and that $-_i$ is an inverse operation for $+_i$. The associative law is verified as for semi-direct products of groups. Secondly, the laws (1),(2) and (3) of Definition 13.1.7 are true for $\square G_{n-1}$. It remains, therefore, to prove the interchange law (4i) (from which (4ii) follows, using the groupoid laws).

Let $1 \leqslant i < j \leqslant n$ and let $x = (\mathbf{x}, \xi), y = (\mathbf{y}, \eta), z = (\mathbf{z}, \zeta), t = (\mathbf{t}, \tau)$ be elements of G_n such that the composite shell

$$\mathbf{w} = \begin{bmatrix} \mathbf{x} & \mathbf{y} \\ \mathbf{z} & \mathbf{t} \end{bmatrix}$$

is defined. Let $g = \partial_1^+\partial_2^+ \cdots \widehat{\imath} \cdots \widehat{\jmath}^+ \cdots \partial_n^+ \mathbf{t} \in G_2$ have boundary

$$\begin{tikzcd} \cdot \arrow[r, "c"] \arrow[d, "b"'] \arrow[dr, phantom, "g"] & \cdot \arrow[d, "d"] \\ \cdot \arrow[r, "a"'] & \cdot \end{tikzcd}$$

Then

$$(x +_i z) +_j (y +_i t) = (\mathbf{w}, \omega), \qquad (x +_j y) +_i (z +_j t) = (\mathbf{w}, \omega'),$$

say, and we have to show that $\omega = \omega'$ in C_n.

If $n = 2$ then $i = 1$ and $j = 2$ and we find that

$$\omega = (\zeta + \xi^b)^a + (\tau + \eta^d), \;\; \omega' = (\zeta^a + \tau) + (\xi^c + \eta)^d.$$

To show that these are equal, it is enough to show that $\xi^{b+a} + \tau = \tau + \xi^{c+d}$. But this follows from the crossed module laws since

$$\delta\tau = \sigma_1\delta\tau = \delta\Phi\mathbf{t} = \delta\Phi g = -a - b + c + d$$

and therefore

$$-\tau + \xi^{b+a} + \tau = (\xi^{b+a})^{\delta\tau} = \xi^{c+d}.$$

If $n > 2$, we find that

$$\omega = (\xi^b + \zeta)^a + \eta^d + \tau, \qquad \omega' = (\xi^c + \eta^d) + \zeta^a + \tau,$$

and since addition is now commutative, the equation $\omega = \omega'$ reduces to $\xi^{a+b} = \xi^{c+d}$, that is, $\xi^{\delta\Phi g} = \xi$. But, by induction hypothesis, we have an isomorphism $\sigma_2 : C_2 \to \gamma G_2$ preserving the crossed module structure, and if $\theta \in C_2$ is the element with $\sigma_2(\theta) = \Phi g$, then $\xi^{\delta\Phi g} = \xi^{\delta\theta} = \xi$ by the crossed complex laws. This completes the proof of the interchange law.

We now have an n-tuple groupoid $(G_n, G_{n-1}, \dots, G_0)$, and we must identify γG_n. For any $\xi \in C_n(p)$, let $\mathbf{d}\xi$ denote the shell $\mathbf{x} \in \square G_{n-1}$ with $x_1^- = \sigma_{n-1}\delta\xi$ and all other x_i^α concentrated at p. Define

$$\sigma_n\xi = (\mathbf{d}\xi, \xi).$$

Clearly $\sigma_n\xi \in \gamma G_n$ and every element of γG_n is of this form. The bijection $\sigma_n : C_n \to \gamma G_n$ is compatible with the boundary maps since $\delta\sigma_n\xi = \partial_1^- \sigma_n\xi = \sigma_{n-1}\delta\xi$. It preserves addition because, for $\xi, \eta \in C_n(p)$,

$$\begin{aligned}(\mathbf{d}\xi, \xi) + (\mathbf{d}\eta, \eta) &= (\mathbf{d}\xi +_n \mathbf{d}\eta, \xi^{u_n \mathbf{d}\eta} + \eta) \\ &= (\mathbf{d}(\xi + \eta), \xi + \eta).\end{aligned}$$

Furthermore, if $\xi \in C_n(p)$ and $a \in C_1(p, q) = G_1(p, q)$, then

$$\begin{aligned}(\sigma_n\xi)^a &= -_n \varepsilon_1^{n-1} a +_n \sigma_n\xi +_n \varepsilon_1^{n-1} a \\ &= (-_n \boldsymbol{\varepsilon}_1 \varepsilon_1^{n-2} a, 0) +_n (\mathbf{d}\xi, \xi) +_n (\boldsymbol{\varepsilon}_1 \varepsilon_1^{n-2} a, 0) \\ &= (\mathbf{y}, \xi^a),\end{aligned}$$

in all cases. Since $(\sigma_n\xi)^a \in \gamma G_n$, it follows that $\mathbf{y} = \mathbf{d}(\xi^a)$, making σ_n an isomorphism of crossed complexes up to dimension n.

This completes the inductive step in our construction, and we therefore obtain an ω-groupoid $G = \lambda C$ and an isomorphism $\sigma : C \to \gamma G$ of crossed complexes. This ω-groupoid has the following universal property: If G' is any ω-groupoid and $\sigma' : C \to \gamma G'$ any morphism of crossed complexes then there is a unique morphism $\theta : G \to G'$ of ω-groupoids making the diagram

$$\begin{tikzcd} C \arrow[r, "\sigma"] \arrow[dr, "\sigma'"'] & \gamma G \arrow[r, hook] \arrow[d, "\gamma\theta"] & G = \lambda C \arrow[d, "\theta"] \\ & \gamma G' \arrow[r, hook] & G' \end{tikzcd}$$

commute.

We define θ inductively, starting with $\theta_0 = \sigma'_0, \theta_1 = \sigma'_1$. For $n \geqslant 2$, each $x' \in G'_n$ is uniquely of the form $\langle \mathbf{x}', \xi' \rangle$ where $\mathbf{x}' \in \square G'_{n-1}, \xi' \in \gamma G_n$ and $\delta\Phi\mathbf{x}' = \delta\xi'$. We define $\theta_n : G_n \to G'_n$ by $(\mathbf{x}, \xi) \mapsto \langle \mathbf{x}', \xi' \rangle$, where $(x')_i^\alpha = \theta_{n-1} x_i^\alpha$ and $\xi' = \sigma'_n \xi$. This definition is forced, and it clearly gives a morphism of ω-groupoids.

From this universal property, it follows that the functor $\lambda : \mathsf{Crs} \to \omega\text{-}\mathsf{Gpds}$ is left adjoint to $\gamma : \omega\text{-}\mathsf{Gpds} \to \mathsf{Crs}$.

The adjunction $\sigma_c : C \to \gamma\lambda C$ is an isomorphism for all C, so $1_{\mathsf{Crs}} \simeq \gamma\lambda$. Also, the adjunction $\lambda\gamma G' \to G'$ is obtained by putting $G = \gamma G', \sigma' =$ identity, in which case θ is an isomorphism $\lambda\gamma G' \to G'$, as is clear from its definition. Hence $\lambda\gamma \simeq 1_{\omega\text{-}\mathsf{Gpds}}$ and we have inverse equivalences λ and γ between Crs and ω-Gpds. □

13.7 The HAL and properties of thin elements

Another very important property of ω-groupoids is that they are Kan cubical sets (see subsection 10.3.1), i.e. that any n-box has a filler.

Moreover the n-boxes have a set of canonical fillers, i.e. the thin elements giving them the structure of T-complexes in the sense of Dakin [Dak83].

The proof of both these facts may be deduced from Proposition 13.5.11 via an algebraic Homotopy Addition Lemma that expresses the only non-trivial face of the folding of a shell in term of the elements of the shell.

Lemma 13.7.1 (Homotopy Addition Lemma) *Let G be an ω-groupoid (or an m-tuple groupoid with $m \geqslant n$). Let $\mathbf{x} \in \square G_n$ and define $\Sigma\mathbf{x} \in C_n = (\gamma G)_n$ by*

$$\Sigma\mathbf{x} = \begin{cases} -x_1^+ - x_2^- + x_1^- + x_2^+ = -\Phi x_1^+ - \Phi x_2^- + \Phi x_1^- + \Phi x_2^+ & \text{if} \quad n = 1, \\ -\Phi x_3^+ - (\Phi x_2^-)^{u_2 x} - \Phi x_1^+ + (\Phi x_3^-)^{u_3 x} + \Phi x_2^+ + (\Phi x_1^-)^{u_1 x} & \text{if} \quad n = 2, \\ \sum_{i=1}^{n+1} (-1)^i \{\Phi x_i^+ - (\Phi x_i^-)^{u_i x}\} & \text{if} \quad n \geqslant 3 \end{cases}$$

where $u_i = \partial_1^+ \partial_2^+ \cdots \widehat{\imath} \cdots \partial_{n+1}^+$ as in Definition 13.4.13. Then $\delta\Phi\mathbf{x} = \Sigma\mathbf{x}$ in all cases. Hence, if t is a thin element of G, then $\Sigma\boldsymbol{\partial} t = 0$.

Proof The case $n = 1$ is trivial, so we assume $n \geqslant 2$. First, notice

$$\begin{aligned} \delta\Phi\mathbf{x} &= \Phi\delta\Phi\mathbf{x} && (\text{because} \quad \delta\Phi\mathbf{x} \in C_n) \\ &= (\Phi\partial_1^- \Phi\mathbf{x})^{u_1 \Phi\mathbf{x}} && (\text{because} \quad u_1 \Phi\mathbf{x} = \varepsilon_1 \beta\mathbf{x}) \\ &= \Sigma\Phi\mathbf{x}. \end{aligned}$$

So, we have to prove $\Sigma\Phi\mathbf{x} = \Sigma\mathbf{x}$. It is enough to show that $\Sigma\Phi_j\mathbf{x} = \Sigma\mathbf{x}$ for $j = 1, 2, \cdots, n$.

To prove that $\Sigma\Phi_j\mathbf{x} = \Sigma\mathbf{x}$, put $\mathbf{y} = \Phi_j\mathbf{x}$ (for fixed j). By Proposition 13.4.2, we have

$$\begin{aligned} y_i^\alpha &= \begin{cases} \Phi_{j-1} x_i^\alpha & (i < j), \\ \Phi_j x_i^\alpha & (i > j+1); \end{cases} \\ y_{j+1}^\alpha &= y_j^+ = \eta_j^+ x_j^+; \\ y_j^- &= [-x_j^+, -x_{j+1}^-, x_j^-, x_{j+1}^+]_j. \end{aligned}$$

Hence, by Proposition 13.4.15 and Proposition 13.4.18,

$$\begin{aligned} \Phi y_i^\alpha &= \Phi x_i^\alpha (i \neq j, j+1), \\ \Phi y_{j+1}^\alpha &= \Phi y_j^+ = 0, \\ \Phi y_j^- &= \Phi[-x_j^+, -x_{j+1}^-, x_j^-, x_{j+1}^+]_j. \end{aligned} \tag{*}$$

We write $a_j = [-x_j^+, -x_{j+1}^-, x_j^-, x_{j+1}^+]_j$ and use Proposition 13.4.14 to compute Φa_j.

First we study the case $(n, j) \neq (2, 1)$. Then

$$\Phi a_j = -(\Phi x_j^+)^{p_j} - (\Phi x_{j+1}^-)^{q_j} + (\Phi x_j^-)^{r_j} + \Phi x_{j+1}^+,$$

where $p_j = u_j a_j$, $q_j = u_j[x_j^+, a_j]_j$, $r_j = u_j x_{j+1}^+$. By the relations in Definition 13.1.7, u_j is a morphism of groupoids from $(G_n, +_j)$ to $(G_1, +)$ so $p_j = -u_j x_j^+ - u_j x_{j+1}^- + u_j x_j^- + u_j x_{j+1}^+$ in G_1, and $q_j = u_j x_j^+ + p_j$. The four terms of p_j are the edges of the square $s_j = \partial_1^+ \partial_2^+ \cdots \widehat{j\,j+1} \cdots \partial_n^+ \mathbf{x}$; hence $p_j = \Sigma\boldsymbol{\partial} s_j = \delta\Phi s_j$. Also $u_j x_j^+ = u_{j+1}\mathbf{x}$ and $u_j x_{j+1}^+ = u_j\mathbf{x}$, so

$$\Phi y_j^- = \Phi a_j = -(\Phi x_j^+)^{\delta\Phi s_j} - (\Phi x_{j+1}^-)^{u_{j+1}\mathbf{x} + \delta\Phi s_j} + (\Phi x_j^-)^{u_j \mathbf{x}} + \Phi x_{j+1}^+. \tag{**}$$

We have to differentiate two subcases.

If $n \geqslant 3$ then $\delta\Phi s_j$ acts trivially on C_n, since $C = \gamma G$ is a crossed complex, and addition is commutative. Hence by (*),

$$\begin{aligned} \Sigma\mathbf{y} &= \sum_{i=1}^{n} (-1)^i \{\Phi y_i^+ - (\Phi y_i^-)^{u_i \mathbf{y}}\} \\ &= \sum_{i \neq j, j+1} (-1)^i \{\Phi x_i^+ - (\Phi x_i^-)^{u_i \Phi_j \mathbf{x}}\} + (-1)^{j+1} (\Phi y_j^-)^{u_j \Phi_j \mathbf{x}}. \end{aligned}$$

But $u_i \Phi_j \mathbf{x} = u_i \mathbf{x}$ if $i \neq j, i \neq j+1$; and $u_j \Phi_j \mathbf{x} = 0$; so substituting from (**) we find $\Sigma \mathbf{y} = \Sigma \mathbf{x}$.

If $n = 2$ and $j = 2$ then $s_2 = \partial_1^+ \mathbf{x} = x_1^+$, and $\delta \Phi s_2 = \delta \Phi x_1^+$ acts on C_2 by $a^{\delta \Phi s_2} = -\Phi x_1^+ + a + \Phi x_1^+$. Hence (**) becomes

$$\Phi y_2^- = -\Phi x_1^+ - \Phi x_2^+ - (\Phi x_3^-)^{u_3 \mathbf{x}} + \Phi x_1^+ + (\Phi x_2^-)^{u_2 \mathbf{x}} + \Phi x_3^+$$

which, together with (*), gives

$$\begin{aligned}\Sigma \mathbf{y} &= -\Phi y_3^+ - (\Phi y_2^-)^{u_2 \Phi_2 \mathbf{x}} - \Phi y_1^+ + (\Phi y_3^-)^{u_3 \Phi_2 \mathbf{x}} + \Phi y_2^+ + (\Phi y_1^-)^{u_1 \Phi_2 \mathbf{x}} \\ &= 0 - \Phi y_2^- - \Phi x_1^+ + 0 + 0 + (\Phi x_1^-)^{u_1 \mathbf{x}} \\ &= \Sigma \mathbf{x}.\end{aligned}$$

Finally, in the case $n = 2, j = 1$, we have

$$\begin{aligned}\Phi y_1^- &= \Phi[-x_1^+, -x_2^-, x_1^-, x_2^+]_1 \\ &= \Phi x_2^+ + (\Phi x_1^-)^{r_1} - (\Phi x_2^-)^{q_1} - (\Phi x_1^+)^{p_1}\end{aligned}$$

by Proposition 13.4.14, where p_1, q_1, r_1 are as defined above. As in the previous cases, this gives

$$\Phi y_1^- = \Phi x_2^+ + (\Phi x_1^-)^{u_1 \mathbf{x}} - \Phi x_3^+ - (\Phi x_2^-)^{u_2 \mathbf{x}} - \Phi x_1^+ + \Phi x_3^+$$

and hence

$$\Sigma \mathbf{y} = -\Phi x_3^+ + (\Phi x_3^-)^{u_3 \mathbf{x}} + \Phi x_2^+ + (\Phi x_1^-)^{u_1 \mathbf{x}} - \Phi x_3^+ - \Phi (x_2^-)^{u_2 \mathbf{x}} - \Phi x_1^+ + \Phi x_3^+.$$

Writing $b = (\Phi x_3^-)^{u_3 \mathbf{x}} + \Phi x_2^+ + (\Phi x_1^-)^{u_1 \mathbf{x}} - \Phi x_3^+$ and $c = -(\Phi x_2^-)^{u_2 \mathbf{x}} - \Phi x_1^+$, it can be verified that $\delta b = -\delta c$, and hence, by the crossed module laws, $b + c = c + b^{\delta c} = c + b^{-\delta b} = c + b$. It follows easily that $\Sigma \mathbf{y} = \Sigma \mathbf{x}$, as required.

To prove the last statement, if t is thin, then $\Sigma \boldsymbol{\partial} t = \delta \Phi \boldsymbol{\partial} t = \partial_1^- \Phi t = 0$ by Lemma 13.5.9 and Proposition 13.4.18. □

Remark 13.7.2 The element $\Sigma \mathbf{x}$ in the case $n = 2$ is in the centre of $C_2(\beta \mathbf{x})$, because conjugation by $\Sigma \mathbf{x} = \delta \Phi \mathbf{x}$ is the same as action by $\delta \delta \Phi \mathbf{x} = 0$. Hence $\Sigma \mathbf{x}$ can be rewritten, for example, by permuting its terms cyclically. □

Proposition 13.7.3 *Let G be an ω-groupoid. Then each box in G has a unique thin filler. In particular, G is a Kan cubical set.*

Proof Let $\mathbf{y}$ be an n-box with missing (γ, k)-face. The result is trivial if $n = 0$, so we assume $n \geqslant 1$. By Corollary 13.5.12, it is enough to prove that there is a unique n-cube y_k^γ which closes the box $\mathbf{y}$ to form an n-shell $\overline{\mathbf{y}}$ with $\delta \Phi \overline{\mathbf{y}} = \Sigma \overline{\mathbf{y}} = 0$.

If $n \geqslant 2$, the edges of the given box $\mathbf{y}$ form the complete 1-skeleton of an $(n+1)$-cube; in particular, $\mathbf{y}$ determines the $n+1$ edges $w_i = u_i \mathbf{y}$ terminating at $\beta \mathbf{y}$. We write $F(s_i^\alpha)$ for the word in the indeterminates $s_i^\alpha (i = 1, 2, \cdots, n+1; \alpha = 0, 1)$ obtained from the formula for $\Sigma \mathbf{x}$ in Lemma 13.7.1 by substituting s_i^α for Φx_i^α and the given edges $w_i = u_i \mathbf{y}$ for $u_i \mathbf{x}$. If $n = 1$, then $F(s_i^\alpha) = -s_1^+ - s_2^- - s_1^- + s_2^+$ does not involve the w_i.

If we put $\mathbf{z}_i^\alpha = \boldsymbol{\partial} y_i^\alpha$ for $(\alpha, i) \neq (\gamma, k)$, then the $\mathbf{z}_i^\alpha$ form a box of $(n-1)$-shells, and there is a unique $(n-1)$-shell $\mathbf{z}_k^\gamma$ which closes this box to form an n-shell $\overline{\mathbf{z}} \in \square^2 G_{n-1}$. Since δ preserves addition and the action of the edges w_i, we find

$$F(\delta \Phi \mathbf{z}_i^\alpha) = \delta F(\Phi \mathbf{z}_i^\alpha) = \delta \Sigma \overline{\mathbf{z}} = \delta^2 \Phi \overline{\mathbf{z}} = 0. \qquad (*)$$

Next, put $\zeta_i^\alpha = \Phi y_i^\alpha$ for $(\alpha, i) \neq (\gamma, k)$ and let $\zeta_k^\gamma \in C_n$ be the unique element determined by the equation $F(\zeta_i^\alpha) = 0$. Then

$$\delta\zeta_i^\alpha = \delta\Phi y_i^\alpha = \delta\Phi \mathbf{z}_i^\alpha \text{ for } (\alpha, i) \neq (\gamma, k),$$

while

$$F(\delta\zeta_i^\alpha) = 0.$$

From these equations and (*) we deduce that $\delta\zeta_k^\gamma = \delta\Phi\mathbf{z}_k^\gamma$ also. Hence, by Proposition 13.5.11, there is a unique $y_k^\gamma \in G_n$ such that $\partial y_k^\gamma = \mathbf{z}_k^\gamma$ and $\Phi y_k^\gamma = \zeta_k^\gamma$; this y_k^γ completes the box $\mathbf{y}$ to form a shell $\overline{\mathbf{y}}$ with $\Sigma\overline{\mathbf{y}} = F(\zeta_i^\alpha) = 0$, as required. □

Proposition 13.7.4 *Let* t *be a thin element in an* ω*-groupoid. If all faces except one of* t *are thin, then the remaining face is also thin.*

Proof Let the faces of t be $t_i^\alpha (i = 1, 2, \ldots, n; \alpha = 0, 1)$. By Proposition 13.4.18, $\Phi t_i^\alpha = 0$ for $(\alpha, i) \neq (\gamma, k)$ say, so $\Sigma\partial t = \pm(\Phi t_k^\gamma)^w$ for some edge w of t. But t is thin so, by the Homotopy Addition Lemma 13.7.1, $\Sigma\partial t = 0$. Hence $\Phi t_k^\gamma = 0$ and t_k^γ is thin. □

The thin elements of an ω-groupoid have another property which is crucial in the proof of the HHvKT in the next chapter; it is used in proving Lemma 14.3.5 in page 396 to show that a constructed element of an ω-groupoid is independent of the choices in the construction. It is also used to relate the fundamental ω-groupoid ρX_* and fundamental crossed complex ΠX_* of a filtered space (Proposition 14.5.1).

Proposition 13.7.5 *Let* G *be an* ω*-groupoid and* x *a thin element of* G_{n+1}*. Suppose that for* $m = 1, \cdots, n$ *and each face operator* $d : G_{n+1} \to G_m$ *not involving*[1] ∂_{n+1}^- *or* ∂_{n+1}^+*, the element* dx *is thin. Then* $x = \varepsilon_{n+1}\partial_{n+1}^- x$ *and hence*

$$\partial_{n+1}^- x = \partial_{n+1}^+ x.$$

Proof The proof is by induction on n, the case $n = 0$ being trivial since a thin element in G_1 is degenerate.

The inductive assumption thus implies that every face $\partial_i^\tau x$ with $i \neq n+1$ is of the form $\varepsilon_n \partial_n^- \partial_i^\tau x$. So the box consisting of all faces of x except $\partial_{n+1}^+ x$ is filled not only by x but also by $\varepsilon_{n+1}\partial_{n+1}^- x$. Since a box in G has a unique thin filler (Proposition 13.7.3), it follows that $x = \varepsilon_{n+1}\partial_{n+1}^- x$. □

Remark 13.7.6 The properties of thin elements in Propositions 13.7.3 and 13.7.4, together with the fact that degenerate cubes are thin, can be taken as axioms for 'cubical T-complexes' or 'cubical sets with thin elements'. (The definition was first given by Dakin [Dak83] in the simplicial case.) Precisely, a (*cubical*) *T-complex* is a cubical set with a distinguished set of elements called 'thin', satisfying:
(i) all degenerate cubes are thin;
(ii) every box has a unique thin filler;
(iii) if a thin cube has all faces except one thin then the last face is also thin.

We have shown that every ω-groupoid is a T-complex, and it is a remarkable fact (see [BH81c, BH81b]) that the converse is also true: all the ω-groupoid structure can be recovered from the set of thin elements using these three assumptions. Thus the category of cubical T-complexes is equivalent (in fact isomorphic) to the category of ω-groupoids; it is therefore, by 13.6.2, equivalent to the category of crossed complexes. Ashley has shown [Ash88] that the category of simplicial

[1] A cubical face operator d is simply a product of various ∂_j^τs. This product may be empty, so that we allow $d = 1$. We say d does not involve ∂_{n+1}^τ if d cannot be written as $d'\partial_{n+1}^\tau$.

T-complexes is also equivalent to the category of crossed complexes. He has also shown that this result generalises the theorem of Dold and Kan [Dol58, Kan58, May67] which gives an equivalence between the category of simplicial abelian groups and the category of chain complexes; the T-complex structure on a simplicial abelian group is obtained by defining the thin elements to be sums of degenerate elements. For more information on the cubical case, see also [BH03]. □

Remark 13.7.7 If G is any ω-groupoid, we may define the fundamental groupoid $\pi_1 G$ and the homotopy groups $\pi_n(G,p)(p \in G_0, n \geqslant 2)$ as follows. For $a, b \in G_1(p,q)$, define $a \sim b$ if there exists $c \in G_2$ such that $\partial_1^- c = a$, $\partial_1^+ c = b$, $\partial_2^- c = \varepsilon_1 p$, $\partial_2^+ c = \varepsilon_1 q$. Then $\sim$ is a congruence relation on G_1 and we define $\pi_1 G = G_1/\sim$. For $n \geqslant 2$ and $p \in G_0$, let $Z_n(G,p) = \{x \in G_n; \partial_1^\alpha x = \varepsilon_1^{n-1} p$ for all $(\alpha, i)\}$. Then the $+_i (i = 1, 2, \cdots, n)$ induce on $Z_n(G,p)$ the same Abelian group structure. Two elements x, y of $Z_n(G,p)$ are *homotopic*, $x \sim y$, if there exists $h \in G_{n+1}$ such that $\partial_{n+1}^- h = x$, $\partial_{n+1}^+ h = y$ and $\partial_i^\alpha h = \varepsilon_1^n p$ for $i \neq n+1$. This is a congruence relation on $Z_n(G,p)$ and we define $\pi_n(G,p)$ to be the quotient group $Z_n(G,p)/\sim$.

Now G is a Kan cubical set, by Proposition 13.7.3, so, there is a standard procedure suggested in Proposition 10.3.26 and in that subsection 10.3.3, for defining $\pi_1 G$ and $\pi_n(G,p)$, without using the compositions $+_i$. As sets they coincide with the definitions above, but their groupoid and group structures are defined by a procedure using only the properties of Kan fillers. □

It is not hard to see that the special properties of thin fillers in G ensure that the groupoid and group structures obtained in this way coincide with those induced by the compositions $+_i$.

It is also clear that the groupoid $\pi_1 G$ and the groups $\pi_n(G,p)$ coincide with the *fundamental groupoid* $\pi_1 \gamma G$ and the *homology groups* $H_n(\gamma G, p)$ of the crossed complex γG (see the definitions in Subsection 7.1.4.

We will later need the following result.

Proposition 13.7.8 *Let G, H be ω-groupoids and let $f : G \to H$ be a morphism of the underlying cubical sets with connections which also preserves the thin structures. Then f is a morphism of ω-groupoids.*

Proof This involves the fact that the compositions can be recovered from the thin structures, which is the main result of [BH81c], showing the equivalence of cubical T-complexes and ω-groupoids. In our terms, this can be shown as follows.

In Proposition 10.3.26 we showed how the fundamental groupoid of a Kan cubical set can be defined. In the case of a cubical T-complex K, with unique Kan fillers, this method actually determines a groupoid structure $+_1$ on K_1.

By using the functor $P^n K$ we can similarly get a composition, and in fact a groupoid structure, $+_n$ derived from the T-structure.

However we showed in Chapter 6, that double groupoids with connection admit rotations, which exchange the two groupoid structures in dimension 2 (Proposition 6.4.4. In higher dimensions, this argument gives an operation of the symmetry group S_n in dimension n of an ω-groupoid G, interchanging the operations $+_i$. Hence any addition $+_i$ is determined by the thin structure.

The same argument applies to H and hence f preserves the compositions $+_i$. □

Remark 13.7.9 It will be shown in Remark 14.6.3 that the crossed complex ΠI^n_* has one generator for each cell of I^n, with defining relations given by the Homotopy Addition Lemma 9.9.4. The corresponding statement for the ω-groupoid ρI^n_* is that it is the free ω-groupoid on a single generator in dimension n; this is the subject of Section **??**, as part of the description of the free ω-groupoid on a cubical set (Proposition 14.6.2). [This remark will need to be examined later after revision of the colim chapter.] □

13.8 Notes

Cubical sets with this, and other, structures have also been considered by Évrard [É]. See also M. Grandis and L.Mauri, [GM03]. This paper deals with normal forms for cubical sets with connections.

Chapter 14

The cubical homotopy ω-groupoid of a filtered space

This chapter contains the construction and applications of the cubical higher homotopy groupoid ρX_* of a filtered space X_*. Without the idea for this construction, the major results of this book would not have been conjectured, let alone proved.

The definition of ρX_* as a cubical set with connections is easy: it is a quotient of RX_*, the filtered cubical singular complex of X_*, by the relation of *filter homotopy rel vertices*. The difficult part is to prove that the compositions on RX_* are inherited by ρX_*, so that it becomes an ω-groupoid: the proof is a generalisation of that in dimension 2, but needs an organisation of the collapsing of cubes necessary to fill some holes starting in low dimensions. It is remarkable that there is exactly enough room to fill these holes as required. This gives one confidence in the correctness of the definitions.

These collapsings were already introduced in subsection 10.3.1 they also enable a proof of a key result, the fibration Theorem 14.2.7 stating that the projection $p : RX_* \to \rho X_*$ is a Kan fibration of cubical sets. Some more precise properties of this fibration are a key to later results. For example, since ρX_* is an ω-groupoid, it has a notion of thin element: these we call *algebraically thin*. There is also a notion of a *geometrically thin*, or deficient, element of $(\rho X_*)_n$, namely those that have a representative $f : I^n_* \to X_*$ such that $f(I^n) \subseteq X_{n-1}$. The precise fibration theorem implies these two notions coincide (see Theorem 14.2.9).

The main part of this chapter gives proofs of a Higher Homotopy van Kampen Theorems (HHvKT) both for ω-groupoids (Section 14.3) and for crossed complexes (Section 14.4). In Section 14.3.1 we prove the result for ω-groupoids. It shows, in succinct terms, that the functor ρ preserves certain colimits of connected filtered spaces.

The proof of the Higher Homotopy van Kampen Theorem (HHvKT) for ω-groupoids (Section 14.3) follows the same structure as the proof of the van Kampen theorem in dimension 1 and 2, given in Parts I and II. It goes as follows.

Let $\mathcal{U} = \{U^\lambda\}_{\lambda\in\Lambda}$ be an open cover of a space X that is filtered X_* and assume that for any finite intersection of elements of $\mathcal{U}$, the induced filtration is connected. The theorem say us that in the induced diagram

$$\bigsqcup_{\nu\in\Lambda^2} \rho U^\nu_* \underset{i_2}{\overset{i_1}{\rightrightarrows}} \bigsqcup_{\lambda\in\Lambda} \rho U^\lambda_* \xrightarrow{i} \rho X_*$$

i is the coequaliser of i_1 and i_2 in the category of ω-groupoids.

To prove the universality condition of the coequaliser, we show that for any ω-groupoid G and morphism of ω-groupoids

$$f : \bigsqcup_{\lambda \in \Lambda} \rho U^{\lambda}_{*} \to G$$

which is compatible with the double intersections (i.e. $fi_1 = fi_2$), there is a unique morphism

$$f' : \rho X_* \to G$$

such that $f'i = f$. This morphism f' is constructed by using choices to construct a map

$$F : RX_* \to G$$

following the same pattern as in the proof of Theorem 6.8.2.

For any element $\alpha \in R_n(X_*)$ we choose a subdivision $\alpha = [\alpha_{(r)}]$ such that each $\alpha_{(r)}$ lies in some element $U^{\lambda(r)} \in \mathcal{U}$. The connectivity conditions imply, as in the 2-dimensional case, that there is are elements $\theta^{\lambda} \in R_n(X_*)$ and a filtered homotopy $h : \alpha \equiv \theta$ such that in the subdivision given by α we have $h_{(r)} : \alpha_{(r)} \equiv \theta_{(r)}$, $\theta_{(r)} \in R_n X_*$ and $h_{(r)}$ lies in $U^{\lambda(r)}$. We define

$$F(\alpha) = [f^{\lambda(r)} \theta_{(r)}]$$

the composite of the array.

The central part of the proof is to show that F is well defined up to homotopy. Here we diverge from the proof of the theorem in dimension 2. There, the Homotopy Addition Lemma was used in dimension 2 to see that any composition of commuting 3-cubes is also a commuting 3-cubes. In higher dimensions, the "commuting n-cubes" are replaced by the thin elements defined in 13.4.17. The geometric characterisation of thin elements already stated is crucial in the proof.

The Higher Homotopy van Kampen Theorem (HHvKT) for crossed complexes (Section 14.3) follows from the HHvKT for ω-groupoids using the equivalence of categories

$$\gamma : \omega\text{-}\mathsf{Gpds} \to \mathsf{Crs}$$

from the category of ω-groupoids to the category of crossed complexes in Section 13.6. In the case of the ω-groupoid ρX_*, we prove that $\gamma\rho X_*$ is naturally isomorphic to the fundamental crossed complex ΠX_* of the filtered space X_* (Theorem 14.4.1). This isomorphism gives the Higher Homotopy van Kampen Theorem (HHvKT) for fundamental crossed complexes (8.1.5) whose applications have been described in Chapter 8, assuming the Higher Homotopy van Kampen Theorem for the ω-groupoid ρX_*.

Section 14.5 shows that every ω-groupoid and every crossed complex arise (up to isomorphism) from our functors from filtered spaces. This shows our axioms for these structures to be optimal. These results are (surprisingly) used in section 14.6 to show that the functor ρ from cubical sets to ω-groupoids is left adjoint to the forgetful functor, and so gives the free ω-groupoid on a cubical set. This result is very useful for our next chapter on monoidal closed structures.

Section 14.7 gives a final link with classical results by showing how these methods help in proving the classical Absolute Hurewicz Theorem, and also relate to an exact sequence of J.H.C. Whitehead which includes the Hurewicz morphism from homotopy to homology. This exact sequence is necessary for applications of our fundamental homotopy classification of maps to the classifying space of a crossed complex, since it gives useful conditions for a space Y to be of the homotopy type of BC for some crossed complex C.

14.1 Construction of the homotopy ω-groupoid of a filtered space

Recall that it is natural to associate to a filtered space X_* its filtered singular cubical set RX_* which in dimension n is the set of filtered maps $I^n_* \to X_*$. The aim is to define a homotopy relation on RX_* which gives an ω-groupoid ρX_* whose associated crossed complex is exactly ΠX_* as used in Part II. The fact that the 0-dimensional part of ΠX_* is just X_0 suggests that the homotopy relation we require is *filter homotopy rel vertices*. It turns out that this works: the 'rel vertices' condition is enough to start the inductive constructions required to prove the compositions on RX_* are inherited by ρX_* .

Definition 14.1.1 Two elements $\alpha, \beta \in R_n X_*$ are *filter homotopic rel vertices* if there is a filter homotopy $f : X \times I \to Y$ from α to β rel vertices, i.e. a homotopy such that $f(X_s \times I) \subseteq Y_s, s = 0, 1, 2, \cdots$, and which is constant on every vertex of I^n.

The set of equivalence classes of elements of $R_n X_*$ under filter-homotopy rel vertices is written $\rho_n X_*$, and the class of $\alpha \in R_n X$ is written $\langle\!\langle \alpha \rangle\!\rangle$. So we have a quotient map $p : RX_* \to \rho_n X_*$. □

It is easy to check that the connections and the face and degeneracy maps of RX_* are inherited by ρX_*, giving it the structure of cubical complex with connections. Let us consider the compositions.

Definition 14.1.2 A *composition* $+_i$ on $\rho_n X_*$ is defined as follows.

Let $\langle\!\langle \alpha \rangle\!\rangle, \langle\!\langle \beta \rangle\!\rangle \in \rho_n X_*$ satisfy $\partial_i^+ \langle\!\langle \alpha \rangle\!\rangle = \partial_i^- \langle\!\langle \beta \rangle\!\rangle$. Then $\partial_i^+ \alpha \equiv \partial_i^- \beta$, so we may choose $h : I^n \to X$, a filter-homotopy in the ith direction, so that $\gamma = [\alpha, h, \beta]_i$ is defined in $R_n X_*$. We let

$$\langle\!\langle \alpha \rangle\!\rangle +_i \langle\!\langle \beta \rangle\!\rangle = \langle\!\langle [\alpha, h, \beta]_i \rangle\!\rangle$$

and prove this composition well-defined. □

To see that the compositions also are well defined in ρX_* we need to check that they are independent of the choices of representatives. The proof requires 'filling a hole' to get a filter homotopy. This is done inductively. In dimension 0, we can fill a map $\dot{I}^2_* \to X_*$ because the homotopy we are using is rel vertices so the map is in fact constant; we then proceed by induction using a retraction procedure based on the collapsings introduced in subsection 10.3.1.

Definition 14.1.3 [should the $\times$ here be $\otimes$?] Let B be a subcomplex of I^n, let $m \geqslant 2$, and let $B \times I^m$ be given the product cell structure, so that the skeletal filtration gives a filtered space $B_* \times I^m_*$. Let

$$h : B \times I^m \to X$$

be a map. Fixing the ith coordinate of I^m at the value t, where $0 \leqslant t \leqslant 1$, we obtain a map

$$\partial_i^t h : B \times I^{m-1} \to X.$$

If X_* is a filtered space, and $\partial_i^t h : B_* \times I^{m-1}_* \to X_*$ is a filtered map for each $0 \leqslant t \leqslant 1$, we say h is a *filter-homotopy in the* ith *direction of* I^m. In such case we write $h : \alpha \equiv_i \beta$ where $\alpha = \partial_i^- h, \beta = \partial_i^+ h$. It is easy to see that the relation $\equiv_i$ defined on filtered maps $B \times I^{m-1} \to X$ by the existence of such an h is an equivalence relation independent of $i, 1 \leqslant i \leqslant m$. □

Definition 14.1.4 A map $h : B_* \times I^2_* \to X_*$ is called a *filter-double-homotopy* if it is a filter-homotopy in each of the two directions of I^2; this is equivalent to $h(B_s \times I^2) \subseteq X_{s+1}, h(B_s \times \partial I^2) \subseteq X_s, s = 0, 1, 2, \cdots$. (If K is a *proper* subcomplex of I^2, and $k : B \times K \to X$ satisfies $k(B_s \times K) \subseteq X_s, s = 0, 1, 2, \cdots$, then by an abuse of language we call k also a filter-double-homotopy). □

Consider now a filtered space X_*.

Proposition 14.1.5 *Let* B, C *be subcomplexes of* I^n *such that* $B \searrow C$. *Let*

$$f : B \times \partial I^2 \to X, \qquad g : C \times I^2 \to X$$

be two filter-double homotopies which agree on $C \times \partial I^2$. *Then their union* $f \cup g$ *extends to a filter-double-homotopy* $h : B \times I^2 \to X$.

Proof It is sufficient to consider the case of an elementary collapse $B \searrow^e C$. Suppose then $B = C \cup a$, $C \cap a = \partial a \setminus b$, where a is an s-cell and b is an $(s-1)$-face of a.

Let $r : a \times I^2 \to (a \times \partial I^2) \cup ((\partial a \setminus b) \times I^2)$ be a retraction. Then r defines an extension $h : B \times I^2 \to X$ of $f \cup g$. Since f is a filter-double-homotopy,

$$h(a \times \partial I^2) = f(a \times \partial I^2) \subseteq X_s,$$

and since g is a filter-double-homotopy

$$h((\partial a \setminus b) \times I^2) = g((\partial a \setminus b) \times I^2) \subseteq X_s.$$

Hence $h(a \times I^2) \subseteq X_s$, and in particular $h(b \times I^2) \subseteq X_s$. These conditions, with those on f and g, imply that h is a filter-double-homotopy. □

Corollary 14.1.6 *Let* X_* *be a filtered space and let* B *be a subcomplex of* I^n *such that* B *collapses to one of its vertices. Then any filter-double-homotopy rel vertices* $f : B_* \times \partial I^2_* \to X_*$ *extends to a filter-double-homotopy rel vertices* $h : B_* \times I^2_* \to X_*$.

Proof Let v be a vertex of B such that $B \searrow \{v\}$. Now $f(\{v\} \times \partial I^2) \subseteq X_0$. Since the homotopies are rel vertices, $f|_{\{v\} \times \partial I^2}$ extends to a constant map $g : \{v\} \times I^2 \to X$ with image in X_0. Thus g is a filter-double-homotopy. By Proposition 14.1.5, $f \cup g$ extends to a filter-double-homotopy $h : B \times I^2 \to X$. □

We now show that the compositions in RX_* are inherited by the quotient to give ρX_* the structure of ω-groupoid. This gives us the definition of the fundamental homotopy groupoid of a filtered space.

Theorem 14.1.7 *If* X_* *is a filtered space, then the compositions on* RX_* *induce compositions on* ρX_* *which, together with the induced face and degeneracy maps and connections, give* ρX_* *the structure of* ω*-groupoid.*

Proof We have to prove that the definition of the composition $+_i$ given in Definition 14.1.2 is independent of the representatives. For this it is sufficient, by symmetry, to suppose $i = n$.

To prove independence of choices, let $\alpha' \in \langle\langle \alpha \rangle\rangle$ and $\beta' \in \langle\langle \beta \rangle\rangle$ be alternative choices. As before let $h' : \partial_i^+ \alpha' \equiv \partial_i^- \beta'$. If we define

$$\gamma' = [\alpha', h', \beta']_n,$$

we have got to prove that

$$\langle\langle \gamma \rangle\rangle = \langle\langle \gamma' \rangle\rangle.$$

By construction there exist filter-homotopies

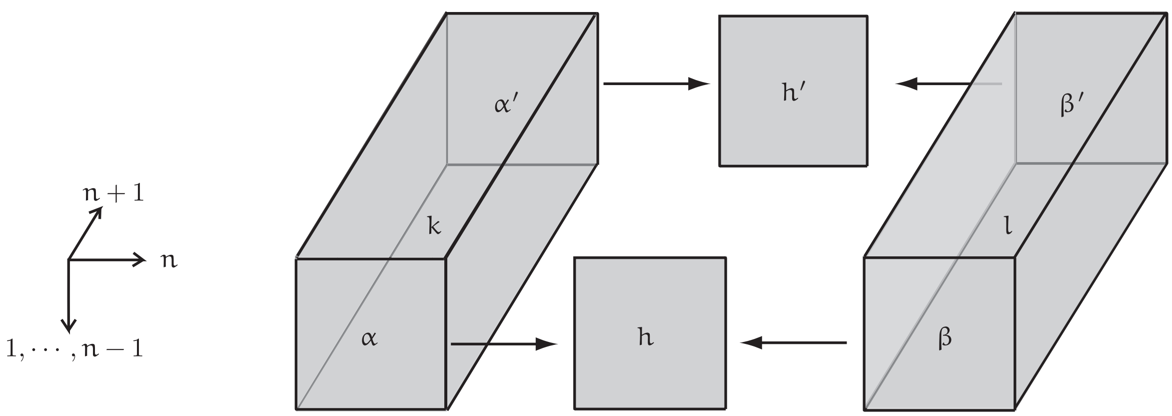
$$k : \alpha \equiv \alpha' \quad l : \beta \equiv \beta'$$

in the $(n+1)$st direction.

We view I^{n+1} as a product $I^{n-1} \times I^2$ and define a filter-double-homotopy rel vertices

$$f : I^{n-1} \times \partial I^2 \to X$$

by $f(x,t,0) = h(x,t)$, $f(x,t,1) = h'(x,t)$, $f(x,0,t) = k(x,1,t)$, $f(x,1,t) = l(x,0,t)$, where $x \in I^{n-1}$ and $t \in I$. The following picture illustrates the situation. [Can the constant edges be emphasised?]



By Corollary 14.1.6 with $B = I^{n-1}$, f extends to a filter-double-homotopy

$$H : I^{n-1} \times I^2 \to X.$$

Then $[k, H, l]_n$ is well defined and is a filter-homotopy $\gamma \equiv \gamma'$. This completes the proof that $+_n$, and by symmetry $+_i$, is well defined.

Suppose now that $\alpha +_i \beta$ is defined in $R_n X_*$. Let $h : \partial_i^+ \alpha \equiv_i \partial_i^- \beta$ be the constant filter-homotopy in the ith direction. Then $\alpha +_i \beta$ is a filter-homotopic to $[\alpha, h, \beta]_i$ and so $\langle\!\langle \alpha +_i \beta \rangle\!\rangle = \langle\!\langle \alpha \rangle\!\rangle +_i \langle\!\langle \beta \rangle\!\rangle$. Thus the operations $+_i$ on $\rho_n X_*$ are induced by those on $R_n X_*$ in the usual algebraic sense.

Further, if $\langle\!\langle \alpha \rangle\!\rangle +_i \langle\!\langle \beta \rangle\!\rangle$ is defined in $\rho_n X_*$, then we may choose representatives α', β' of $\langle\!\langle \alpha \rangle\!\rangle$, $\langle\!\langle \beta \rangle\!\rangle$ such that $\alpha' +_i \beta'$ is defined and represents $\langle\!\langle \alpha \rangle\!\rangle +_i \langle\!\langle \beta \rangle\!\rangle$ (for example we may take $\alpha' = \alpha$, $\beta' = h +_i \beta'$ where $h : \partial_i^+ \alpha \equiv_i \partial_i^- \beta$).

Defining $-_i \langle\!\langle \alpha \rangle\!\rangle = \langle\!\langle -_i \alpha \rangle\!\rangle$, one easily checks that $+_i$ and $-_i$ make $\rho_n X_*$ a groupoid with initial, final and identity maps ∂_i^-, ∂_i^+ and ε_i.

The laws for ε_j, ∂_j^τ, Γ_j of a composite $\langle\!\langle \alpha \rangle\!\rangle +_i \langle\!\langle \beta \rangle\!\rangle$ follow from the laws in $R_n X_*$ by choosing the representatives α, β so that $\alpha +_i \beta$ is defined.

Finally, we must verify the interchange law for $+_i$, $+_j$ $(i \neq j)$. By symmetry, it is sufficient to assume $i = n-1$, $j = n$.

Suppose that $\langle\!\langle \alpha \rangle\!\rangle +_{n-1} \langle\!\langle \beta \rangle\!\rangle$, $\langle\!\langle \gamma \rangle\!\rangle +_{n-1} \langle\!\langle \delta \rangle\!\rangle$, $\langle\!\langle \alpha \rangle\!\rangle +_n \langle\!\langle \gamma \rangle\!\rangle$, $\langle\!\langle \beta \rangle\!\rangle +_n \langle\!\langle \delta \rangle\!\rangle$ are defined in $\rho_n X_*$. We choose the representatives $\alpha, \beta, \gamma, \delta$ and are going to construct in $R_n X_*$ a composite

$$\begin{bmatrix} \alpha & k & \gamma \\ h & H & h' \\ \beta & k' & \delta \end{bmatrix}_{n-1,n} \qquad (*)$$

in which the filter-homotopies h, h' in the $(n-1)$st direction and the filter-homotopies k, k' in the nth direction already exist, because the appropriate composites are defined and H has to be defined (we are 'filling the hole').

To construct H, we define a filter-double-homotopy

$$f : I^{n-2} \times \partial I^2 \to X$$

by $f(x, 0, t) = k(x, 1, t)$, $f(x, 1, t) = k'(x, 0, t)$, $f(x, t, 0) = h(x, t, 1)$, $f(x, t, 1) = h'(x, t, 0)$ where $x \in I^{n-2}$, and $t \in I$. By Corollary 14.1.6, f extends to a filter-double-homotopy

$$H : I^{n-2} \times I^2 \to X.$$

Then the composite (*) is defined in $R_n X$ and the interchange law

$$(\langle\langle\alpha\rangle\rangle +_{n-1} \langle\langle\beta\rangle\rangle) +_n (\langle\langle\gamma\rangle\rangle +_{n-1} \langle\langle\delta\rangle\rangle) = (\langle\langle\alpha\rangle\rangle +_n \langle\langle\gamma\rangle\rangle) +_{n-1} (\langle\langle\beta\rangle\rangle +_n \langle\langle\delta\rangle\rangle)$$

is readily deduced by evaluating (*) in two ways.

This completes the proof that ρX_* is an ω-groupoid. $\square$

Definition 14.1.8 We call ρX_* the *homotopy ω-groupoid* (or the *fundamental ω-groupoid*) of the filtered space X_*.

A filtered map $f : X_* \to Y_*$ clearly defines a map $Rf : RX_* \to RY_*$ of cubical complexes with connections and compositions, and a map $\rho f : \rho X_* \to \rho X_*$ of ω-groupoids. So we have a functor

$$\rho : \mathsf{FTop} \to \omega\text{-}\mathsf{Gpds}. \qquad \square$$

The behaviour of ρ with regard to filtered homotopies will be studied in the next chapter. At this stage we can use standard results in homotopy theory to prove:

Proposition 14.1.9 *Let $f : X_* \to Y_*$ be a filtered map of filtered spaces such that each $f_n : X_n \to Y_n$ is a homotopy equivalence. Then $\rho f : \rho X_* \to \rho Y_*$ is an isomorphism of ω-groupoids.*

Proof This is immediate from [tDKP70, 10.11]. The background to this is discussed in the Notes. $\square$

14.2 The fibration and deformation theorems

In this Section we provide all the technical results on extensions of filtered homotopies needed for the further development of the theory.

The main result is the Deformation Theorem 14.2.5 which explains how to extend a filtered homotopy from a special kind of subcomplex $B \subseteq I^n$ to the full I^n.

To get this Deformation Theorem we use some consequences of Corollary 14.1.6 about construction of filter-double homotopies. Particularly useful is the filter homotopy extension property of Proposition 14.2.4. Also important is the method of elementary collapses already seen at work in Proposition 14.1.5.

We finish the Section with some consequences of the deformation Theorem, the most used [I am not sure it is the most used, but it is intuitively important. Maybe it is the fibration theorem which is most important.] being the possibility of lifting arrays of homotopy classes of filtered maps to arrays of maps.

Let us begin with consequences of Corollary 14.1.6.

Proposition 14.2.1 *Let* $B \subseteq A$ *be subcomplexes of* I^n *such that* B *collapses to one of its vertices. Let* X_* *be a filtered space. Let* $\alpha, \beta : A_* \to X_*$ *be filtered maps and let*

$$\psi : \alpha \equiv \beta, \quad \phi : \alpha|_B \equiv \beta|_B$$

be filter homotopies rel vertices. Then there is a filter-double-homotopy

$$H : A \times I^2 \to X$$

such that H *is a homotopy rel end maps of* ψ *to a filter-homotopy*

$$H_1 : \alpha \equiv \beta$$

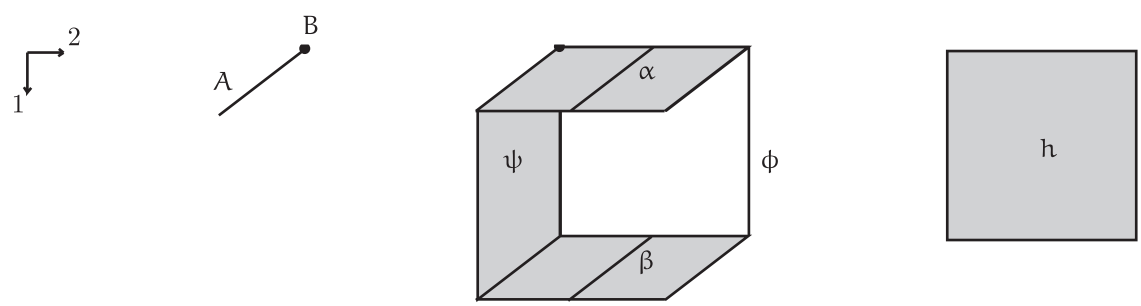
extending ϕ.

Proof Let $L = (I \times \{0\}) \cup (\partial I \times I)$. Define

$$l : (A \times L) \cup (B \times I \times \{1\}) \to X$$

by $l(x, t, 0) = \psi(x, t), l(x, 0, t) = \alpha(x), l(x, 1, t) = \beta(x), l(y, t, 1) = \phi(y, t), x \in A, y \in B, t \in I$. Then $f = l\,|_{B \times \partial I^2}$ and $k = l\,|_{A \times L}$ are filter-double-homotopies.

By Corollary 14.1.6, f extends to a filter-double homotopy $h : B \times I^2 \to X$.



[Can one mark in some way the edges on which the map is constant?]

We are going to extend the map

$$k \cup h : (A \times L) \cup (B \times I^2) \to X$$

to a filter-double-homotopy $H : A \times I^2 \to X$ by induction on the dimension of $A \setminus B$.

Suppose that H^s is a filter-double-homotopy defined on $(A \times L) \cup ((A^s \times B) \times I^2)$, extending $H^{-1} = k \cup h$. For each $(s+1)$-cell a of $A \setminus B$, choose a retraction

$$r_a : a \times I^2 \to (a \times L) \cup (\partial a \times I^2).$$

These retractions extend H^s to H^{s+1} defined also on $A^{s+1} \times I^2$. Since $r_a(a \times I^2) \subseteq X_{s+1}$, it follows that H^{s+1} is also a filter-double-homotopy.

Clearly $H = H^n$ is a filter-double-homotopy as required. □

Corollary 14.2.2 *Let* B, A, X_* *be as in Proposition 14.2.1. If* $\alpha, \beta : A_* \to X_*$ *are maps which are filter-homotopic rel vertices, then any filter-homotopy rel vertices* $\alpha|_B \equiv \beta|_B$ *extends to a filter-homotopy* $\alpha \equiv \beta$. □

We need to pay attention to the filtered maps which 'drop filtration' by at least one; we call these deficient.

Definition 14.2.3 If $f : Y_* \to X_*$ is a filtered map, where Y_* is a CW-complex with its skeletal filtration, we say that f is *deficient on a cell* a *of* Y if $\dim a = s$ but $f(a) \subseteq X_{s-1}$. In particular, a filtered map $I^n_* \to X_*$ is *deficient* if it is deficient on the top dimensional cell of I^n. □

[it is quite good to define 'dim' as an operator name]

Proposition 14.2.4 (filter-homotopy extension property) *Let* B, A *be subcomplexes of* I^n *such that* $B \subseteq A$. *Let*

$$f : A \times \{0\} \cup B \times I \to X$$

be a map such that $f|_{A\times\{0\}}$ *is a filtered map and* $f|_{B\times I}$ *is a filter-homotopy rel vertices. Then* f *extends to a filter-homotopy*

$$h : A \times I \to X.$$

Further, h *can be chosen so that if* f *is deficient on a cell* $a \times \{0\}$ *of* $(A \setminus B) \times \{0\}$, *then* h *is deficient on* $a \times \{1\}$.

Proof The proof of this proposition is an easy induction on the dimension of the cells of $A \setminus B$, using retractions $a \times I \to a \times \{0\} \cup \partial a \times I$ for each cell a of $A \setminus B$. □

Now we can proceed to the proof of the deformation theorem which is needed as a technical tool for the results of the next Section. The proof uses the results on partial boxes from section 10.3.1.

Theorem 14.2.5 (the deformation theorem) *Let* X_* *be a filtered space, and let* $\alpha \in R_n X_*$. *Any filtered map*

$$\gamma : B_* \to X_*$$

defined in a partial box $B \subseteq I^n$ *such that for each* $(n-1)$*-face* a *of* B, *the maps* $\alpha|_a$, $\gamma|_a$ *are filter-homotopic rel vertices has an extension to a filtered map*

$$\beta : I^n \to X$$

that is filter-homotopic to α.

Further, if α *is deficient (i.e.* $\alpha(I^n) \subseteq X_{n-1}$*), then* β *may be chosen to be deficient.*

Proof Let B_1 be any $(n-1)$-cell contained in B. We choose a chain $B = B_s \searrow B_{s-1} \searrow \cdots \searrow B_1$ of partial boxes and $(n-1)$-cells $a_1, a_2, \cdots, a_{s-1}$ as in Theorem 10.3.5.

We construct filter-homotopies $\phi_i : \alpha|_{B_i} \equiv \gamma|_{B_i}$ by induction on i, starting with ϕ_1 any filter-homotopy $\alpha|_{B_1} \equiv \gamma|_{B_1}$. Suppose ϕ_i has been constructed and extends ϕ_{i-1}. Then $\phi_i|_{(a_i \cap B_i)}$ is defined. Since $a_i \cap B_i$ is a partial box, it collapses to any of its vertices. Since $\alpha|_{a_i} \equiv \gamma|_{a_i}$, the homotopy $\phi_i|_{(a_i\cap B_i)}$ extends, by Corollary 14.2.2, to a filter-homotopy $\alpha|_{a_i} \equiv \gamma|_{a_i}$; this, with ϕ_i, defines ϕ_{i+1}.

Finally, we apply the filter-homotopy extension property (Proposition 14.2.4) to extend $\phi_s : \alpha|_B \equiv \gamma$ to a filter-homotopy $\alpha \equiv \beta$, for some β extending γ. The last part of Proposition 14.2.4 gives the final part of this Theorem. □

For some applications of the deformation theorem, it is convenient to work in the category of cubical sets. Recall that we write $\mathbb{I}^n$ for the free cubical set on one generator c^n of dimension n (See Definition 10.1.6). Then an element γ of dimension n of a cubical set C determines a unique cubical map $\hat{\gamma} : \mathbb{I}^n \to C$ such that $\hat{\gamma}(c^n) = \gamma$ (Proposition 10.1.7). As a useful abuse of notation we are going to 'drop the hat'.

In particular, a filtered map $\gamma : I^n_* \to X_*$ determines a unique cubical map $\gamma : \mathbb{I}^n \to RX_*$ such that $\gamma(c^n) = \gamma$. Also, if B is a subcomplex of the geometric n-cube B then B determines a cubical subset, also written B, of the cubical set $\mathbb{I}^n$, and a filtered map $\gamma : B_* \to X_*$ determines uniquely a cubical map $\hat{\gamma} : B \to RX_*$. The same may be said about homotopy classes of maps $[\gamma]$.

We can now rewrite the deformation theorem in the category of cubical sets as follows:

Corollary 14.2.6 *Let B be a box in I^n and let $i : B \to I^n$ be the inclusion. Let X_* be a filtered space, and suppose given a commutative diagram of cubical maps*

$$\begin{tikzcd} B \arrow[r, "\gamma"] \arrow[d, "i"'] & RX_* \arrow[d, "p"] \\ I^n \arrow[r, "{[\alpha]}"'] & \rho X_* \end{tikzcd}$$

Then there is a cubical map

$$\beta : \mathbb{I}^n \to RX_*$$

such that $\beta i = \gamma$, $p\beta = [\alpha]$, i.e. extends γ and induces $[\alpha]$.

Further, if $[\alpha](c^n)$ has a deficient representative, then β may be chosen so that $\beta(c^n)$ is deficient. □

The following result is an easy and memorable consequence of the first part of Corollary 14.2.6. We shall use this in applications to the homotopy classification of maps in section ???.

Theorem 14.2.7 (the fibration theorem) *Let X_* be a filtered space. Then the quotient map*

$$p : RX_* \to \rho X_*$$

is a Kan fibration. □

Another application of Corollary 14.2.6 is to the lifting of subdivisions from $\rho_n X_*$ to $R_n X_*$. For the proof of this, and of the Higher Homotopy van Kampen Theorem 14.3.1, we require the following construction.

Let $(m) = (m_1, \cdots, m_n)$ be an n-tuple of positive integers. The subdivision of I^n with small n-cubes $c_{(r)}, (r) = (r_1, \cdots, r_n), 1 \leqslant r_i \leqslant m_i$, where $c_{(r)}$ lies between the hyperplanes $x_i = (r_i - 1)/m_i$ and $x_i = r_i/m_i$ for $i = 1, \cdots, n$, is called the subdivision of I^n of *type* (m).

Proposition 14.2.8 (lifting arrays of homotopy classes) *Let X_* be a filtered space and*

$$\langle\langle \alpha \rangle\rangle = [\langle\langle \alpha_r \rangle\rangle]$$

a subdivision of an element $\langle\langle \alpha \rangle\rangle \in \rho_n X_*$. Then there is an element $\beta \in R_n X_*$ and a subdivision

$$\beta = [\beta_{(r)}]$$

of β, where all $\beta_{(r)}$ lie in $R_n X_*$ such that $\langle\langle \beta \rangle\rangle = \langle\langle \alpha \rangle\rangle$ and $\langle\langle \beta_{(r)} \rangle\rangle = \langle\langle \alpha_{(r)} \rangle\rangle$ for all (r).

Further, if each $\langle\langle \alpha_{(r)} \rangle\rangle$ has a deficient representative, then the $\beta_{(r)}$, and hence also β, may be chosen to be deficient.

Proof Let K be the cell complex of the subdivision of I^n of the same type as the given subdivision of $\langle\!\langle \alpha \rangle\!\rangle$. Then K collapses to a vertex, [This is clear intuitively if one draws a subdivided square, but perhaps more should be made of this?.] so that there is a chain

$$K = A_s \searrow A_{s-1} \searrow \cdots \searrow A_1 = \{v\}$$

of elementary collapses, where $A_{i+1} = A_i \cup a_i$ for some cell a_i of K, and $A_i \cap a_i$ is a box in a_i.

We now work in terms of the corresponding cubical sets $K = A_s, A_{s-1}, \ldots, A_1$, where K has unique nondegenerate elements $c_{(r)}$ of dimension n. The subdivision of $\langle\!\langle \alpha \rangle\!\rangle$ determines a unique cubical map

$$g : K \to \rho X_*$$

such that $g(c_{(r)}) = \langle\!\langle \alpha_{(r)} \rangle\!\rangle$. We construct inductively maps

$$f_i : A_i \to RX_*,$$

for $i = 1, \cdots, s$, such that f_i extends f_{i-1}, produces $g|_{A_i}$, and $f_{i+1}(a_i)$ is deficient if $g(a_i)$ has a deficient representative. The induction is started by choosing $f_1(v)$ to be any element such that $pf_1(v) = g(v)$. The inductive step is given by Corollary 14.2.6.

Let

$$f = f_s : K \to RX_*,$$

and let $\beta_{(r)} = f(c_{(r)})$ for all (r). Then the $\beta_{(r)}$ compose in R_nX_* to give an element $\beta = [\beta_{(r)}]$ as required. □

Recall that in any ω-groupoid G, an element $x \in G_n$ is thin if it can be written as a composite $x = [x_{(r)}]$ with each entry of the form $\varepsilon_j y$ or of the form a repeated negative of $\Gamma_j y$ (see Definition 13.4.17). The following characterisation of thin elements of $\rho_n X_*$ is essential for later work.

Theorem 14.2.9 (Geometric characterisation of thin elements) *Let X_* be a filtered space and let $n \geqslant 2$. Then an element of $\rho_n X_*$ is thin if and only if it has a deficient representative.*

Proof We suppose $n \geqslant 2$ and that α in R_nX_* is deficient. Define $\Psi_i\alpha \in R_nX_*$ by

$$\Psi_i\alpha = [-\varepsilon_i\partial_i^+\alpha, -\Gamma_i\partial_{i+1}^-\alpha, \alpha, \Gamma_i\partial_{i+1}^+\alpha]_{i+1}$$

where $-$ denotes $-_{i+1}$. Let $\Psi\alpha = \Psi_1 \cdots \Psi_{n-1}\alpha$; then $\Psi\alpha$ also is deficient.

In Section 13.4 we defined for any ω-groupoid, and hence also for $\rho_n X_*$, a 'folding operation' Φ. The above formula for Ψ is the same as that for Φ. It follows that $p\Psi = \Phi p$, where $p : RX_* \to \rho X_*$ is the quotient map. So by Proposition 13.4.9, $\partial_1^\tau \Phi p(\alpha) = \varepsilon_1^{n-1}[x]$ for some $[x] \in \rho_0 X = \pi_0 X_0$, if $(\tau, j) \neq (-, 1)$.

Thus if B is the box in I^n with base $\partial_1^+ I^n$, then for each $(n-1)$-cell a of B, $\Psi\alpha\mid_a$ is filter-homotopic to the constant map at x. By the Deformation Theorem 14.2.5, $\Psi\alpha$ is filter-homotopic to an element β such that $\beta(B) = \{x\}$, and such that β is deficient. Therefore, the homotopy of β to the constant map at x, defined by a strong deformation retraction of I^n onto B, is a filter-homotopy giving $p\Psi\alpha = p\beta = 0$. So $\Phi p\alpha = 0$. By Proposition 13.4.18, $\langle\!\langle \alpha \rangle\!\rangle = p\alpha$ is thin.

For the other implication, suppose that $\langle\!\langle \alpha \rangle\!\rangle$ is thin. Then $\langle\!\langle \alpha \rangle\!\rangle$ has a subdivision $\langle\!\langle \alpha \rangle\!\rangle = [\langle\!\langle \alpha_{(r)} \rangle\!\rangle]$ in which each $\alpha_{(r)}$ is deficient. By Proposition 14.2.8, $\langle\!\langle \alpha \rangle\!\rangle$ has a deficient representative. □

14.3 The HHvKT theorem for ω-groupoids

Suppose for the rest of this section that X_* is a filtered space. We suppose given a cover $\mathcal{U} = \{U^\lambda\}_{\lambda\in\Lambda}$ of X such that the interiors of the sets of $\mathcal{U}$ cover X. For each $\nu \in \Lambda^n$ we set $U^\nu = U^{\nu_1} \cap \cdots \cap U^{\nu_n}$, $U^\nu_i = U^\nu \cap X_i$. Then $U^\nu_0 \subseteq U^\nu_1 \subseteq \cdots$ is called the *induced filtration* U^ν_* of U^ν. So the fundamental ω-groupoids in the following ρ-*diagram* of the cover are well defined:

$$\bigsqcup_{\nu\in\Lambda^2} \rho U^\nu_* \underset{i_2}{\overset{i_1}{\rightrightarrows}} \bigsqcup_{\lambda\in\Lambda} \rho U^\lambda_* \xrightarrow{i} \rho X_*$$

Here $\bigsqcup$ denotes disjoint union (which is the same as coproduct in the category of ω-groupoids); i_1, i_2 are determined by the inclusions $i_{1\nu} : U^\lambda \cap U^\mu \to U^\lambda$, $i_{2\nu} : U^\lambda \cap U^\mu \to U^\mu$ for each $\nu = (\lambda, \mu) \in \Lambda^2$; and i is determined by the inclusions $i_\lambda : U^\lambda \to X$.

Theorem 14.3.1 (HHvKT theorem for ω-groupoids) *Suppose that for every finite intersection U^ν of elements of $\mathcal{U}$, the induced filtration U^ν_* is connected. Then*

(Con) X_* *is connected;*

(Iso) *in the above ρ-diagram i is the coequaliser of i_1, i_2 in the category of ω-groupoids.*

Proof The proof of (Con) will be made on the way to verifying the universal property which proves (Iso).

Suppose we are given a morphism

$$f' : \bigsqcup_{\lambda\in\Lambda} \rho U^\lambda_* \to G \tag{*}$$

of ω-groupoids such that $f' i_1 = f' i_2$. We have to show there is a unique morphism $f : \rho X_* \to G$ of ω-groupoids such that $fi = f'$. It is clear that if the morphism f satisfying $f' i_1 = f' i_2$ exists, then it must be given by the following recipe. The problem is to show that this recipe gives a well defined morphism.

Let i_λ be the inclusion of $\rho U^\lambda_* \to \bigsqcup_{\lambda\in\Lambda} \rho U^\lambda_*$. Let $p_\lambda : RU^\lambda_* \to \rho U^\lambda_*$ be the quotient map, and let $F_\lambda = f' i_\lambda p_\lambda : RU^\lambda_* \to G$. We can use these F_λ to construct $F\theta$ for certain kinds of elements θ in $R_n X_*$.

1.- Suppose that θ in $R_n X_*$ is such that θ lies in some set U^λ of $\mathcal{U}$. Then θ determines uniquely an element θ^λ of $R_n U^\lambda_*$, and the rule $f' i_1 = f' i_2$ implies that an element of G_n

$$F\theta = F_\lambda \theta^\lambda$$

is determined by θ.

2.- Suppose given a subdivision $[\theta_{(r)}]$ of an element θ of $R_n X_*$ such that each $\theta_{(r)}$ is in $R_n X_*$ and also lies in some $U^{\lambda(r)}$ of $\mathcal{U}$. Since the composite $\theta = [\theta_{(r)}]$ is defined, it is easy to check, again using $f' i_1 = f' i_2$, that the elements $F\theta_{(r)}$ form a composable array in G_n. We write the composition as $F\theta$,

$$F\theta = [F\theta_{(r)}]$$

although *a priori* it could depend on the subdivision chosen.

3.- Suppose now that α is an arbitrary element of $R_n X_*$. The construction from α of an element g in G_n and the proof that g depends only on the class of α in $\rho_n X_*$ are based on the following Lemma which generalises Lemma 6.8.3 of Part I.

Lemma 14.3.2 *Let $\alpha : I^n \to X$ and let $\alpha = [\alpha_{(r)}]$ be a subdivision of α such that each $\alpha_{(r)}$ lies in some set $U^{\lambda(r)}$ of $\mathcal{U}$. Then there is a homotopy $h : \alpha \simeq \theta$ with $\theta \in R_n X_*$ such that in the subdivision $h = [h_{(r)}]$ determined by that of α, each homotopy $h_{(r)} : \alpha_{(r)} \simeq \theta_{(r)}$ satisfies:*

(i) *$h_{(r)}$ lies in $U^{\lambda(r)}$;*

(ii) *$\theta_{(r)}$ belongs to $R_n X_*$;*

(iii) *if some m-dimensional face of $\alpha_{(r)}$ lies in X_j, so also do the corresponding faces of $h_{(r)}$ and $\theta_{(r)}$;*

(iv) *if v is a vertex of I^n and $\alpha(v) \in X_0$ then h is the constant homotopy on v.*

Proof Let K be the cell-structure on I^n determined by the subdivision $\alpha = [\alpha_{(r)}]$. Let $L_m = K^m \times I \cup K \times \{0\}$. We construct maps

$$h_m : L_m \to X$$

for $m = 0, \dots, n$ such that h_m extends h_{m-1}, starting with $h_{-1} = \alpha$. Further we construct h_m to satisfy the following conditions, for each m-cell e of K:

$(\mathrm{i})_m$ if e is contained in the domain of $\alpha_{(r)}$, then $h_m(e \times I) \subseteq U^{\lambda(r)}$;

$(\mathrm{ii})_m$ $h_m|_{e \times \{1\}}$ is an element of $R_m(X_*)$;

$(\mathrm{iii})_m$ if α maps e into X_j, then $h_m(e \times I) \subseteq X_j$;

$(\mathrm{iv})_m$ if $\alpha|_e : e \to X$ is a filtered map, then h is constant on e.

For an m-cell e of K, let j be the smallest integer such that α maps e into X_j. Let U^e be the intersection of all the sets $U^{\lambda(s)}$ such that e is contained in the domain of $\alpha_{(s)}$.

Let $h_m|_{K \times 0}$ be given by α, and for those cells e of K such that $\alpha|_e$ is filtered, let h_m be the constant homotopy on $e \times I$.

Let e be a 0-cell of K. If $\alpha(e)$ does not lie in X_0, then, since U^e_* is connected, there is be a path in U^e joining e to a point of X_0. We define h_0 on $e \times I$ busing this path.

Let $m \geqslant 1$. The construction of h_m from h_{m-1} is as follows on those m-cells e such that $\alpha|e$ is not filtered. If $j \leqslant m$, then h_{m-1} can be extended to h_m on $e \times I$ by means of a retraction $\alpha \times I \to e \times \{0\} \cup \partial e \times I$. If $j > m$ the restriction of h_{m-1} to the pair $(e \times \{0\} \cup \partial e \times I, \partial e \times I)$ determines an element of $\pi_m(U^e_j, U^e_{m-1})$. By $\phi(X_*, m)$, h_{m-1} extends to h_m on $e \times I$ mapping into U^e_j and such that $e \times \{1\}$ is mapped into U^e_m. □

Corollary 14.3.3 *Let $\alpha \in R_n X_*$. Then there is a filter-homotopy rel vertices $h : \alpha \equiv \theta$ such that $F\theta$ is defined in G_n.*

Proof Choose a subdivision $\alpha = [\alpha_{(r)}]$ such that $\alpha_{(r)}$ lies in some set $U^{\lambda(r)}$ of $\mathcal{U}$. Lemma 14.3.2 gives a filter-homotopy $h : \alpha \equiv \theta$ and subdivision $\theta = [\theta_{(r)}]$ as required. □

We will show in Lemma 14.3.5 below that this element $F\theta$ depends only on the class of α in $\rho_n X_*$. But first we can now prove that X_* is connected.

Proof of (Con)

The condition $\phi(X_*, 0)$ is clear since each point of X_j belongs to some U^λ and so may be joined in U^λ to a point of X_0.

Let $J^{m-1} = I \times \partial I^{m-1} \cup \{1\} \times I^{m-1}$. Let $j > m > 0$, $x \in X_0$ and let $[\alpha] \in \pi_m(X_j, X_{m-1}, x)$, so that $\alpha : (I^m, \{0\} \times I^{m-1}, J^{m-1}) \to (X_j, X_{m-1}, x)$. By Lemma 14.3.2, α is deformable as a map of triples into X_m.

This proves X_* is connected.

Remark 14.3.4 Up to this stage, our proof of the union theorem is very like the proof for the 2-dimensional case given in 6.8. We now diverge from that proof for two reasons. First, the form of the homotopy commutativity lemma given in 6.7.6 is not so easily stated in higher dimensions. So we employ thin elements, since these are elements with 'commuting boundary'. Second, we can now arrange that the proof is nearer in structure to the 1-dimensional case, for example the proof of the classical van Kampen theorem given in 1.6. □

Two facts about ω-groupoids which made the proof work are that composites of thin elements are thin (as is obvious from Definition 13.4.17), and Proposition 13.7.5.

Suppose now that $h' : \alpha \equiv \alpha'$ is a filter-homotopy between elements of $R_n X_*$, and $h : \alpha \equiv \theta$, $h'' : \alpha' \equiv \theta'$ are filter-homotopies constructed as in Corollary 14.3.3, so that $F\theta$, $F\theta'$ are defined. From the given filter-homotopies we can obtain a filter-homotopy $H : \theta \equiv \theta'$. So to prove $F\theta = F\theta'$ it is sufficient to prove the following key Lemma. In fact, it could be said that the previous machinery has been developed in order to give expression to this proof.

Lemma 14.3.5 *Let $\theta, \theta' \in R_n X_*$ and let $H : \theta \equiv \theta'$ be a filter-homotopy. Suppose $\theta = [\theta_{(r)}]$, $\theta' = [\theta'_{(s)}]$ are subdivisions into elements of $R_n X_*$ each of which lies in some set of $\mathcal{U}$. Then in G_n*

$$[F\theta_{(r)}] = [F\theta'_{(s)}].$$

Proof Suppose $\theta_{(r)}$ lies in $U^{\lambda(r)} \in \mathcal{U}$, $\theta'_{(s)}$ lies in $U^{\lambda'(s)} \in \mathcal{U}$, for all $(r), (s)$. Now $\theta = \partial^-_{n+1} H$, $\theta' = \partial^+_{n+1} H$. We choose a subdivision $H = [H_{(t)}]$ such that each $H_{(t)}$ lies in some set $V^{(t)}$ of $\mathcal{U}$ and so that on $\partial^-_{n+1} H$ and $\partial^+_{n+1} H$ it induces refinements of the given subdivisions of θ and θ' respectively. Further, this subdivision can be chosen fine enough so that $\partial^-_{n+1} H_{(t)}$, if it is a part of $\theta_{(r)}$, lies in $U^{\lambda(s)}$, and $\partial^+_{n+1} H_{(t)}$, if it is part of $\theta'_{(s)}$, lies in $U^{\lambda'(s)}$. So we can and do choose $V^{(t)} = U^{\lambda(r)}$ in the first instance, $V^{(t)} = U^{\lambda'(s)}$ in the second instance (and avoid both cases holding together by choosing, if necessary, a finer subdivision).

We now apply Lemma 14.3.2 with the substitution of $n+1$ for n, H for α, K for θ, and (t) for (r), to obtain in $R_{n+2} X_*$ a filter-homotopy $h : H \equiv K$ such that in the subdivision $h = [h_{(t)}]$ determined by that of H, each homotopy $h_{(t)} : H_{(t)} \simeq K_{(t)}$ satisfies

(i) $h_{(t)}$ lies in $V^{(t)}$

(ii) $K_{(t)}$ belongs to $R_{n+1} X_*$,

(iii) if some m-dimensional face of $H_{(t)}$ lies in X_j, so also do the corresponding faces of $h_{(t)}$ and $K_{(t)}$.

Now $k = \partial^-_{n+1} h$, $k' = \partial^+_{n+1} h$ are filter-homotopies $k : \theta \equiv \phi$, $k' : \theta' \equiv \phi'$, say. Further, the previous choices ensure that in the subdivision $k = [k_{(r)}]$ induced by that of θ, $k_{(r)}$ is a filter-homotopy $\theta_{(r)} \equiv \phi_{(r)}$ (by (iii)) and lies in $U^{\lambda(r)}$ (by (i)). It follows that $F\theta_{(r)} = F\phi_{(r)}$ in G_n and hence $F\theta = F\phi$. Similarly $F\theta' = F\phi'$, so it is sufficient to prove $F\phi = F\phi'$.

We have a filter-homotopy $K : \phi \equiv \phi'$ and a subdivision $K = [K_{(t)}]$ such that each $K_{(t)}$ belongs to $R_{n+1} X_*$ and lies in some $V^{(t)}$ of $\mathcal{U}$. Thus $FK = [FK_{(t)}]$ is defined in G_{n+1}. Further, the induced subdivisions of $\partial^-_{n+1} FK$, $\partial^+_{n+1} FK$ refine the subdivisions $[F\phi_{(r)}]$, $[F\phi'_{(s)}]$ respectively. Hence $\partial^-_{n+1} FK =$

$F\phi$, $\partial^+_{n+1}FK = F\phi'$, and it is sufficient to prove $\partial^-_{n+1}FK = \partial^+_{n+1}FK$. For this we apply Proposition 13.7.5.

Let d be a face operator from dimension $n+1$ to dimension m, and not involving ∂^-_{n+1} or ∂^+_{n+1}. Let $\sigma = d(H)$, $\tau = d(K)$. Then σ is deficient (since H is a filter homotopy) and so by the choice of h in accordance with (iii), τ is deficient. In the subdivision $\tau = [\tau_{(u)}]$ induced by the subdivision $K = [K_{(t)}]$, $\tau_{(u)} \in R_m X_*$ and is deficient. By Theorem 14.2.9, the $F\tau_{(u)} \in G_m$ are thin, and hence their composite $F\tau \in G_m$ is thin. But $FK = [FK_{(t)}]$ has, by its construction, the property that $dFK = F\tau$. So dFK is thin. By Proposition 13.7.5, $\partial^-_{n+1}FK = \partial^+_{n+1}FK$. □

Proof of (Iso)

We have completed the proof that there is a well-defined function $f : \rho_n X_* \to G_n$ given by $f(\langle\langle \alpha \rangle\rangle) = F(\theta)$, where θ is constructed as in Corollary 14.3.3. These maps $f : \rho_n X_* \to G_n$, $n \geqslant 0$, determine a morphism $f : \rho X_* \to G$ of ω-groupoids. By its construction, f satisfies $fi = f'$ and is the only such morphism. Thus the proof of Theorem 14.3.1 is complete. □

14.4 The HHvKT for crossed complexes

In order to interpret the HHvK Theorem 14.3.1, we relate the ω-groupoid ρX_* to the fundamental crossed complex ΠX_* of part II.

In Section 13.3 we have defined a functor

$$\gamma : \omega\text{-}\mathsf{Gpds} \to \mathsf{Crs}$$

associating a crossed complex γG to any ω-groupoid G.

Now we prove that for any filtered space the crossed complex $\gamma\rho X_*$ is canonically isomorphic to ΠX_* the fundamental crossed complex used throughout Part II.

Thus we can translate Theorem 14.3.1 getting the HHvK Theorem for crossed complexes (Theorem 8.1.5) whose consequences we have studied in Part II.

Theorem 14.4.1 *If X_* is a filtered space then $\gamma\rho X_*$ is naturally isomorphic to ΠX_*.*

Proof It is clear that the dimension 1 groupoids in both structures are the same.

Let $n \geqslant 2$, and $x \in X_0$. We construct an isomorphism

$$\theta_n : \pi_n(X_n, X_{n-1}, x) \to (\gamma\rho X_*)_n.$$

The elements of $\pi_n(X_n, X_{n-1}, x)$ are homotopy classes of maps of triples

$$\alpha : (I^n, \partial^-_1 I^n, B) \to (X_n, X_{n-1}, x),$$

where B is the box in I^n with base $\partial^+_1 I^n$. Such a map α defines a filtered map

$$\theta'\alpha : I^n_* \to X_*$$

with the same values as α, and $\theta'\alpha$ is constant on B.

If α is homotopic to β (as maps of triples), then $\theta'\alpha$ is filter-homotopic to $\theta'\beta$, and so θ' induces a map $\theta_n : \pi_n(X_n, X_{n-1}, x) \to (\gamma\rho X_*)_n$. But addition in the relative homotopy group $\pi_n(X_n, X_{n-1}, x)$ is defined using any $+_i$, $i \geqslant 2$. So θ_n is a morphism of groups.

Suppose α represents in $\pi_n(X_n, X_{n-1}, x)$ an element mapped to 0 by θ_n. Then there is a filter homotopy rel vertices

$$H : \theta'\alpha \equiv x*,$$

where $x*$ is the constant map at x. Now we want a map of triples

$$F : (I^n \times I, \partial_1^- I^n \times I, B \times I) \to (X_n, X_{n-1}, x)$$

with $F_0 = \alpha$ and $F_1 = x*$. We know that $\alpha|_B$ is constant. By Corollary 14.2.2 and since B collapses to a vertex (by Corollary 10.3.7), the constant filter-homotopy $\theta'\alpha|_B \equiv x * |_B$ extends to a filter-homotopy $\theta'\alpha \equiv x *$. This filter-homotopy defines a homotopy $F : \alpha \simeq x *$. So θ_n is injective.

We now prove θ_n surjective. Let $\langle\!\langle \gamma \rangle\!\rangle \in (\gamma\rho X_*)_n$. Then for each $(n-1)$-face a of B, $\gamma|_a$ is filter-homotopic to $\tilde{x}|_a$ (where $\tilde{x}$ is the constant map $B \to X_*$ at x). By the deformation Theorem 14.2.5, γ is filter-homotopic to a map $\gamma' : I^n \to X_*$ extending $\tilde{x}$. Hence θ_n is surjective.

The isomorphism θ also preserves the boundary maps δ. To complete the proof, we only have to show that θ preserves the action of C_1 on C.

Let α represent an element of $\pi_n(X_n, X_{n-1}, x)$, and let ξ represent an element of $\pi_1 X_1(x, y)$. A standard method of constructing $\beta = \alpha^\xi$ representing an element of $\pi_n(X_n, X_{n-1}, y)$ (as seen in section 2.1) is to use the homotopy extension property as follows. Let $\xi' : B \times I \to X_*$ be $(x, t) \mapsto \xi(t)$. Then ξ' is a homotopy of $\alpha|_B$ which extends to a homotopy $h : \alpha \simeq \beta$, and we set $\alpha^\xi = \beta$. We want to prove that $\theta_n[\alpha^\xi] = \theta_n[\beta] = (\theta_n[\alpha])^{[\xi]}$. So, if we recall that h is constructed by extending ξ' over $\partial_1^- I^n \times I$ using a retraction of $\partial_1^- I^n \times I$ to its box with base $\partial_1^- I^n \times \{0\}$, and then extending again using a retraction of $I^n \times I$ to its box with base $I^n \times \{0\}$. Thus h is a filtered map $I_*^{n+1} \to X_*$ with h and $\partial_i^\tau h$ $(i \neq n+1)$ deficient; hence $[h]$ and $\partial_i^\tau [h]$ $(i \neq n+1)$ are thin (Theorem 14.2.9). Therefore the folding map $\Phi : \rho_n X_* \to \rho_n X_*$ defined in Section 13.4 vanishes on these elements by Proposition 13.4.18 and so the homotopy addition Lemma 13.7.1 reduces to

$$\Phi\partial_{n+1}^+[h] = (\Phi\partial_{n+1}^-[h])^{u_{n+1}[h]}.$$

By Corollary 13.4.10, Φ is the identity on D_n, [Where was D_n defined?] to which belong both $\partial_{n+1}^+[h] = \theta_n[\beta]$ and $\partial_{n+1}^-[h] = \theta_n[\alpha]$. Further $u_{n+1}[h] = [\xi]$. So

$$\theta_n[\beta] = (\theta_n[\alpha])^{[\xi]}.$$

Thus θ preserves the operations.

Finally, the naturality of θ is clear. □

Proof of Theorem 8.1.5

Since the functor γ is an equivalence of categories, we obtain immediately from the previous Theorem and the HHvK Theorem 14.3.1 for ω-groupoids, the HHvK Theorem 8.1.5 for crossed complexes.

Proposition 14.4.2 *Let $n \geqslant 2$ and let $c^n \in \rho_n I_*^n$ be the class of the identity map $I_*^n \to I_*^n$. Then $\pi_n(I^n, \partial I^n, 1)$ is isomorphic to $\mathbb{Z}$ and is generated by $\theta^{-1}\Phi c^n$.*

Proof There is an alternative definition of relative homotopy groups, namely $\pi_n'(X, Y, x)$ is the set of homotopy classes of maps $(I^n, \partial I^n, 1) \to (X, Y, x)$, with addition induced by a map $I^n \to I^n \bigvee I^n$. An isomorphism $\xi : \pi_n(X, Y, x) \to \pi_n'(X, Y, x)$ is induced by $\alpha \mapsto \alpha'$ where (in the notation of the proof of Theorem 14.4.1) $\alpha : (I^n, \partial_1^- I^n, B) \to (X, Y, x)$, and $\alpha' : (I^n, \partial I^n, 1) \to (X, Y, x)$ has the same values as α. (Here $1 = (1, \cdots, 1)$ is the base point of I^n.)

Let $\rho_n(I^n_*, 1)$ be the set of y in $\rho_n I^n_*$ such that $(\partial_1^+)^n y = 1$. Then a map

$$\eta : \rho_n(I^n_*, 1) \to \pi'_n(I^n, \partial I^n, 1)$$

is induced by $\beta \mapsto \beta'$ where $\beta : I^n_* \to I^n_*$ satisfies $\beta(1) = 1$, and β' has the same values as β. Clearly $\eta\theta = \xi$.

A standard deduction from the results of Section 8 is that $\pi'_n(I^n, \partial I^n, 1)$ is isomorphic to $\mathbb{Z}$ and is generated by a^n, the class of the identity map. Now clearly $\eta c^n = a^n$. Also, it is easily checked that for any $y \in \rho_n(I^n_*, 1)$ and $j = 1, \cdots, n-1$, we have $\eta\Phi_j y = \eta y$. Hence $\eta\Phi c^n = \eta c^n = a^n$. The result now follows. □

From now on, we identify ΠX_* with $\theta\Pi X_* = \gamma\rho X_*$ for any filtered space X_*.

14.5 Realisation properties of ω-groupoids and crossed complexes

In this section, we show that each of the functors ρ and Π from FTop to respectively ω-groupoids and crossed complexes are *representative functors*, i.e. all ω-groupoids and all crossed complexes are, up to isomorphism, values of these functors. An implication of this is that the axioms for these structures well reflect the properties of these functors.

Let G be any ω-groupoid and define G^m to be the ω-subgroupoid of G generated by all elements of dimension $\leqslant m$. Then G^m has only thin elements in dimension greater than m and is the largest such ω-groupoid. In fact,

$$G^m \cong \mathsf{Sk}^m G = \mathsf{sk}^m(\mathsf{tr}^m G)$$

as described in Section 13.5, and by abuse of language we call it the m-*skeleton of* G (not to be confused with the m-skeleton of G considered as a cubical set). We define the *skeletal filtration* of G to be

$$G^* : G^0 \subseteq G^1 \subseteq \cdots .$$

The elements of G^m_n are the same as those of G_n for $n \leqslant m$; and for $n > m$, G^m_n can be described inductively as the set of thin elements of G_n whose faces are in G^m_{n-1}.

Since G^m is an ω-groupoid, it is a Kan complex. Therefore if $x \in G_0$, and $0 < l < m$, the rth relative homotopy group $\pi_r(G^m, G^l, x)$ is defined for $r \geqslant 2$. So there is a crossed complex ΠG^* which in dimension $n \geqslant 2$ is the family of groups $\pi_n(G^n, G^{n-1}, x)$, $x \in G_0$, and in dimension 1 is the groupoid $\pi_1 G^1$.

Proposition 14.5.1 *If* G^* *is the skeletal filtration of an* ω*-groupoid* G *then the crossed complex* ΠG^* *is naturally isomorphic to* γG*. Further,* G^* *is connected.*

Proof The elements of $\pi_n(G^n, G^{n-1}, p)$, $p \in G_0$, $n \geqslant 2$, are classes of elements x of G_n such that $\partial_i^\tau x = \varepsilon_1^{n-1} p$ for $(\tau, i) \neq (0, 1)$, two such elements x, y being equivalent if there is an $h \in G^n_{n+1}$ such that $\partial^-_{n+1} h = x$, $\partial^+_{n+1} h = y$, $\partial_i^\tau h = \varepsilon_1^n p$ for $(\tau, i) \neq (0, 1)$ and $i \neq n+1$, and $\partial_1^- h \in G^{n-1}_n$. Then h is thin, as is dh for any face operator d not involving ∂^-_{n+1} or ∂^+_{n+1}. It follows from Proposition 13.7.5 that $x = y$. Thus $\pi_n(G^n, G^{n-1}, p)$ can be identified with $C_n(p) = (\gamma_n G^*)(p)$.

The identification of the groupoid $\pi_1 G^1$ with G_1 is simple, as is the identification of the boundary maps. The identification of the operations may be carried out in a similar manner to the proof of Theorem 14.4.1.

Finally, that G^* is connected follows from the fact that $G^r_n = G_n$ for $r \geqslant n$. □

We now use the geometric realisation $|A\,|$ of a cubical set A as described in subsection 10.1.3. If G is an ω-groupoid, then $|G\,|$ denotes the geometric realisation of the underlying cubical set of G.

Proposition 14.5.2 *Let G be an ω-groupoid, G^* its skeletal filtration, and let $X_* = |G^*|$ be the filtration of $X = |G|$ given by $X_n = |G^n|$. Then there is a natural isomorphism of ω-groupoids*

$$G \cong \rho|G^*|.$$

Proof By the previous remarks and Proposition 14.5.1 we have natural isomorphisms

$$\gamma G \cong \Pi G^* \cong \Pi|G^*|.$$

The result follows since $\Pi|G^*| \cong \gamma\rho|G^*|$ and γ is an equivalence. □

Corollary 14.5.3 *If C is a crossed complex, there is a filtered space X_* such that C is isomorphic to ΠX_*.*

Proof Let G be the ω-groupoid λC (cf. 13.6) and let $X = |G|$. By Proposition 14.5.2, $C \cong \Pi X_*$. □

Remark 14.5.4 This result contrasts with Whitehead's example of a crossed complex C which is of dimension 5, has $\pi_1 C = Z_2$, is free in each dimension but is not isomorphic to ΠX_* for the skeletal filtration X_* of any CW-complex X (see [Whi49b]). □

Remark 14.5.5 Note also that when $X = |\lambda C|$, the absolute homotopy groups $\pi_n(X, x)$ are isomorphic to $\pi_1(C, x)$ for $n = 1$, $H_n(C, x)$ for $n \geqslant 2$ by Remark 13.7.7 of 13.7. Thus Corollary 14.5.3 generalises a cubical version of the construction of Eilenberg-Mac Lane spaces. □

14.6 Free properties

Proposition 14.6.1 *For any cubical set K, the natural cubical map $i_K : K \to \rho\,|K_*|$ makes $\rho\,|K_*|$ the free ω-groupoid on K.*

Proof Let G be an ω-groupoid, and let $f : K \to UG$ be a cubical map. Then f induces a filtered map $|K_*| \to |U_*G|$, which composes with the inclusion $|U_*G| \to |UG_*|$ to give $|f| : |K_*| \to BG_*$. The natural isomorphism $i_G : G \to \rho BG_*$ and the natural map $i' : K \to U\rho|K_*|$ give a commutative diagram

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d] & UG \arrow[d, "Ui_G"] \\
U\rho|K| \arrow[r, "U\rho|f|"'] \arrow[ur, dashed, "U\tilde{f}"'] & U\rho BG
\end{tikzcd}$$

Thus $\tilde{f} = (i_G)(\rho|f|) : \rho|K_*| \to UG$ is a morphism of ω-groupoids extending f. Its uniqueness follows if we can show that $\rho|K_*|$ is generated, as an ω-groupoid, by $i'(K)$. But $\rho|K_*|$ is generated by the crossed complex $\gamma\rho|K_*| = \Pi|K_*|$ which it contains (see 13.5.14). Also $\Pi|K_*|$ is generated, as crossed complex, by the cells of $|K_*|$, i.e. by non-degenerate elements of K, by 9.6.4. So uniqueness is proved. □

Corollary 14.6.2 *The homotopy ω-groupoid ρI^n_* is the free ω-groupoid on the class $c^n \in \rho_n I^n_*$ of the identity map.*

Remark 14.6.3 We now describe the crossed complex ΠI^n_*. The cell complex I^n has one cell for each cubical face operator d from dimension n to r, $0 \leqslant r \leqslant n$, and d determines a characteristic map $\tilde{d} : I^r_* \to I^n_*$ for this cell. Then $\tilde{d}$ induces $\rho(\tilde{d}) : \rho I^r_* \to \rho I^n_*$ and $\rho(\tilde{d})(c^r) = dc^n$. Since $\rho(\tilde{d})$ is a morphism of ω-groupoids, it follows that $\rho(\tilde{d})(\Phi c^r) = \Phi dc^n$. Hence ΠI^n_* has generators Φdc^n for each face operator d from dimension n to r, $0 \leqslant r \leqslant n$. The boundary $\delta\Phi dc^n$ is given by the HAL 13.7.1.

Corollary 14.6.4 *If* G *is an ω-groupoid, then* G_n *is naturally isomorphic to* $\mathsf{Crs}(\Pi I^n_*, \gamma G)$.

Proof $G_n \cong \mathsf{Gpds}(\rho I^n_*, G) \cong \mathsf{Crs}(\Pi I^n_*, \gamma G)$. □

Remark 14.6.5 This corollary gives another description of the functor $\lambda : \mathsf{Crs} \to \omega\text{-}\mathsf{Gpds}$, the inverse equivalence of γ, namely that λ is naturally equivalent to $C \mapsto \mathsf{Crs}(\Pi I^n_*, C)$. In view of the explicit description of ΠI^n_* given above, a morphism $f : \Pi I^n_* \to C$ of crossed complexes is describable as a family $\{f(d)\}$ where d runs through all the cubical face operators from dimension n to dimension r $(0 \leqslant r \leqslant n)$, $f(d) \in C_r$, and the elements $f(d)$ are required to satisfy the relations (cf. 13.7.1)

$$\delta f(d) = \begin{cases} \sum_{i=1}^{r}(-1)^i\{f(\partial_i^+ d) - f(\partial_i^- d)^{f(u_i d)}\} & (r \geqslant 4),\\ -f(\partial_3^+ d) - f(\partial_2^- d)^{f(u_2 d)} - f(\partial_1^+ d) + f(\partial_3^- d)^{f(u_3 d)} + f(\partial_2^+ d) + f(\partial_1^- d)^{f(u_1 d)} & (r = 3),\\ -f(\partial_1^+ d) - f(\partial_2^- d) + f(\partial_1^- d) + f(\partial_2^+ d) & (r = 2),\end{cases}$$

and $\delta^\tau f(d) = f(\partial_1^\tau d)$ $(r = 1)$. (These relations imply that $f(d) \in C_r(p)$ where $p = f(\beta d)$). □

Corollary 14.6.6 *For any cubical set* K*, there is a natural isomorphism* $\gamma\rho(K) \cong \Pi|K_*|$.

By virtue of this corollary we identify these two crossed complexes and write either as $\Pi(K)$. So we have a functor $\Pi : \mathsf{Cub} \to \mathsf{Crs}$.

Corollary 14.6.7 *The functor* $\Pi : \mathsf{Cub} \to \mathsf{Crs}$ *is left adjoint to the nerve functor* $N : \mathsf{Crs} \to \mathsf{Cub}$.

Proof This follows from the fact that $\rho : \mathsf{Cub} \to \omega\text{-}\mathsf{Gpd}$ is left adjoint to $U : \omega - \mathsf{Gpd} \to \mathsf{Cub}$, that $\Pi = \gamma\rho$, that $N = \lambda U$, and that γ and λ give the equivalence of the categories of ω-groupoids and crossed complexes. □

Remark 14.6.8 The fact that the functor $\rho : \mathsf{Cub} \to \omega$-groupoids is a left adjoint implies that it preserves all colimits. However, the Higher Homotopy van Kampen Theorem (Theorem 14.3.1) is not an immediate consequence of this fact since that theorem is about the functor $\rho : \mathsf{FTop} \to \omega$-groupoids from *filtered spaces* to ω-groupoids, and one of the conditions for $\rho \operatorname{colim} U_* \cong \operatorname{colim} \rho U_*$ is that each filtered space U_* should be connected, in the sense of 14.3. It would be interesting to know whether this Higher Homotopy van Kampen Theorem can be deduced from the fact that $\rho : \mathsf{Cub} \to \omega$-groupoids preserves all colimits. □

14.7 Homology and homotopy

The homology groups of a cubical set K are defined as follows. First we form the chain complex $C'(K)$ where $C'_n(K)$ is the free abelian group on K_n, and with boundary

$$\partial k = \sum_{i=1}^{n} (-1)^i (\partial_i^- k - \partial_i^+ k). \tag{14.7.1}$$

It is easily verified that this gives a chain complex, i.e. $\partial\partial = 0$. However if K is a point, i.e. K_n is a singleton for all n, then the homology groups of $C'(K)$ are $\mathbb{Z}$ in even dimensions, whereas we want the homology of a point to be zero in dimensions > 0. We therefore normalise, i.e. factor $C'(K)$ by the subchain complex generated by the degenerate cubes. This gives the chain complex $C_*(K)$ of K, and the homology groups of this chain complex are defined to be the homology groups of K. A full exposition of this cubical theory is in [Mas80].

In particular the homology groups of $S^{\square}X$ are the (cubical) singular homology groups of the space X. It is proved in [EM53] using acyclic models that the cubical singular homology groups are isomorphic to the simplicial singular homology groups.

Let X_* be a filtered space. Then RX_* is a Kan complex and ρX_* is an ω-groupoid, and hence a Kan complex (by Proposition 13.7.3). (A direct proof that ρX_* is a Kan complex can be given using Theorem 14.2.5.)

The following proposition is one step towards the Hurewicz theorem. It should be compared with a special case discussed in [Mas80, Section III.7]. In the proof, a useful lemma is that if (Y, Z) is a cofibred pair, and $f : (Y, Z) \to (X, A)$ is a map of pairs which is deformable (as a map of pairs) into A, then f is deformable into A rel Z ([Bro06, 7.4.4]).

Proposition 14.7.1 *Let X_* be a filtered space such that the following conditions $\psi(X_*, m)$ hold for all $m \geqslant 0$:*

$\psi(X_*, 0)$: *The map $\pi_0 X_0 \to \pi_0 X$ induced by inclusion is surjective;*

$\psi(X_*, 1)$: *Any path in X joining points of X_0 is deformable in X rel end points to a path in X_1;*

$\psi(X_*, m)(m \geqslant 2)$: *For all $\nu \in X_0$, the map*

$$\pi_m(X_m, X_{m-1}, \nu) \to \pi_m(X, X_{m-1}, \nu)$$

induced by inclusion is surjective.

Then the inclusion $i : RX_* \to KX = S^{\square}X$ is a homotopy equivalence of cubical sets.

Proof There exist maps $h_m : K_m X \to K_{m+1} X$, $r_m : K_m X \to K_m X$ for $m \geqslant 0$ such that

(i) $\partial^-_{m+1} h_m = 1, \partial^+_{m+1} h_m = r_m$,

(ii) $r_m(KX) \subset R_m X_*$ and $h_m \mid R_m X_* = \varepsilon_{m+1}$,

(iii) $\partial^\tau_i h_m = h_{m-1} \partial^\tau_i$ for $1 \leqslant i \leqslant m$ and $\tau = 0, 1$,

(iv) $h_m \varepsilon_j = \varepsilon_j h_{m-1}$ for $1 \leqslant j \leqslant m$.

Such r_m, h_m are easily constructed by induction, starting with $h_{-1} = \emptyset$, and using $\psi(X_*, m)$ to define $h_m\alpha$ for elements α of K_mX which are not degenerate and do not lie in R_mX_*. Here is a picture for h_1:

$$\begin{tikzcd}
r\partial_1^- k \arrow[r, dash, "h_0\partial_1^- k"] \arrow[d, dash, dashed, "rk"'] \arrow[dr, phantom, "h_1 k"] & \partial_1^- k \arrow[d, dash, "k"] \\
r\partial_1^+ k \arrow[r, dash, "h_0\partial_1^+ k"'] & \partial_1^+ k
\end{tikzcd}$$

These maps define a retraction $r : KX \to RX_*$ and a homotopy $h \simeq ir$ rel RX_*. □

Corollary 14.7.2 *If the conditions $\psi(X_*, m)$ of the proposition hold for all $m \geqslant 0$, then the inclusion $i : RX_* \to KX$ induces a homotopy equivalence of chain complexes and hence an isomorphism of all homology and homotopy groups.*

Proof The result on homotopy is standard, and that on homology follows from the development in [Mas80]. □

Corollary 14.7.3 *If X_* is the skeletal filtration of a CW-complex, then the inclusion $RX_* \to S^\square X$ is a homotopy equivalence of Kan cubical sets.*

Definition 14.7.4 Let $C_*(X)$ denote the chain complex of normalised cubical singular chains of the space X. We now coin a term: for a subspace A of X, let $C_*(X \operatorname{rel}_0 A)$ denote the chain complex generated by singular cubes $f : I^n \to X$ which map the vertices of I^n into A, for $n \geqslant 1$, and in which $C_0(X \operatorname{rel}_0 A) = 0$, so that all elements of $C_1(X \operatorname{rel}_0 A)$ are cycles. We write $H_*(X \operatorname{rel}_0 A)$ for the homology of this chain complex □

Theorem 14.7.5 *Let A be a subspace of the space X. Then a Hurewicz morphism*

$$\omega : \pi_1(X, A) \to H_1(X \operatorname{rel}_0 A)$$

is defined and induces an isomorphism

$$\omega' : \pi_1(X, A)^{\mathrm{ab}} \to H_1(X \operatorname{rel}_0 A).$$

Proof For each path class $[f] \in \pi_1(X, A)$ the representative f determines a generator of $C_1(X \operatorname{rel}_0 A)$. Differing choices of f yield homologous elements of $C_1(X \operatorname{rel}_0 A)$, so this defines ω as a function. If $f \circ g$ is a composite of paths with vertices in A then the diagram

$$\begin{tikzcd}
{} \arrow[r, "f \circ g"] \arrow[d, "f"'] & {} \arrow[d, "1"] \\
{} \arrow[r, "g"'] & {}
\end{tikzcd} \tag{14.7.2}$$

extends to a map of $I^2 \to X$ with vertices mapped to A whose boundary shows that ω is a morphism to $H_1(X \operatorname{rel}_0 A)$. It hence defines $\omega' : \pi_1(X, A)^{\mathrm{totab}} \to H_1(X \operatorname{rel}_0 A)$.

Now $C_1(X \operatorname{rel}_0 A)$ is free abelian on the non degenerate paths $f : I \to X$ with vertices in A. So a morphism $\eta : C_1(X \operatorname{rel}_0 A) \to \pi_1(X, A)^{\mathrm{ab}}$ is defined by sending f to its class in $\pi_1(X, A)^{\mathrm{ab}}$. It is easy to check that $\eta\partial_2 = 0$, so that η defines a morphism $H_1(X \operatorname{rel}_0 A) \to \pi_1(X, A)^{\mathrm{totab}}$, and that η is inverse to ω'. □

Next we relate $H_*(X \operatorname{rel}_0 A)$ to the standard relative homology.

For a subspace A of X, we define the filtered space X_A to be A in dimension 0 and X in dimensions > 0. Our next result generalises a classical case when X is path connected and A consists of a single point.

Proposition 14.7.6 *If A meets each path component of X, then the inclusion $C_*(X_A) \to C_*(X)$ is a chain equivalence.*

Proof This is an immediate consequence of Corollary 14.7.2. □

We say $C_*(A)$ is *concentrated in dimension* 0 if $C_i(A) = 0$ for $i > 0$. This occurs for example if A is totally path disconnected, and so if A is discrete.

Theorem 14.7.7 (Relative Hurewicz Theorem: dimension 1) *If A is totally path disconnected and meets each path component of X then $H_1(X, A) \cong H_1(X \operatorname{rel}_0 A)$.*

Proof We define A_* to be the constant filtered space with value A. So we regard A_* as a sub-filtered space of X_A.

We consider the morphism of exact sequences of chain complexes

$$\begin{tikzcd} 0 \arrow[r] & C_*(A) \arrow[r] \arrow[d, "="] & C_*(X) \arrow[r] \arrow[d, "i"] & C_*(X, A) \arrow[r] \arrow[d, "j"] & 0 \\ 0 \arrow[r] & C_*(A_*) \arrow[r] & C_*(X_A) \arrow[r] & C_*(X_A, A_*) \arrow[r] & 0 \end{tikzcd} \tag{14.7.3}$$

where classically the first sequence defines relative homology $H_*(X, A)$, and the second sequence defines $H_*(X_A, A_*)$. Under our assumptions, the morphism i is a homotopy equivalence and hence so also is j (since all the chain complexes are free in each dimension).

Our assumption that A is totally path disconnected implies that $C_i(A) = 0$ for $i > 0$. This implies that $C_*(X_A, A_*) \cong C_*(X \operatorname{rel}_0 A)$. □

Remark 14.7.8 We now outline a proof of the Absolute Hurewicz Theorem using Corollary 14.7.2 and the homotopy addition lemma in the following form. Let $n \geqslant 2$, and let $\beta : (I^{n+1}, I^{n+1}_{n-1}) \to (X, \nu)$ be a map. Then each $\partial_i^\tau \beta$ represents an element β_i^τ of $\pi_n(X, \nu)$, and we have

$$\sum_{i=1}^{n+1} (-1)^i (\beta_i^- - \beta_i^+) = 0. \tag{14.7.4}$$

This follows from the form of the homotopy addition lemma given in (13.7.1) applied to the ω-groupoid ρX_* where X_* is the filtered space with $X_i = \{\nu\}, i < n, X_i = X, i \geqslant n$.

Theorem 14.7.9 (The Absolute Hurewicz Theorem) *If $n \geqslant 2$ and X is an $(n-1)$-connected pointed space, then $H_i X = 0$ for $0 < i < n$ and the Hurewicz map $\omega_n : \pi_n X \to H_n X$ is an isomorphism.*

Proof Let X_* be the filtered space defined immediately above. Then X_* satisfies $\psi(X_*, m)$ for all $m \geqslant 0$ and so $i : RX_* \to KX$ is a homotopy equivalence. But $H_i RX_* = 0$ for $0 < i < n$; hence $H_i X = H_i KX = 0$ for $0 < i < n$,

For $m \geqslant 0$ let $C_m X_*$ denote the group of (normalised) m-chains of RX_*. Then every element of $C_n X_*$ is a cycle, and the basis elements $\alpha \in R_n X_*$ of $C_n X_*$ are maps $I^n \to X$ with $\alpha(\dot{I}^n) = \{\nu\}$. So they determine elements $\tilde{\alpha}$ of $\pi_n(X, \nu)$, and $\alpha \mapsto \tilde{\alpha}$ determines a morphism $C_n X_* \to \pi_n(X, \nu)$. But by equation (14.7.4), this morphism annihilates the group of boundaries. So it induces a map $H_n X \to \pi_n(X, \nu)$ which is easily seen to be inverse to the Hurewicz map. □

We know that if X_* is a filtered space, then $p : RX_* \to \rho X_*$ is a Kan fibration of cubical sets. Notice that is $nu \in X_0$, then ν also belongs to RX_* and to the fibre of p over ν.

Theorem 14.7.10 *Let X_* be a filtered space, and let $\nu \in X_0$. Let F_ν be the fibre of $p : RX_* \to \rho X_*$ over ν. Then:*

(i) *there is an exact sequence*

$$\cdots \to \pi_n(F_\nu, \nu) \to \pi_n(RX_*, \nu) \to \pi_n(\rho X_*, \nu) \to$$
$$\cdots \to \pi_1(F_\nu, \nu) \to \pi_1(RX_*, \nu) \to \pi_1(\rho X_*, \nu) \to$$

(ii) *$\pi_n(F, \nu)$ is isomorphic to the image of the morphism*

$$i_n : \pi_n(X_{n-1}, \nu) \to \pi_n(X_n, \nu)$$

induced by inclusion.

(iii) *if X_* is the skeletal filtration of a CW-complex X, then the above exact sequence is equivalent to one of the form*

$$\cdots \to \mathbf{\Gamma}_n(X, \nu) \to \pi_n(X, \nu) \xrightarrow{\omega} H_n(\widetilde{X}_\nu) \to$$

where ω is called the Hurewicz morphism.

Proof (i) This is just the exact sequence of a Kan fibration of cubical sets, whose proof is entirely analogous to that for the topological case.

(ii)We define a map $\theta : \pi_n(F, \nu) \to \pi_n(X_n, \nu)$.

Let $\alpha \in F_n$ have all its faces at the base point ν. Then α determined $\alpha' : (I^n, I^n) \to (X_n, \nu)$ with the same values as α, and $\alpha \mapsto \alpha'$ induces θ.

If $\alpha \in F_n$, then $p\alpha = \varepsilon_1^n \bar{\nu}$ in $\rho_n X$, and so α is filter-homotopic to $\bar{\nu}$, the constant map at ν. Suppose further that α has all its faces at the base point. Let B be the box in I^n with base $\partial_n^- I^n$. By Corollary 14.2.2, the constant filter-homotopy $\bar{\nu} \mid B \equiv \alpha \mid B$ extends to a filter-homotopy $h : \bar{\nu} \equiv \alpha$. Let $\beta = \partial_n^+ h, k = \Gamma_n \beta$. Then $h +_n k$ is a filter-homotopy $\bar{\nu} +_n \beta \simeq \alpha +_n \bar{\nu}$, rel $\dot{I}^n$. Let $\beta' : (I^n, \dot{I}^n) \to (X_{n-1}, \nu)$ be the map with the same values as β. Then $\alpha' \simeq i\beta'$. This proves $\operatorname{Im}\theta \subset \operatorname{Im} i_n$.

Let $\alpha' : (I^n, \dot{I}^n) \to (X_{n-1}, \nu)$ represent an element of $\pi_n(X_{n-1}, \nu)$. Let $\alpha : I^n_* \to X_*$ have the same values as α'. Then $\Gamma_n \alpha$ is a filter-homotopy $\alpha \equiv \bar{\nu}$, so that $\alpha \in F_n$. Clearly $\theta\bar{\alpha} = i_n \alpha'$, and this proves $\operatorname{Im} i_n \subset \operatorname{Im}\theta$.

Finally, we prove θ injective. Suppose $\theta\bar{\alpha} = 0$. Then there is a homotopy $h : \alpha' \simeq \bar{\nu}$ of maps $(I^n, \dot{I}^n) \to (X_n, \nu)$. Clearly $h \in R_{n+1} X_*$. However, $\Gamma_{n+1} h$ is a filter-homotopy $h \equiv \bar{\nu}$. Therefore $h \in F_{n+1}$, and so $\bar{\alpha} = 0$.

(iii) We have proved in Proposition **??** that in the case of a CW-filtration, and $n \geqslant 2$, $H_n(\Pi X_*, \nu) \cong H_n(\widetilde{X}_\nu)$, and in Corollary 14.7.3, that the inclusion $RX_* \to S^\square X$ is a homotopy equivalence. □

Remark 14.7.11 The reason for choosing the notation $\mathbf{\Gamma}$ in the above theorem is that we have essentially derived a certain exact sequence considered in the paper [Whi50a]. However we have used the bold symbol Γ so as not to confuse with our connections.

Definition 14.7.12 We say X_* is a J_n*-filtered space* if for $0 \leqslant i < n$ and $\nu \in X_0$, the map

$$\pi_{i+1}(X_i, \nu) \to \pi_{i+1}(X_{i+1}, \nu)$$

induced by inclusion is trivial. □

Corollary 14.7.13 *If X_* is a J_n-filtered space, then each fibre of $p : RX_* \to \rho X_*$ is n-connected, and the induced maps $\pi_i RX_* \to \pi_i \rho X_*, H_i RX_* \to H_i \rho X_*$, of homotopy and homology, are isomorphisms for $i \leqslant n$ and epimorphisms for $i = n + 1$.* □

14.8 Notes

14.8.1 Notes to section 14.7

The history of classical papers on singular homology and the Hurewicz Theorem shows the use of deformation theorems of the type of Theorem 14.7.1, as for example in Blakers [Bla48]. However the use of simplicial rather than cubical methods, and of chain complexes, does seem to complicate the proof. Cubical methods are easier for constructing homotopies, as in [Mas80, Section III.7].

The methods of Whitehead in [Whi50a] for his exact sequence are more direct and he also proves a remarkable determination of $\Gamma_3 X$ as the value of a 'universal quadratic functor' on $\pi_2(X)$. This is related to results in [BL87b].

Further, the condition that X_* be a J_n-filtered space is in the CW-complex case precisely the condition that X is a J_n-complex in the sense of [Whi49b], and is also by Theorem 14.7.10 equivalent to $p : RX_* \to \rho X_*$ being an n-equivalence. Thus these results are related to the results of [Ada56] which give necessary and sufficient conditions for X to be a J_n-complex.

Chapter 15

Tensor products and homotopies

We now explain the final piece of algebraic structure which gives power to the machinery of crossed complexes, particularly the homotopy classification Theorem 10.4.17. A vital part of this machinery is the monoidal closed structure on crossed complexes, and its properties, which were stated in Chapter 9. Our justification of these properties is in terms of the category ω-Gpds of ω-groupoids, where the corresponding monoidal closed structure has a simple and convenient expression. In this category it is also easy to construct the natural transformation

$$\eta : \rho X_* \otimes \rho Y_* \to \rho(X_* \otimes Y_*)$$

and so this may be transferred to the category of crossed complexes and the functor Π via the equivalence of categories γ and Theorem 14.4.1.

The structure of this chapter is as follows. In Section 15.1 we extend to ω-groupoids the structure of monoidal closed category constructed for cubical sets in Chapter 10. The extension is quite straightforward.

Then, in Section 15.2 we study the transition from ω-groupoids to crossed complexes using the details of the inverse equivalences

$$\gamma : \omega\!-\!\mathsf{Gpd} \rightleftarrows \mathsf{Crs} : \lambda$$

getting a fairly complicated description of the closed category structure for closed complexes. In some sense this difficulty is an advantage, since the results of the story are easy to use (see Chapter 9), and when we do use these results, we know we have a powerful machine in the background, so that the applications have the potential of being highly non trivial, without this machinery.

In Section 15.3, we define a natural transformation of the Eilenberg-Zilber type

$$\theta' : \rho X_* \otimes \rho Y_* \to \rho(X_* \otimes Y_*)$$

proving that it is an isomorphism when X and Y are CW-complexes. Again this result can be transferred to crossed complex giving the very important Theorem 9.8.1. In Section 15.4 we establish the symmetry of the tensor product which, by contrast with the other results, is easier to prove for crossed complexes than for ω-groupoids. (It is interesting to note that the tensor product of cubical sets is not symmetric; the extra structure of ω-groupoids is needed to define the symmetry map $G \otimes H \to H \otimes G$). In Section 15.5 we give a brief account of the case of ω-groupoids with base-point.

In the last two sections we give a dense subcategory of the category of ω-groupoids, and use this to show certain covering crossed complexes of tensor product of crossed complexes is also a tensor product of coverings. We use this to prove that the tensor product of two free aspherical crossed complexes is also aspherical. This is a useful result for our earlier chapter on homotopy classification of maps and cohomology (Chapter 12).

15.1 The monoidal closed structure on omega-groupoids

The category ω-GPDS of ω-groupoids is a convenient algebraic model for certain geometric constructions. In particular it is well-suited for the discussion of homotopies and higher homotopies and their composition.

The precise definition of ω-groupoid is in a previous chapter, Section 13.2; recall that an ω-groupoid is a cubical set with extra structures of connections and compositions, the latter giving groupoid structures. The internal hom functor for cubical sets developed in Subsection 10.2.3 generalises immediately to ω-groupoids as follows.

Definition 15.1.1 Any ω-groupoid G has an underlying cubical set and we have given in Definition 10.2.10 the n-fold left path cubical set P^nG. It is

$$(P^nG)_r = G_{n+r},$$

with cubical operators

$$\begin{aligned} \partial^\alpha_{n+1}, \partial^\alpha_{n+2}, \ldots, \partial^\alpha_{n+r} : (P^nG)_r \to (P^nG)_{r-1}, \quad \alpha = +, - \qquad \text{and,}\\ \varepsilon_{n+1}, \varepsilon_{n+2}, \ldots, \varepsilon_{n+r} : (P^nG)_{r-1} \to (P^nG)_r. \end{aligned}$$

Now, we can define connections

$$\Gamma_{n+1}, \Gamma_{n+2}, \ldots, \Gamma_{n+r-1} : (P^nG)_{r-1} \to (P^nG)_r$$

and compositions

$$+_{n+1}, +_{n+2}, \ldots, +_{n+r} \;\text{ on }\quad (P^nG)_r.$$

They make P^nG an ω-groupoid since the laws to be checked are just a subset of the ω-groupoid laws of G. We call P^nG the *n-fold (left) path ω-groupoid* of G. □

The operators of G not used in P^mG give maps

$$\begin{aligned} \partial^\alpha_1, \ldots, \partial^\alpha_m : P^mG \to P^{m-1}G,\\ \varepsilon_1, \ldots, \varepsilon_m : P^{m-1}G \to P^mG,\\ \Gamma_1, \ldots, \Gamma_{m-1} : P^{m-1}G \to P^mG \end{aligned}$$

which are morphisms of ω-groupoids and obey the cubical laws. The unused additions of G define partial compositions $+_1, +_2, \ldots, +_m$ on P^mG which, by the ω-groupoid laws for G, are compatible with the ω-groupoid structure of P^mG.

Definition 15.1.2 The *'internal hom' ω-groupoid* $\omega\text{-GPDS}(G, H)$ is defined for any ω-groupoids G, H by:

$$\omega\text{-GPDS}_m(G, H) = \omega\text{-Gpds}(G, P^mH),$$

with cubical operators

$$\partial_1^\alpha, \dots, \partial_m^\alpha : \omega\text{-}\mathsf{GPDS}(G, H)_m \to \omega\text{-}\mathsf{GPDS}(G, H)_{m-1};$$
$$\varepsilon_1, \dots, \varepsilon_m : \omega\text{-}\mathsf{GPDS}(G, H)_{m-1} \to \omega\text{-}\mathsf{GPDS}(G, H)_m,$$

connections

$$\Gamma_1, \dots, \Gamma_{m-1} : \omega\text{-}\mathsf{GPDS}(G, H)_{m-1} \to \omega\text{-}\mathsf{GPDS}(G, H)_m$$

and compositions

$$+_1, \cdots, +_m \text{ on } \omega\text{-}\mathsf{GPDS}(G, H)_m$$

all induced by the similarly numbered operations on H.

We make ω-$\mathsf{GPDS}(G, H)$ a functor in G and H (contravariant in G) in the obvious way: if $g : G \to G'$ and $h : H \to H'$ are morphisms, the corresponding morphism

$$\omega\text{-}\mathsf{GPDS}(g, h) : \omega\text{-}\mathsf{GPDS}(G, H) \to \omega\text{-}\mathsf{GPDS}(G', H')$$

is given, in dimension r, by

$$\omega\text{-}\mathsf{GPDS}(g, h)_r(f) = (P^r h) \circ f \circ g,$$

for each $f : G \to P^r H$. □

Remark 15.1.3 Thus in dimension 0, ω-$\mathsf{GPDS}(G, H)$ consists of all morphisms $G \to H$, while in dimension n it consists of n-*fold (left) homotopies* $G \to H$.

The definition of tensor products of ω-groupoids is harder. We require that $- \otimes G$ be left adjoint to ω-$\mathsf{GPDS}(G, -)$ as a functor from ω-GPDS to ω-GPDS, and this determines $\otimes$ up to natural isomorphism.

One way of getting the tensor product is using the power of generalities, because the representability of the functor ω-$\mathsf{GPDS}(F, \omega\text{-}\mathsf{GPDS}(G, -))$ can be asserted on general grounds. The point is that ω-GPDS is an equationally defined category of many sorted algebras in which the domains of the operations are defined by finite limit diagrams. General theorems on such algebraic categories (see Notes) imply that ω-GPDS is complete and cocomplete and that it is monadic over the category Cub of cubical sets.

We are going to follow an alternative path strengthening the bicubical maps of subsection 10.2.1 to bimorphisms. The definition requires for any ω-groupoid H the *transposition* TH (see definition 10.2.19): here we just say that TH has the same elements as H but has its cubical operations, connections and compositions numbered in reverse order.

Definition 15.1.4 For any ω-groupoids F, G, H a *bimorphism* $f : (F, G) \to H$ is a family of maps

$$f_{pq} : F_p \times G_q \to H_{p+q} \ (p, q \geqslant 0)$$

such that

(i) for each $x \in F_p$, the map

$$f_x = f(x, -) : G \to P^p H$$

given by $y \mapsto f(x, y)$ is a morphism of ω-groupoids;

(ii) for each $y \in G_q$ the map

$$f_y = f(-, y) : F \to TP^q TH$$

given by $x \mapsto f(x, y)$ is a morphism of ω-groupoids. □

These bimorphisms may be reinterpreted in terms of morphisms.

Proposition 15.1.5 *There is a natural one-one correspondence between*

1.- Bimorphisms $(F, G) \to H$*, and*

2.- Morphisms $f : F \to \omega\text{-GPDS}(G, H)$.

Proof The conditions in the definition of a bimorphism from (F, G) to H, may be interpreted as saying that condition (i) gives maps $F_p \to \omega\text{-GPDS}_p(G, H)$ for each p, and condition (ii) states that these combine to give a morphism of ω-groupoids $F \to \omega\text{-GPDS}(G, H)$. □

Definition 15.1.6 We define the *ω-groupoid tensor product* $F \otimes G$ as given by the bimorphism

$$\chi : (F, G) \to F \otimes G$$

universal with respect to bimorphisms $(F, G) \to H$. We shall denote $\chi(x, y)$ by $x \otimes y$.

The universality condition says of course that every bimorphism $f : (F, G) \to H$ factors uniquely as $(x, y) \mapsto \hat{f}(x \otimes y)$ where $\hat{f} : F \otimes G \to H$ is a morphism of ω-groupoids. □

Proposition 15.1.7 *The tensor product is associative: i.e. for all ω-groupoids E, F, G there is a natural isomorphism*

$$(E \otimes F) \otimes G \cong E \otimes (F \otimes G).$$

Proof Both sides of the above equation are determined by a universal property with respect to 'trimorphisms' from E, F, G. □

Proposition 15.1.8 (exponential law for ω-groupoids) *For any ω-groupoid G, the functor $\omega\text{-GPDS}(G, -)$ is right adjoint to the functor $- \otimes G$; so there are bijections*

$$\omega\text{-Gpds}(F \otimes G, H) \cong \omega\text{-Gpds}(F, \omega\text{-GPDS}(G, H))$$

natural with respect to ω-groupoids F, G, H.

Proof We get the bijection just by putting together the previous definitions and the universality condition. □

This proposition can be strengthened in a standard way:

Proposition 15.1.9 *For ω-groupoids F, G, H there is a natural equivalence*

$$\omega\text{-GPDS}(F \otimes G, H) \cong \omega\text{-GPDS}(F, \omega\text{-GPDS}(G, H)).$$

Proof We can use proposition 15.1.8 repeatedly and the associativity of the tensor product to give for any ω-groupoid E a natural isomorphism

$$\omega\text{-Gpds}(E, \omega\text{-GPDS}(F \otimes G, H)) \cong \omega\text{-Gpds}(E, \omega\text{-GPDS}(F, \omega\text{-GPDS}(G, H)).$$

The result follows. □

We will show in Section 15.4 that the tensor product of ω-groupoids is symmetric, although the isomorphism $G \otimes H \cong H \otimes G$ is not an obvious one.

We now show that, as in the tensor product of R-modules, the tensor product for ω-groupoids may also be given by a presentation.

We may specify an ω-groupoid by a *presentation*, that is, by giving a set of generators in each dimension and a set of defining relations of the form $u = v$, where u, v are well-formed formulae of the same dimension made from generators and the operators $\partial_i^\alpha, \varepsilon_i, \Gamma_i, +_i, -_i$.

Now, given ω-groupoids F, G, we give an alternative, but equivalent, definition of $F \otimes G$ by giving a presentation of it as an ω-groupoid. The universal property of the presentation will then give the required adjointness.

Definition 15.1.10 Let F, G be ω-groupoids. We define $F \otimes G$ to be the ω-groupoid generated by elements in dimension $n \geqslant 0$ of the form $x \otimes y$ where $x \in F_p$, $y \in G_q$ and $p + q = n$, subject to the following defining relations (plus, of course, the laws for ω-groupoids)

(i) $\partial_i^\alpha(x \otimes y) = \begin{cases} (\partial_i^\alpha x) \otimes y & \text{if } 1 \leqslant i \leqslant p, \\ x \otimes (\partial_{i-p}^\alpha y) & \text{if } p+1 \leqslant i \leqslant n; \end{cases}$

(ii) $\varepsilon_i(x \otimes y) = \begin{cases} (\varepsilon_i x) \otimes y & \text{if } 1 \leqslant i \leqslant p+1, \\ x \otimes (\varepsilon_{i-p} y) & \text{if } p+1 \leqslant i \leqslant n+1; \end{cases}$

(iii) $\Gamma_i(x \otimes y) = \begin{cases} (\Gamma_i x) \otimes y & \text{if } 1 \leqslant i \leqslant p, \\ x \otimes (\Gamma_{i-p} y) & \text{if } p+1 \leqslant i \leqslant n; \end{cases}$

(iv) $(x +_i x') \otimes y = (x \otimes y) +_i (x' \otimes y)$ if $1 \leqslant i \leqslant p$ and $x +_i x'$ is defined in F;

(v) $x \otimes (y +_j y') = (x \otimes y) +_{p+j} (x \otimes y')$ if $1 \leqslant j \leqslant q$ and $y +_j y'$ is defined in G. □

Remark 15.1.11 There are quite a few relations that can be deduced from this Definition. In particular

(vi) $-_i(x \otimes y) = \begin{cases} (-_i x) \otimes y & \text{if } 1 \leqslant i \leqslant p, \\ x \otimes (-_{i-p} y) & \text{if } p+1 \leqslant i \leqslant n \end{cases}$ and

(vii) $(\varepsilon_{p+1} x) \otimes y = x \otimes (\varepsilon_1 y)$. □

15.1.1 Relations between the internal homs for cubes and for omega-groupoids.

We now use the free ω-groupoid ρK on a cubical set K, which gives the left adjoint

$$\rho : \mathsf{Cub} \to \omega\text{-}\mathsf{Gpds}$$

to the forgetful functor

$$U : \omega\text{-}\mathsf{Gpds} \to \mathsf{Cub},$$

to relate the monoidal closed structures of Cub and ω-Gpds. This will enable us to tie in the theory with results in Sections 10.2 10.4 on the nerve of a crossed complex.

It is easy to see that $\rho(K)$ is the ω-groupoid generated by elements $[k]$ for all $k \in K$ with defining relations given by $\partial_i^\alpha[k] = [\partial_i^\alpha k]$ and $\varepsilon_i[k] = [\varepsilon_i k]$ for all $n \geqslant 1$ and face and degeneracy maps $\partial_i^\alpha : K_n \to K_{n-1}$ and $\varepsilon_i : K_{n-1} \to K_n$.

This notation is consistent with our previous use of $\rho(K)$ as the fundamental ω-groupoid of the filtered space $|K_*|$ because, for any cubical set K, $\rho(K) \cong \rho(|K_*|)$, by the HHvKT, as a deduction from Theorem 14.3.1. In particular we will write $\mathbf{I}^n$ for the ω-groupoid $\rho(\mathbb{I}^n)$, which is also the free ω-groupoid on one generator of dimension n.

Proposition 15.1.12 *For a cubical set* L *and an* ω*-groupoid* G*, there is a natural isomorphism of cubical sets*

$$U(\omega\text{-}\mathsf{GPDS}(\rho(L), G)) \cong \mathsf{CUB}(L, UG).$$

Proof Let us get first the bijections at every dimension, i.e.

$$\omega\text{-}\mathsf{GPDS}_r(\rho(L), G) \cong \mathsf{CUB}_r(L, UG)$$

for all $r \geqslant 0$.

They follow from the adjointness since the bijections

$$\omega\text{-}\mathsf{GPDS}_r(\rho(L), G) = \omega\text{-}\mathsf{Gpds}(\rho(L), P^r G) \cong \mathsf{Cub}(L, UP^r G) = \mathsf{CUB}_r(L, UG).$$

are compatible with the cubical operators. □

From this Property we easily deduce that the free ω-groupoid functor preserves the tensor product.

Proposition 15.1.13 *If* K, L *are cubical sets, there is a natural isomorphism of* ω*-groupoids*

$$\rho K \otimes \rho L \cong \rho(K \otimes L).$$

Proof From the previous Proposition 15.1.12 and the closed category structures of Cub and ω-Gpds, we get the bijection of cubical sets

$$\begin{array}{ll} U(\omega\text{-}\mathsf{GPDS}(\rho(K \otimes L), G)) \cong & \\ \cong \mathsf{CUB}(K \otimes L, UG) & \text{by 15.1.12} \\ \cong \mathsf{CUB}(K, \mathsf{CUB}(L, UG)) & \text{since } \mathsf{Cub} \text{ is monoidal closed} \\ \cong \mathsf{CUB}(K, U(\omega\text{-}\mathsf{GPDS}(\rho(L), G))) & \text{by 15.1.12} \\ \cong U(\omega\text{-}\mathsf{GPDS}(\rho(K), \omega\text{-}\mathsf{GPDS}(\rho(L), G))) & \text{by 15.1.12} \\ \cong U(\omega\text{-}\mathsf{GPDS}(\rho(K) \otimes \rho(L), G)) & \text{since } \omega\text{-}\mathsf{Gpds} \text{ is monoidal closed.} \end{array}$$

The proposition follows from the bijection in dimension 0, namely

$$\omega\text{-}\mathsf{Gpds}(\rho(K \otimes L), G) \cong \omega\text{-}\mathsf{Gpds}(\rho(K) \otimes \rho(L), G).$$

□

We get as a consequence the following relation among $\rho(\mathbb{I}^n)$ the ω-groupoid freely generated by one element in dimension n.

Corollary 15.1.14 *There are natural isomorphisms of* ω*-groupoids*

$$\rho(\mathbb{I}^m) \otimes \rho(\mathbb{I}^n) \cong \rho(\mathbb{I}^{m+n}).$$

Proposition 15.1.15 (i) $\rho(\mathbb{I}^n) \otimes -$ *is left adjoint to* $P^n : \omega\text{-}\mathsf{GPDS} \to \omega\text{-}\mathsf{GPDS}$.

(ii) $- \otimes \rho(\mathbb{I}^n)$ *is left adjoint to* $\omega\text{-}\mathsf{GPDS}(\rho(\mathbb{I}^n), -)$.

(iii) ω-GPDS$(\rho(\mathbb{I}^n), -)$ *is naturally isomorphic to* $\mathsf{T}\mathsf{P}^n\mathsf{T}$.

Proof (i) There are natural bijections

$$\begin{aligned}\omega\text{-Gpds}(\rho(\mathbb{I}^n) \otimes H, K) &\cong \omega\text{-Gpds}(\rho(\mathbb{I}^n), \omega\text{-GPDS}(H, K)) \\ &\cong \omega\text{-GPDS}_n(H, K) = \omega\text{-Gpds}(H, P^n K).\end{aligned}$$

(ii) This is a special case of Proposition 15.1.8.

(iii) It follows from (i) that $\mathsf{T}\mathsf{P}^n\mathsf{T} : \omega\text{-GPDS} \to \omega\text{-GPDS}$, has left adjoint $\mathsf{T}(\rho(\mathbb{I}^n) \otimes \mathsf{T}(-)) \cong - \otimes \mathsf{T}\rho(\mathbb{I}^n)$. But the obvious isomorphism $\mathsf{T}\mathbb{I} \to \mathbb{I}$ induces an isomorphism $\mathsf{T}\rho(\mathbb{I}^n) \cong \rho(\mathbb{I}^n)$, so $- \otimes \mathsf{T}\rho(\mathbb{I}^n)$ is naturally isomorphic to $- \otimes \rho(\mathbb{I}^n)$. The result now follows from (ii). □

Remark 15.1.16 It was proved in Section **??** that $\rho(\mathbb{I}^n)$ is the fundamental ω-groupoid $\rho(I^n_*)$ of the n-cube with its skeletal filtration. We will show, by similar methods, that for any cubical set K, there is a natural isomorphism $\rho(K) \cong \rho(|K_*|)$, where $|K_*|$ is the geometric realisation of K, with its skeletal filtration. Thus Proposition 15.1.13 gives an isomorphism

$$\rho(|K_*| \otimes |L_*|) \cong \rho(|K_*|) \otimes \rho(|L_*|)$$

which can be generalised to an isomorphism

$$\rho(X \otimes Y) \cong \rho(X) \otimes \rho(Y)$$

for arbitrary CW-complexes X, Y. □

15.2 The monoidal closed structure on crossed complexes revisited

It is an easy exercise to prove that given a monoidal closed category $\mathcal{C}$ and a equivalent category $\mathcal{C}'$, we can use the equivalence to transfer the closed category structure from $\mathcal{C}$ to $\mathcal{C}'$. Thus the monoidal closed structure defined on ω-GPDS in Section 15.1 can be transferred to the category Crs by defining

$$C \otimes D = \gamma(\lambda C \otimes \lambda D) \quad \text{and} \quad \mathsf{Crs}(C, D) = \gamma(\omega\text{-GPDS}(\lambda C, \lambda D)),$$

for arbitrary crossed complexes C and D.

Remark 15.2.1 There is one aspect of the notion of monoidal categories which should in principle be given more coverage than we are giving, namely the various coherence laws which are part of the standard definition, see [ML71, Chapter VII]. These laws will not be important for our purposes, and so we leave their investigation in our cases to the reader. Because the tensor product is defined in the various cases by a universal property of 'bi'-morphisms of carious types, coherence properties reduce to those of the usual cartesian product of sets, which, as an example on p.160 of [ML71] shows, cannot be be taken to give a strict monoidal structure. The coherence laws should be taken into account when making constructions such as 'free internal monoids with respect to tensor', as discussed for crossed complexes in [BB93]. A discussion of free monoids in the general case is in [Lac08].

Our goal in this Section is to derive this monoidal closed structure on the category Crs from that on ω-groupoids and so arrive at the Definitions already given in Section 9.3.

We begin with the translation of the internal hom functor that is the most direct. Then we translate the concept of bimorphism since it is essentially a 'morphism of morphisms'. The least direct is the tensor product, that can be done in terms of a presentation.

The difficulty in passing from presentations in ω-GPDS to presentations in Crs may be illustrated by the example $\rho(\mathbb{I}^n)$. In ω-GPDS, this is free on one generator in dimension n; however, the corresponding crossed complex $\gamma\rho(\mathbb{I}^n) \cong \pi(\rho(I^n_*))$ requires, for each r-dimensional face d of I^n, a generator $x(d)$ in dimension r, with defining relations of the form

$$\delta(x(d)) = \sum_{(\alpha,i)} \{x(\partial_i^\alpha d)\},$$

where the formula for the 'sum of the faces' on the right is given by the Homotopy Addition Lemma 13.7.1.

15.2.1 The internal hom on crossed complexes

As we have seen we could define

$$\mathsf{CRS}(C, D) = \gamma(\omega\text{-}\mathsf{GPDS}(\lambda C, \lambda D))$$

for any crossed complexes C, D and get a closed category structure on Crs. We want to describe the structure of $\mathsf{CRS}(C, D)$ in terms internal to the crossed complexes C, D and arrive to the definition of left (or right) m-fold homotopy for crossed complexes given in 9.3.6, i.e. a pair (F, f) where f is a morphism of crossed complexes and F has degree m over f satisfying some conditions.

So, we have to study $\gamma(\omega\text{-}\mathsf{GPDS}(G, H))_m$ for two ω-groupoids G, H. It is clear that its elements are m-fold homotopies of ω-groupoids which satisfy an extra degeneracy condition (almost all faces are degenerate). Thus we want to examine these homotopies.

The main technical tool for changing a cube to another one with extra degeneracies is the folding map Φ. Thus we are going to use the folding map to relate both kinds of m-fold homotopies.

Proposition 15.2.2 *Let G, H be ω-groupoids, let $\psi : G \to H$ be an m-fold left homotopy. We may define*

(i) *A morphism of crossed complexes*

$$f : \gamma G \to \gamma H$$

defined by $f = \partial_1^+ \partial_2^+ \cdots \partial_m^+ \psi$.

(ii) *An homotopy over* f

$$F : \gamma G \to \gamma H$$

given by $F = \Phi\psi$.

This m-fold left homotopy of crossed complexes (F, f) is said to be associated to ψ.

Proof The part (i) is clear since

$$\partial_1^+ \partial_2^+ \cdots \partial_m^+ \psi : G \to H$$

is a morphism of ω-groupoids. Thus it maps γG to γH and restricts to a morphism

$$f : \gamma G \to \gamma H$$

of crossed complexes.

Part (ii) is much longer since we have to check all conditions for a homotopy in Definition 9.3.6.

- Let us begin with the **base point**. Let us see that F is a map over f. For any $c \in (\gamma G)_n$, the base point is

$$\beta F(c) = \beta\Phi\psi(c) = \beta\psi(c) = \partial_1^+\partial_2^+\cdots\partial_{m+n}^+\psi(c) = \partial_1^+\cdots\partial_m^+\psi(\partial_1^+\cdots\partial_n^+c) = f(\beta c).$$

Thus $F(c) \in f_0\beta(c)$.

The other conditions for (F, f) to be a homotopy follow from the formulae for $\Phi(x +_i y)$ in Proposition 13.4.14.

- First the **operations**. Recall that in previous notation, for a k-dimensional cube x,

$$u_i x = \partial_1^+\cdots\partial_{i-1}^+\partial_{i+1}^+\cdots\partial_k^+ x.$$

- If $c + c'$ is defined in $(\gamma G)_1 = G_1$, then

$$F(c + c') = \Phi\psi(c +_1 c') = \Phi(\psi(c) +_{m+1} \psi(c')) = (\Phi\psi(c))^u + \Phi\psi(c') = h(c)^u + h(c'),$$

where $u = u_{m+1}\psi(c') = \partial_1^+\cdots\partial_m^+\psi(c') = f(c')$.

- Similarly, if $n \geqslant 2$ and $c + c'$ is defined in $(\gamma G)_n$, then

$$F(c + c') = F(c)^u + F(c'),$$

where $u = u_{m+n}\psi(c') = \partial_1^+\cdots\partial_m^+\partial_{m+1}^+\cdots\partial_{m+n-1}^+\psi(c') = \partial_1^+\cdots\partial_m^+\psi(\partial_1^+\cdots\partial_{n-1}^+c')$.

But since $c' \in (\gamma G)_n$, the element $\partial_1^+\cdots\partial_{n-1}^+c'$ of $(\gamma G)_1$ is the identity element $\varepsilon_1\beta c'$; so $u = f(\varepsilon_1\beta c')$ is also an identity element and $F(c + c') = F(c) + F(c')$.

- Now the **action**. If c^t is defined, where $c \in (\gamma G)_n$, $(n \geqslant 2)$ and $t \in (\gamma G)_1$, then

$$\begin{aligned}
F(c^t) = \Phi\psi(c^t) &= \Phi\psi(-_n\varepsilon_1^{n-1}t +_n c +_n \varepsilon_1^{n-1}t)\\
&= -_{m+n}\varepsilon_{m+1}^{n-1}\psi(t) +_{m+n} \psi(c) +_{m+n} \varepsilon_{m+1}^{n-1}\psi(t)\\
&= -(\Phi\varepsilon_{m+1}^{n-1}\psi(t))^u + (\Phi\psi(c))^v + \Phi\varepsilon_{m+1}^{n-1}\psi(t)
\end{aligned}$$

for certain edges $u, v \in (\gamma H)_1$.

But $n \geqslant 2$, so $\varepsilon_{m+1}^{n-1}\psi(t)$ is degenerate and $\Phi\varepsilon_{m+1}^{n-1}\psi(t) = 0$ for Proposition 13.4.18. Hence

$$F(c^t) = F(c)^v,$$

where $v = u_{m+n}(\varepsilon_{m+1}^{n-1}\psi(t)) = \partial_1^+\cdots\partial_{m+n-1}^+\varepsilon_{m+1}^{n-1}\psi(t) = \partial_1^+\cdots\partial_m^+\psi(t) = f(t)$ giving the result. □

Remark 15.2.3 Notice that given an m-fold left homotopy $\psi : G \to H$ of ω-groupoids, the m-fold left homotopy of crossed complexes associated to this, (F, f), satisfies an extra condition with respect to the folding map, namely:

$$F(\Phi x) = \Phi\psi(\Phi x) = \Phi\psi(\Phi_1\cdots\Phi_{n-1}x) = \Phi\Phi_{m+1}\cdots\Phi_{m+n-1}\psi(x) = \Phi\psi(x)$$

using Proposition 13.4.15. We call this extra condition

(FOLD) $F(\Phi x) = \Phi\psi(x)$. □

So we have associated to any m-fold left homotopy between ω-groupoids an m-fold left homotopy between the associated crossed complexes satisfying the extra condition (FOLD). Now we prove that the former homotopy between ω-groupoids may be reconstructed from the homotopy between the associated crossed complexes.

Proposition 15.2.4 *Let* G, H *be* ω*-groupoids, and* F *be any* m*-fold left homotopy from* γG *to* γH *beginning at* f *then there is a unique* m*-fold left homotopy* $\psi : G \to H$ *such that* F *is the associated homotopy and satisfies the extra condition about degeneration of the faces*

(DEG) $\partial_i^\alpha \psi(x) = \varepsilon_1^{m-1}\hat{f}(x)$

for $1 \leqslant i \leqslant m$, $\alpha = 0, 1$ *and* $(\alpha, i) \neq (0, 1)$ *and all* $x \in G$*, where* $\hat{f} : G \to H$ *denotes the unique morphism of* ω*-groupoids extending the morphism* $f : \gamma G \to \gamma H$ *of crossed complexes.*

Proof We are looking for the existence and uniqueness of an m-fold left homotopy $\psi : G \to H$ having F as associated homotopy and satisfying the extra conditions

(DEG) $\partial_i^\alpha \psi(x) = \varepsilon_1^{m-1}\hat{f}(x)$ for $i \leqslant m$, $(\alpha, i) \neq (0, 1)$, and

(FOLD) $\Phi\psi(x) = F(\Phi x)$.

Using these conditions we construct ψ inductively.

- When $n = 0$, all faces but one of $\psi(x)$ are specified by (DEG). The elements $z_i^\alpha = \varepsilon_1^{m-1}\hat{f}(x) = \varepsilon_1^{m-1}f(x)$ of H_{m-1} for $(\alpha, i) \neq (0, 1)$ form a box and the Homotopy Addition Lemma (13.7.1) gives a unique last face z_1^- such that $\delta\Phi\mathbf{z} = \Sigma\mathbf{z}$ has the value $\delta F(\Phi x) \in (\gamma H)_{m-1}$. Proposition 13.5.11 then gives a unique filler $\psi(x)$ for the box such that $\Phi(\psi(x))$ has the value $F(\Phi x)$. (Of course, one must verify that $\delta F(\Phi x) = \delta F(x)$ has the same basepoint as the given box, but this is clear since $\beta F(x) = \beta f(x)$).

- Now suppose that $n \geqslant 1$ and assume that $\psi(x)$ is already defined for all x of dimension $< n$ and that it satisfies (DEG) and (FOLD) for all such x. Assume further that ψ satisfies all the conditions for an m-fold left homotopy in so far as they apply to elements of dimension $< n$.

Then, for $x \in G_n$ we need to find $\psi(x) \in H_{m+n}$ satisfying (amongst others) the conditions

$$\begin{cases} \partial_j^\alpha \psi(x) = \varepsilon_1^{m-1}\hat{f}(x) & \text{for } 1 \leqslant j \leqslant m, (\alpha, j) \neq (0, 1), \\ \partial_{m+j}^\alpha \psi(x) = \psi(\partial_j^\alpha x) & \text{for } 1 \leqslant j \leqslant n, \\ \Phi\psi(x) = F(\Phi x). \end{cases} \tag{*}$$

It is direct to verify that the specified faces of $\psi(x)$ form a box whose basepoint is $\beta\hat{f}(x) = f(\Phi x) = F(\Phi x)$ and therefore, as in the case $n = 0$, there is a unique $\psi(x)$ satisfying these conditions.

To complete the induction we need only to verify that this $\psi(x)$ has all the defining properties of an m-fold homotopy.

For example, to prove that

$$\psi(x +_i y) = \psi(x) +_{m+i} \psi(y),$$

we first note that $\partial_{m+i}^+\psi(x) = \psi(\partial_i^+ x) = \psi(\partial_i^- y) = \partial_{m+i}^-\psi(y)$ so that $z = \psi(x) +_{m+i} \psi(y)$ is defined. We then verify easily, using the induction hypotheses, that the faces of z other than $\partial_1^- z$ are given by

$$\begin{cases} \partial_j^\alpha z = \varepsilon_1^{m-1}\hat{f}(x +_i y) & \text{for } 1 \leqslant j \leqslant m, \\ (\alpha, j) \neq (0, 1), & \\ \partial_{m+j}^\alpha z = \psi(\partial_j^\alpha(x +_i y)) & \text{for } 1 \leqslant j \leqslant n. \end{cases}$$

Also

$$\Phi z = \Phi(\psi(x) +_{m+i} \psi(y)) = (\Phi\psi(x))^u + \Phi\psi(y),$$

by Proposition 13.4.14, where $u = u_{m+i}\psi(y) = \partial_1^+ \cdots \partial_m^+\psi(u_i y) = \hat{f}(u_i y) = f(u_i y)$. But it may be verified that

$$F(\Phi(x +_i y)) = F(\Phi x)^{f(u_i y)} + F(\Phi y)$$

using the defining properties of h and formulae of Proposition 13.4.14. (In the case $n = 1, i = 1$ one needs to observe that addition in $(\gamma H)_{m+n}$ is commutative). Hence

$$\Phi z = F(\Phi(x +_i y))$$

in all cases.

It follows, by the uniqueness of $\psi(x)$ satisfying conditions (*), that $z = (x +_i y)$. The other properties of ψ are proved in a similar way. □

These propositions set up for $m \geqslant 1$ a bijection between m-fold left homotopies $\gamma G \to \gamma H$ and elements of $\gamma(\omega\text{-GPDS}(G, H))_m$, namely m-fold left homotopies $\psi : G \to H$ that satisfying the extra degeneracy condition

(DEG) $\partial_i^\alpha \psi(x) = \varepsilon_1^{m-1} \partial_1^+ \partial_2^+ \cdots \partial_m^+ \psi(x)$ for $i \leqslant m$, $(\alpha, i) \neq (0, 1)$.

(Note that if $\partial_i^\alpha u = \varepsilon_1^{m-1} v$, then v must be $\partial_1^+ \cdots \partial_m^+ u$).

We complete this correspondence by defining a 0-fold left (or right) homotopy of crossed complexes $C \to D$ to be a morphism $f : C \to D$. We then have:

Proposition 15.2.5 *The elements of* $\mathsf{CRS}(C, D)$ *in dimension* $m \geqslant 0$ *are in natural one-one correspondence with the* m*-fold left homotopies from* C *to* D.

In view of this result we will, from now on, identify $\mathsf{CRS}(C, D)$ with the collection of morphisms and left homotopies from C to D. The operations which give this collection the structure of a crossed complex can be deduced from the above correspondence. They will also be described later in internal terms.

15.2.2 Bimorphisms on crossed complexes

Next, we want to relate the concepts of bimorphism of ω-groupoids given in Definition 15.1.4 with that of bimorphism of crossed complexes introduced in Definition 9.3.11 of Part II.

We are going to use extensively the previous Subsection since in both cases a bimorphism may be interpreted by fixing the first variable as a family of m-fold left homotopies one for each element of dimension m (see Definition 15.1.4 and 9.3.11) and we know from the previous Subsection how both kinds of m-fold homotopies are related (essentially by the folding map).

Before entering in the proof of this correspondence let us state a result that is going to be used later.

Lemma 15.2.6 *If* $\chi : (F, G) \to H$ *is a bimorphism of* ω*-groupoids, then* $\chi(x, y)$ *is thin whenever* x *or* y *is thin.*

Proof We have just remarked that $\chi_y : F \to P^n H$ is a morphism of ω-groupoids. If x is thin in F, it follows that $\chi(x, y)$ is a thin element of $P^n H$. But the thin elements of $P^n H$ are a subset of the thin elements of H. □

Proposition 15.2.7 *Let* F, G, H *be* ω*-groupoids with associated crossed complexes* γF*,* γG*,* γH*. If*

$$\chi : (F, G) \to H$$

is any bimorphism of ω*-groupoids, then we have an associated bimorphism of crossed complexes*

$$\theta : (\gamma F, \gamma G) \to \gamma H$$

defined by $\theta(c,d) = \Phi\chi(c,d)$ *for any* $c \in \gamma F$ *and* $d \in \gamma G$.

Proof To check that θ is a bimorphism of crossed modules we have to see the behaviour with respect to source and target, actions and operations and boundary maps is according to Definition 9.3.11.

- With respect to the **base point**

$$\beta\theta(c,d) = \beta\Phi\chi(c,d) = \beta\chi(c,d) = \chi(\beta c, \beta d) = \theta(\beta c, \beta d).$$

- With respect to **actions and operations**.

For $c \in (\gamma F)_0$, the map $\chi_c : G \to H$ is a morphism of ω-groupoids. Thus

$$\Phi\chi_c : \gamma G \to \gamma H$$

is a morphism of crossed complexes.

Similarly, by Proposition 15.2.2, if $c \in (\gamma F)_m$ is fixed, then the map $\chi_c : G \to H$ is an m-fold left homotopy of ω-groupoids. Thus the map

$$\theta_c = \Phi\chi_c : \gamma G \to \gamma H$$

is an m-fold left homotopy $\gamma G \to \gamma H$ over the morphism $\theta_{\beta c} = \Phi\chi_{\beta c}$.

The morphism $\chi_{\beta c}$ maps γG into γH, so $\theta(\beta c, d) = \Phi\chi(\beta c, d) = \chi(\beta c, d)$ and θ_c is an m-fold homotopy over $\theta_{\beta c}$.

Now we repeat the same process with respect to the second variable. (Note that in this version for n-fold right homotopies $\gamma F \to \gamma H$ the formula $f(d) = \partial_1^+ \cdots \partial_m^+ \psi(d)$ is replaced by $f(c) = \partial_{n+1}^+ \cdots \partial_{n+m}^+ \psi(c)$. Hence, if $d \in (\gamma G)_n$, the right homotopy $c \mapsto \Phi\chi(c,d) : \gamma F \to \gamma H$ has base morphism $c \mapsto \partial_{n+1}^+ \cdots \partial_{n+m}^+ \chi(c,d) = \chi(c, \beta d)$.)

- With respect to **boundary maps** we use the Homotopy Addition Lemma 13.7.1; in order to compute $\delta\theta(c,d) = \delta\Phi\chi(c,d)$ we need to compute $\Phi\partial_i^\alpha\chi(c,d)$ for each face of $\chi(c,d)$ and sum them according to the formulae in the Lemma 13.7.1.

To compute $\delta\theta(c,d)$ in the case $m \geqslant 2, n \geqslant 2$ we note that the faces of c and d other than $\partial_1^- c, \partial_1^- d$ are all thin, so all but two faces of $\chi(c,d)$ are thin by Lemma 15.2.6, and we conclude that $\Phi\partial_i^\alpha\chi(c,d) = 0$ except when $\alpha = 0$ and $i = 1$ or $m+1$. The appropriate formula of the Homotopy Addition Lemma 13.7.1 now gives

$$\delta\theta(c,d) = \delta\Phi\chi(c,d) = (\Phi\chi(\partial_1^- c, d))^\nu + (-1)^m(\Phi\chi(c, \partial_1^- d))^w = \theta(\delta c, d)^\nu + (-1)^m\theta(c, \delta d)^w,$$

where $\nu = u_1\chi(c,d) = \chi(u_1 c, \beta d)$ and $w = u_{m+1}\chi(c,d) = \chi(\beta c, u_1 d)$. Since $c \in \gamma F$, $d \in \gamma G$, both $u_1 c$ and $u_1 d$ are identities, so ν, w act trivially and we obtain the formula

$$\delta\theta(c,d) = \theta(\delta c, d) + (-1)^m\theta(c, \delta d).$$

The other formulae of Definition 9.3.11 are proved in the same way using the different forms of the Homotopy Addition Lemma 13.7.1 in various cases. Thus θ is a bimorphism of crossed complexes. □

Proposition 15.2.8 *Let* F, G, H *be* ω*-groupoids with corresponding crossed complexes* γF, γG, γH. *Given any bimorphism*

$$\theta : (\gamma F, \gamma G) \to \gamma H$$

of crossed complexes, there is a unique bimorphism

$$\chi : (F, G) \to H$$

of ω*-groupoids satisfying* $\theta(c,d) = \Phi\chi(c,d)$ *for* $c \in \gamma F$ *and* $d \in \gamma G$.

Proof For each $c \in (\gamma F)_m$ we have an m-fold left homotopy

$$(\theta_c, f_c) : \gamma G \to \gamma H.$$

By Proposition 15.2.4, there is a unique m-fold left homotopy

$$\psi_c : G \to H$$

satisfying the conditions

$$\begin{cases} \Phi\psi_c(d) = \theta_c(d) = \theta(c, d) & \text{for } d \in D, \\ \psi_c \in \gamma(\omega\text{-GPDS}(G, H)). \end{cases} \qquad (**)$$

The required bimorphism χ must yield such an n-fold left homotopy $y \mapsto \chi(c, y)$, so the definition $\chi(c, y) = \psi_c(y)$ is forced. Furthermore, since γF generates F as ω-groupoid by Corollary 13.5.14 13.5.14 and $\chi(x, y)$ must preserve first variable x, for fixed y, the values $\chi(c, y)$ for $c \in \gamma F$, $y \in G$ determine χ completely. Thus χ is unique if it exists.

To prove that the required bimorphism χ exists we first note that we have a map $c \mapsto \psi_c$ from γF to $\gamma(\omega\text{-GPDS}(G, H))$ of degree 0 and we will show that it is a morphism of crossed complexes where the crossed complex structure of $\gamma(\omega\text{-GPDS}(G, H))$ has been given in Definition 9.3.8

We need to show that $\psi_{c+c'} = \psi_c + \psi_{c'}, \psi_{c^t} = \psi_c^{\psi_t}, \psi_{\delta c} = \delta\psi_c$ if $c \in (\gamma F)_m (m \geqslant 2)$, and $\psi_{\delta^\alpha c} = \delta^\alpha \psi_c$ if $c \in (\gamma F)_1$. Using the characterisation (**) of ψ_c and the fact that $\psi_c + \psi_{c'}, \psi_c^{\psi_t}$, etc. are all elements of $\gamma(\omega\text{-GPDS}(G, H))$, it is enough to prove that, for $d \in \gamma G$,

(i) $\Phi(\psi_c(d) +_m \psi_{c'}(d)) = \theta(c + c', d)$ if $c + c'$ is defined in $(\gamma F)_m$,

(ii) $\Phi(-_m \varepsilon_1^{m-1}\psi_t(d) +_m \psi_c(d) +_m \varepsilon_1^{m-1}\psi_t(d)) = \theta(c^t, d)$ if $t \in A_1$ and c^t is defined in $(\gamma F)_m (m \geqslant 2)$,

(iii) $\Phi(\partial_1^- \psi_c(d)) = \theta(\delta c, d)$ if $c \in (\gamma F)_m, m \geqslant 2$,

(iv) $\Phi(\partial_1^\alpha \psi_c(d)) = \theta(\delta^\alpha c, d)$ if $c \in (\gamma F)_1, \alpha = \pm$.

The calculations for (i) and (ii) are similar to calculations done in the proof of Proposition 15.2.4. For example, in (ii), if $c \in (\gamma F)_m, d \in (\gamma G)_n$, then $\Phi(\varepsilon_1^{m-1}\psi_t(d)) = 0$, so

$$\Phi(-_m \varepsilon_1^{m-1}\psi_t(d) +_m \psi_c(d) +_m \varepsilon_1^{m-1}\psi_t(d)) = (\Phi\psi_c(d))^\nu = \theta(c, d)^\nu$$

where

$$\begin{aligned} \nu = u_m \varepsilon_1^{m-1}\psi_t(d) &= \partial_1^+ \cdots \partial_{m-1}^+ \partial_{m+1}^+ \cdots \partial_{m+n}^+ \varepsilon_1^{m-1}\psi_t(d) \\ &= \partial_2^+ \cdots \partial_{n+1}^+ \psi_t(d) = \psi_t(\partial_1^+ \cdots \partial_n^+ b) \\ &= \psi_t(\beta d) = \theta(t, \beta d) \quad (\text{since } \Phi = \text{id in dimension } 1). \end{aligned}$$

Hence $\theta(c, d)^\nu = \theta(c, d)^{\theta(t, \beta d)} = \theta(c^t, d)$ since $c \mapsto \theta(c, d)$ is an n-fold right homotopy with base morphism $c \mapsto \theta(c, d)$.

The calculations for (iii) and (iv) use the Homotopy Addition Lemma 13.7.1 and the behaviour of θ with respect to the boundary map. For example, to prove (iii) we observe that $\Phi\psi_c(d) = \theta(c, d)$ and $\delta\Phi\psi_c(d) = \Sigma\{\Phi\partial_i^\alpha\psi_c(d)\}$, the sum of the folded faces on the right being calculated by the appropriate formula of the Homotopy Addition Lemma 13.7.1, depending on the dimensions of c and d. Now $c \in \gamma F$ and $d \in \gamma G$ so most terms in this sum are 0. In the case $m \geqslant 2, n \geqslant 2$, two terms survive and one of these, $\Phi\partial_{m+1}^- \psi_c(d)$, we can calculate: because ψ_c is an m-fold left homotopy of

ω-groupoids, $\Phi\partial^-_{m+1}\psi_c(d) = \Phi\psi_c(\partial^-_1 d) = \theta(c, \delta d)$. Hence the Homotopy Addition Lemma 13.7.1 says

$$\delta\theta(c, d) = \Phi\partial^-_1\psi_c(d) + (-1)^m\theta(c, \delta d).$$

Comparing this with the defining property

$$\delta\theta(c, d) = \theta(\delta c, d) + (-1)^m\theta(c, \delta d)$$

we obtain (iii). The other cases are similar. This proves that $c \mapsto \psi_c$ is a morphism of crossed complexes from γF to $\gamma(\omega\text{-GPDS}(G, H))$.

It therefore extends uniquely to a morphism of ω-groupoids $x \mapsto \psi_x$, say, from F to ω-GPDS(G, H). But now the definition $\chi(x, y) = \psi_x(y)$ gives a bimorphism of ω-groupoids $\chi : (F, G) \to H$ such that $\Phi\chi(c, d) = \Phi\psi_c(d) = \theta(c, d)$ for $c \in \gamma F$, $d \in \gamma G$, and this completes the proof. □

15.2.3 The tensor product of crossed complexes

Last, we want to describe tensor products of crossed complexes. Let C, D be crossed complexes. If we choose ω-groupoids F, G such that $C = \gamma F, D = \gamma G$, we should have

$$C \otimes D = \gamma(F \otimes G).$$

If we consider the universal bimorphism of ω-groupoids

$$\chi : (F, G) \to F \otimes G,$$

it is clear that the bimorphism of crossed complexes

$$\theta : (C, D) \to C \otimes D$$

given by the restriction of the composition $\Phi\chi$ is universal with respect to bimorphisms of crossed complexes from (C, D).

By the universality of the bimorphism of crossed complexes

$$\theta : (C, D) \to \gamma(F \otimes G),$$

it is clear that $C - \otimes D$ is the left adjoint to $\mathsf{Crs}(D, -)$.

A warning about notation. For any $c \in C = \gamma F$ and $d \in D = \gamma G$, we have already defined their tensor product by

$$c \otimes d = \chi(c, d) \in F \otimes G.$$

Clearly we have good reason for calling also

$$c \otimes d = \theta(c, d) \in C \otimes D.$$

We shall keep $c \otimes d$ for this last definition, while calling $c \hat{\otimes} d$ its tensor product in $F \otimes G$.

The Definition 9.3.11 of a bimorphism now gives the presentation of $C \otimes D$ described in Definition 9.3.13.

This completes the derivation of the monoidal closed structure on the category Crs.

15.2.4 Another description of the internal hom in Crs

We now go back to $\mathsf{CRS}(C,D)$ and produce a description of its crossed complex structure in terms of the crossed complex structures of C and D.

Recall from Definition 7.4.8 that $\mathbb{F}(m)$ is the crossed complex freely generated by one generator a in dimension m. Any element of $\mathsf{CRS}_m(C,D)$ corresponds to a morphism $\mathbb{F}(m) \to \mathsf{CRS}(C,D)$, or, equivalently, to a bimorphism $\theta : (\mathbb{F}(m), C) \to D$. If $m = 0$ the given element is the morphism

$$\psi_a : C \to D$$

defined by $\psi_a(c) = \theta(a,c)$.

If $m \geqslant 1$ then $\psi_a(c) = \theta(a,c),\ f_a(c) = \theta(\beta a, c)$ defines the m-fold left homotopy $\psi_a = (\psi_a, f_a)$.

Similarly, if two elements of $\mathsf{CRS}(C,D)$ are given, we may choose A to be the free crossed complex on two generators of appropriate dimensions and represent both the given elements as induced by the same bimorphism $\theta : (A,C) \to D$ for suitable fixed values of the first variable. We have seen that the map $a \mapsto \psi_a$ from A to $\mathsf{CRS}(C,D)$ given in this way by θ is a morphism of crossed complexes, so we can now read off the crossed complex operations on $\mathsf{CRS}(C,D)$ from the bimorphism laws of Definition 9.3.11 for θ.

For example, given $(F,f) \in \mathsf{CRS}_m(C,D)(m \geqslant 2)$ we determine $\delta(F,f)$ as follows. Write $(F,f) = (F_a, f_a)$ for suitable $a \in A$ as above, where $F_a(c) = \theta(a,c), f_a(c) = \theta(\beta a, c)$. Then $\delta(F,f) = (F_{\delta a}, f_{\delta a})$. We note that $f_{\delta a} = f$ since $\delta\beta a = \beta a$. We write δF for $F_{\delta a}$, so that $\delta(F,f) = (\delta F, f)$. Now $(\delta F)(c) = \theta(\delta a, c)$ is given by the formula in Definition 9.3.11 in terms of known elements, namely (assuming $m \geqslant 2$)

$$\theta(\delta a, c) = \begin{cases} \delta(\theta(a,c)) + (-1)^{m+1}\theta(a, \delta c) & \text{if } c \in C_n\ (n \geqslant 2), \\ (-1)^{m+1}\theta(a, \delta^- c)^{\theta(\beta a, c)} + (-1)^m \theta(a, \delta^+ c) + \delta(\theta(a,c)) & \text{if } c \in C_1, \\ \delta(\theta(a,c)) & \text{if } c \in C_0. \end{cases}$$

In other words

$$(\delta F)(c) = \begin{cases} \delta(F(c)) + (-1)^{m+1}F(\delta c) & \text{if } c \in C_n\ (n \geqslant 2), \\ (-1)^{m+1}F(\delta^- c)^{f(c)} + (-1)^m F(\delta^+ c) + \delta(F(c)) & \text{if } c \in C_1, \\ \delta(h(c)) & \text{if } c \in C_0. \end{cases} \qquad (***)$$

This automatic procedure gives the crossed complex structure of $\mathsf{CRS}(C,D)$ as stated in Definition 9.3.8.

15.2.5 Crossed complexes and cubical sets

[We need to be clear about the relation of this to our later written sections on cubical sets in the classifying space chapter in Part II. There ΠK is defined as a coend. We have to show that this definition is recovered. Maybe this is where the notion of density also comes in! See wikipedia or the (downloadable) book by Adamek, Herrlich, Rosiscki (TAC reprint), and Mac Lane.]

For any cubical set K we define the *fundamental crossed complex of* K to be $\Pi(K) = \gamma\rho(K)$. Propositions 15.1.13 and 15.1.14 then give immediately

Theorem 15.2.9 *If* K, L *are cubical sets, there is a natural isomorphism of crossed complexes*

$$\Pi(K \otimes L) \cong \Pi(K) \otimes \Pi(L).$$

In particular

$$\Pi(\mathbb{I}^m) \otimes \Pi(\mathbb{I}^n) \cong \Pi(\mathbb{I}^{m+n}).$$

For any crossed complex C we define the *cubical nerve* of C to be $NC = U\lambda C$, which is a cubical set. Since ρ is left adjoint to $U, \Pi = \gamma\rho$ is left adjoint to $N = U\lambda$, but we now prove a stronger result. We observe that, for any ω-groupoid G and any cubical set L, $\mathsf{Cub}(L, UG)$ has a canonical ω-groupoid structure induced by the structure of G (see Proposition 15.1.12). In particular $\mathsf{Cub}(L, NC)$ is an ω-groupoid and Proposition 15.1.12 gives

Theorem 15.2.10 *For any cubical set* L *and any crossed complex* C*, there are natural isomorphisms of crossed complexes*

$$\mathsf{Crs}(\Pi L, C) \cong \gamma(\omega\text{-}\mathsf{GPDS}(\rho L, \lambda C)) \cong \gamma(\mathsf{Cub}(L, NC)).$$

By taking cubical nerves and connected components we obtain

Corollary 15.2.11 *Let* L *be a cubical set and* C *be a crossed complex.*

(i) *There is a natural isomorphism of cubical sets*

$$\mathsf{Cub}(L, NC) \cong N(\mathsf{Crs}(\Pi L, C)).$$

(ii) *There is a natural bijection*

$$[L, NC] \cong [\Pi L, C],$$

where $[-,-]$ *denotes the set of homotopy classes of morphisms in* Cub *or in* Crs*, as the case may be.*

15.3 The Eilenberg-Zilber natural transformation

We now prove the important Theorem 9.8.1 that if X_*, Y_* are filtered spaces, then there is a natural transformation

$$\theta : \Pi(X_*) \otimes \Pi(Y_*) \to \Pi(X_* \otimes Y_*)$$

which is an isomorphism if X_*, Y_* are CW-complexes (and in fact more generally as is proved in [BB93]).

In view of the previous Sections, it is sufficient to prove a similar result for ω-groupoids.

Theorem 15.3.1 *If* X_* *and* Y_* *are filtered spaces, then there is a natural morphism*

$$\theta' : \rho X_* \otimes \rho Y_* \to \rho(X_* \otimes Y_*)$$

such that:

i) θ' *is associative;*

ii) *if $*$ denotes a singleton space or crossed complex, then the following diagrams are commutative*

$$\begin{tikzcd} \rho X_* & (\rho X_*) \otimes * \arrow[l, "\cong"'] \arrow[d, "\theta'"] \\ & \rho(X_* \otimes *) \arrow[from=1-1, "\cong"'] \end{tikzcd} \qquad \begin{tikzcd} * \otimes \rho X_* \arrow[r, "\cong"] \arrow[d, "\theta'"'] & \rho X_* \arrow[dl, "\cong"] \\ \rho(* \otimes X_*) & \end{tikzcd}$$

iii) θ' *is commutative in the sense that if* $T_c : C \otimes D \to D \otimes C$ *is the transposition and* $T_t : X_* \otimes Y_* \to Y_* \otimes X_*$ *is the twisting map, then the following diagram is commutative*

$$\begin{tikzcd} \rho X_* \otimes \rho Y_* \arrow[r, "\theta'"] \arrow[d, "T_c"'] & \rho(X_* \otimes Y_*) \arrow[d, "\rho(T_t)"] \\ \rho Y_* \otimes \rho X_* \arrow[r, "\theta'"] & \Pi(Y_* \otimes X_*); \end{tikzcd}$$

Proof To construct a natural morphism

$$\theta' : \rho X_* \otimes \rho Y_* \to \rho(X_* \otimes Y_*)$$

all we need is to construct a bimorphism of ω -groupoids

$$\theta'' : (\rho X_*, \rho Y_*) \to \rho(X_* \otimes Y_*).$$

Let $f : \mathbf{I}_*^p \to X_*$, $g : \mathbf{I}_*^q \to Y_*$ be representatives of elements of $\rho_p X_*$, $\rho_q Y_*$ respectively. We define $\theta''([f],[g])$ to be the class of the composite

$$\mathbf{I}_*^{p+q} \cong \mathbf{I}_*^p \otimes \mathbf{I}_*^q \xrightarrow{f \otimes g} X_* \otimes Y_*.$$

It is easy to check that $\theta''([f],[g])$ is independent of the choice of representatives. Also, the conditions that θ'' be a bimorphism are almost automatic. Thus, we have a natural morphism θ'.

The proofs of (i) (associativity) and (ii) (preserves base point) are clear.

The proof of (iii) (symmetry) follows from the description of the isomorphism $G \otimes H \to H \otimes G$ of ω-groupoids as given by $x \otimes y \mapsto (y^* \otimes x^*)^*$ where, in the geometric case $G = \rho X_*$, $x \mapsto x^*$ is induced by the map $(t_1, \ldots, t_p) \mapsto (t_p, \ldots, t_1)$ of the unit cube. □

This gives conditions (i)-(iii) of Theorem 9.8.1

To prove (iv) (it is an isomorphism for CW-filtrations) recall that $X_* \otimes Y_*$ is a CW-filtration, and so the crossed complex $\Pi(X_* \otimes Y_*)$ is of free type, with basis the characteristic maps of the product cells $e^p \times e^q$ of $X_* \otimes Y_*$. So the theorem follows from Theorem 9.6.3 that the tensor product of free crossed complexes is free.

15.4 The symmetry of tensor products

We have seen that in the category Cub, the map $x \otimes y \mapsto y \otimes x$ does not give an isomorphism $K \otimes L \to L \otimes K$; indeed it is easy to construct examples of cubical sets K, L such that $K \otimes L$ and $L \otimes K$ are not isomorphic. However, in ω-GPDS, and Crs the situation is different. Although the map $x \otimes y \mapsto y \otimes x$ still does not give an isomorphism $K \otimes L \to L \otimes K$, there is a less obvious map which does. This is easiest to see in Crs.

Theorem 15.4.1 *Let C, D be crossed complexes. Then there is a natural isomorphism $C \otimes D \to D \otimes C$ which, for $c \in C_m, d \in D_n$, sends the generator $c \otimes d$ to $(-1)^{mn} d \otimes c$. This isomorphism, combined with the structure studied until now, makes the category of crossed complexes a symmetric monoidal closed category.*

Proof One merely checks that the relations defining the tensor product are preserved by the map $c \otimes d \mapsto (-1)^{mn} d \otimes c$. The necessary coherence and naturality conditions are obviously satisfied. □

Remark 15.4.2 This proof is unsatisfactory because, although it is clear that $c \otimes d \mapsto d \otimes c$ does not preserve the relations of the tensor product, the fact that $c \otimes d \mapsto (-1)^{mn} d \otimes c$ does preserve them seems like a happy accident. A better explanation is provided by the transposing functor T (see Sections 10.2 and 15.1). □

For a cubical set K, TK is not in general isomorphic to K. But for any ω-groupoid G and any crossed complex C we will construct isomorphisms $G \to TG$ and $C \to TC$. Since in all these categories we have obvious natural isomorphisms $T(X \otimes Y) \cong TY \otimes TX$, this implies the symmetry $X \otimes Y \cong Y \otimes X$.

For an ω-groupoid G, the *transpose* TG has the same elements as G but has all its operations ∂_i^α, ε_i, Γ_i, $+_i$, $-_i$ numbered in reverse order with respect to i (but not with respect to $\alpha = \pm$). For a crossed complex C, TC is defined, of course, as $\gamma(T, \lambda C)$. The calculation expressing this crossed complex in terms of the crossed complex structure of C is straightforward (though it needs a clear head).

Proposition 15.4.3 *The crossed complex TC is defined, up to natural isomorphism, in the following way:*

(i) $(TC)_0 = C_0$ *as a set;*

(ii) $(TC)_2 = C_2^{op}$ *as a groupoid;*

(iii) $(TC)_n = C_n$ *as a groupoid for* $n = 1$ *and* $n \geqslant 3$*;*

(iv) *the action of* $(TC)_1$ *on* $(TC)_n (n \geqslant 2)$ *is the same as the action of* C_1 *on* C_n*;*

(v) *the boundary map* $T\delta : (TC)_{n+1} \to (TC)_n$ *is given by*

$$T\delta = (-1)^n \delta : C_{n+1} \to C_n.$$

We note that $-\delta : C_2 \to C_1$ is an anti-homomorphism, that is a homomorphism $C_2^{op} \to C_1$, as required; the map $+\delta : C_3 \to C_2^{op}$ is also a homomorphism because the image is in the centre of C_2. In higher dimensions the groupoids C_n and C_n^{op} are the same.

Corollary 15.4.4 *For any crossed complex C there is a natural isomorphism $\tau : C \to TC$ given by*

$$\tau(c) = (-1)^{[n/2]} c \quad \textit{for } c \in C_n.$$

Remark 15.4.5 The somewhat surprising sign $(-1)^{[n/2]}$ is forced by the signs in Proposition 15.4.3; it is less surprising when one notices that it is the signature of the permutation which reverses the order of $(1, 2, \ldots, n)$. The symmetry map of Theorem 15.4.1 now comes from the map

$$c \otimes d \to \tau^{-1}(\tau d \otimes \tau c) = (-1)^k d \otimes c,$$

where $k = [m/2] + [n/2] - [(m+n)/2]$, which is 0 if m or n is even, and 1 if both are odd. □

Let G be an ω-groupoid and $C = \gamma G$. Then $G = \lambda C$ and the isomorphism $\tau : C \to TC$ extends uniquely to an isomorphism $\tau : G \to TG$. This isomorphism can be viewed as a 'reversing automorphism' $x \mapsto x^*$ of G, that is, a map of degree 0 from G to itself which preserves the operations while reversing their order (e.g. $(x +_i y)^* = x^* +_{n-i+1} y^*$ in dimension n). The isomorphism $G \otimes H \to H \otimes G$ for ω-groupoids is then given by

$$x \otimes y \mapsto (y^* \otimes x^*)^*.$$

The element x^* should be viewed as a transpose of the cube x.

Remark 15.4.6 In the geometric case $G = \rho(X_*)$, x^* is induced from x by the map $(t_1, \dots, t_n) \mapsto (t_n, \dots, t_1)$ of the unit n-cube. □

The operation $*$ is preserved by morphisms of ω-groupoids, because of the naturalness of $\tau : 1 \to T$. It follows (Is it that clear? NO) that the operation $*$ can be written in terms of the ω-groupoid operations ∂_i^α, ε_i, Γ_i, $+_i$, $-_i$, but the formulae needed for this are rather complicated.

15.5 The pointed case

We consider briefly the notions of tensor product and homotopy in the categories ω-GPDS$_*$ and Crs$_*$ of pointed ω-groupoids and pointed crossed complexes. Here the objects have a distinguished element $*$ in dimension 0 and all morphisms are to preserve the base points.

Definition 15.5.1 For any ω-groupoid H with basepoint $*$, the ω-groupoid $P^m H$ has basepoint $0_* = \varepsilon_1^m(*)$, the constant cube at $*$. An m*-fold pointed (left) homotopy* $h : G \to H$ is a morphism $h : G \to P^m H$ preserving basepoints, that is, a homotopy h with $h(*) = 0_*$. Clearly, all such pointed homotopies form an ω-subgroupoid ω-GPDS$_*(G, H)$ of ω-GPDS(G, H) since $0_* = \varepsilon_1^m(*)$ is an identity for all the compositions $+_i (1 \leqslant i \leqslant m)$. This ω-subgroupoid has as basepoint the trivial morphism $G \to H$ which sends each element of dimension n to $0_* = \varepsilon_1^n(*)$. Thus we have an internal hom functor ω-GPDS$_*(G, H)$ in the pointed category ω-GPDS$_*$. The pointed morphisms from F to ω-GPDS$_*(G, H)$ are in one-one correspondence with the *pointed bimorphisms* $\chi : (F, G) \to H$, that is, bimorphisms χ satisfying the extra conditions

$$\begin{cases} \chi(x, *) = 0_* & \text{for all } x \in F, \\ \chi(*, y) = 0_* & \text{for all } y \in G. \end{cases} \tag{i}$$

To retain the correspondence between bimorphisms $(F, G) \to H$ and morphisms $F \otimes G \to H$, we must therefore add corresponding relations to the definition of the tensor product. Thus, for pointed ω-groupoids F, G, we define $F \otimes_* G$ to be the ω-groupoid with generators $x \otimes_* y$, $(x \in F, y \in G)$, basepoint $* = * \otimes_* *$, and defining relations the same as in Definition 10.2.5 together with

$$\begin{cases} x \otimes_* * = 0_* & \text{for all } x \in F, \\ * \otimes_* y = 0_* & \text{for all } y \in G. \end{cases} \tag{ii}$$

These equations are to be interpreted as $x \otimes_* * = * \otimes_* y = *$ when x, y have dimension 0, so that $(F \otimes_* G)_0 = F_0 \wedge G_0$. □

Theorem 15.5.2 *The pointed tensor product and hom functor described above define a symmetric monoidal closed structure on the pointed category* ω-GPDS$_*$.

15.6 Dense subcategories

Our aim in this section is to explain and prove the theorem:

Theorem 15.6.1 *The full subcategory* $\widehat{\mathbf{I}}$ *of* ω-Gpds *on the objects* $\mathbf{I}^n$ *is dense in* ω-Gpds*.*

We recall the definition of a dense subcategory. First, in any category C, a morphism $f : C \to D$ induces a natural transformation $f_* : C(-, C) \to C(-, D)$ of functors $C^{op} \to \mathsf{Set}$. Conversely, any such natural transformation is induced by a (unique) morphism $C \to D$.

Again, if $\mathcal{I}$ is a subcategory of C, then $f : C \to D$ induces a natural transformation of functors $f^* : \mathcal{I}^{op} \to \mathsf{Set}$. The subcategory $\mathcal{I}$ is *dense* in C if every such natural transformation arises from a morphism. More precisely, there is a functor $\eta : C \to \mathsf{Fun}(\mathcal{I}^{op}, \mathsf{Set})$ defined in the above way, and $\mathcal{I}$ is *dense* in C if η is full and faithful.

Example 15.6.2 Let $\mathbb{Z}$ be the cyclic group of integers. Then $\{\mathbb{Z}\}$ is a generating set for the category Ab of abelian groups, but the full subcategory of Ab on this set is not dense in Ab. In order for a natural transformation to specify not just a function $f : A \to B$ but a morphism in Ab, we have to enlarge this subcategory to include $\mathbb{Z} \oplus \mathbb{Z}$. □

Example 15.6.3 Consider the Yoneda embedding

$$\Upsilon : C \to C^{op}\text{-}\mathsf{Set} = \mathsf{Fun}(C^{op}, \mathsf{Set})$$

where C is a small category. Then each object $K \in C^{op}$-Set is a colimit of objects in the image of Υ and this is conveniently expressed in terms of coends as that the natural morphism

$$\int^{c} (C^{op}\text{-}\mathsf{Set}(\Upsilon c, K) \times \Upsilon c) \to K$$

is an isomorphism. Thus the Yoneda image of C is dense in C^{op}-Set. □

Proof of Theorem 15.6.1 Let G, H be ω–groupoids and let $\hat{f} : \omega\text{-}\mathsf{Gpds}_{\mathcal{I}}(-, G) \to \omega\text{-}\mathsf{Gpds}_{\mathcal{I}}(-, H)$ be a natural transformation. We define $f : G \to H$ as follows.

Let $x \in G_n$. Then x defines $\hat{x} : \mathbf{I}^n \to G$. We set $f(x) = \hat{f}(\hat{x})(c^n) \in H_n$. We have to prove f preserves all the structure.

For example, we prove that $f(\partial_i^{\pm} x) = \partial_i^{\pm} f(x)$. Let $\bar{\partial}_i^{\pm} : \mathbf{I}^{n-1} \to \mathbf{I}^n$ be given by having value $\partial_i^{\pm} c^n$ on c^{n-1}. The natural transformation condition implies that $\hat{f}(\bar{\partial}_i^{\pm})^* = (\bar{\partial}_i^{\pm})^* \hat{f}$. On evaluating this on $\hat{x}$ we obtain $f(\partial_i^{\pm} x) = \partial_i^{\pm} f(x)$ as required. In a similar way, we prove that f preserves the operations ϵ_i, Γ_i.

Now suppose that $t \in G_n$ is thin in G. We prove that $f(t)$ is thin in H.

Consider the morphism of ω-groupoids $\hat{t} : \mathbf{I}^n \to G$. Let S be the shell consisting of all faces but one of c_n. Then S has a unique thin filler b_t. Now $\hat{t}(S)$ consists of all faces but one of t, and so is filled by t. Since $\hat{t}$ preserves thin elements, we must have $\hat{t}(b_t) = t$. Let $\bar{b} : \mathbf{I}^n \to \mathbf{I}^n$ be the unique morphism such that $\bar{b}(c^n) = b_t$. Then the natural transformation condition implies $f(t) = \hat{f}(\hat{t})(c^n) = \hat{f}(\hat{t})(b_t)$. Since b_t is thin, it follows that $f(t)$is thin. Thus f preserves the thin structure.

Now Proposition 13.7.8 implies that the operations $+_i$ are preserved by f.

We can now conveniently represent each ω–groupoid as a coend.

Corollary 15.6.4 *The subcategory* $\widehat{\mathbf{I}}$ *of* ω-Gpds *is dense and for each object* G *of* ω-Gpds *the natural morphism*

$$\int^n \omega\text{-}\mathsf{Gpds}(\mathbf{I}^n, G) \times \mathbf{I}^n \to G$$

is an isomorphism.

Proof This is a standard consequence of the property of $\widehat{\mathbf{I}}$ being dense. □

Corollary 15.6.5 *The full subcategory of* Crs *generated by the objects* $\Pi \mathfrak{I}^n_*$ *is dense in* Crs.

Proof This follows from the fact the equivalence $\gamma : \omega\text{-}\mathsf{Gpds} \to \mathsf{Crs}$ takes $\mathbf{I}^n$ to $\Pi \mathfrak{I}^n_*$ (Theorem 14.4.1). □

15.7 Fibrations and coverings of ω-groupoids

We use the definitions of fibration and covering morphism of crossed complexes as given earlier. We now give corresponding conditions for ω-groupoids.

Theorem 15.7.1 *Let* $p : G \to H$ *be a morphism of* ω-Gpdss. *Then the morphism of crossed complexes* $\gamma(p) : \gamma(G) \to \gamma(H)$ *is a fibration (covering morphism) if and only if* $p : G \to H$ *is a Kan fibration (covering map) of cubical sets.*

Proof As regards fibrations this is the result of Proposition 10.5.10. The restriction to covering morphisms follows in a similar way. □

Corollary 15.7.2 *Let* $p : K \to L$ *be a morphism of* ω-Gpdss *such that the underlying map of cubical sets is a Kan fibration. Then the pullback functor*

$$f_* : \omega\text{-}\mathsf{Gpds}/L \to \omega\text{-}\mathsf{Gpds}/K$$

has a right adjoint and so preserves colimits.

Proof This is immediate from Theorem 15.7.1 and results of Howie stated as Theorem 11.2.9. □

Remark 15.7.3 It seems likely that a covering ω-groupoid is also free.

15.8 Application to the tensor product of covering morphisms

First we know from [BH87] that the tensor product of ω-groupoids satisfies $\mathbf{I}^m \otimes \mathbf{I}^n \cong \mathbf{I}^{m+n}$. It follows that the tensor product $G \otimes H$ of ω-Gpdss G, H satisfies

$$G \otimes H \cong \int^{m,n} \omega\text{-}\mathsf{Gpds}(\mathbf{I}^m, G) \times \omega\text{-}\mathsf{Gpds}(\mathbf{I}^n, H) \times (\mathbf{I}^m \otimes \mathbf{I}^n). \tag{15.8.1}$$

We suppose G, H are connected. Choose a base point $(p, q) \in G_0 \times H_0$. Now let $p : C \to G \otimes H$ be a covering morphism determined by a pair of normal subgroups $M \trianglelefteq \pi_1(G, p), N \trianglelefteq \pi_1(H, q)$. Let

$\tilde{G} \to G, \tilde{H} \to H$ be the covering morphisms determined by these subgroups. By corollary 15.7.2, pullback p^* by p preserves colimits. Hence

$$\begin{aligned} C &\cong p^*\left(\int^{m,n} \omega\text{-}\mathsf{Gpds}(\mathbf{I}^m, G) \times \omega\text{-}\mathsf{Gpds}(\mathbf{I}^n, H) \times (\mathbf{I}^m \otimes \mathbf{I}^n)\right) \\ &\cong \int^{m,n} p^*(\omega\text{-}\mathsf{Gpds}(\mathbf{I}^m, G) \times \omega\text{-}\mathsf{Gpds}(\mathbf{I}^n, H)) \times (\mathbf{I}^m \otimes \mathbf{I}^n) \end{aligned}$$

which, because of the construction of C by the specified subgroups:

$$\begin{aligned} &\cong \int^{m,n} \omega\text{-}\mathsf{Gpds}(\mathbf{I}^m, \tilde{G}) \times \omega\text{-}\mathsf{Gpds}(\mathbf{I}^m, \tilde{H}) \times (\mathbf{I}^m \otimes \mathbf{I}^n) \\ &\cong \tilde{G} \otimes \tilde{H}. \end{aligned}$$

This finally enables us to prove Theorem 11.1.14.

Corollary 15.8.1 *If F, F' are free and aspherical crossed complexes, then so also is $F \otimes F'$.*

Proof It is sufficient to assume F, F' are connected. Then the universal covers $\tilde{F}, \tilde{F'}$ are free and acyclic and hence contractible. Therefore $\tilde{F} \otimes \tilde{F'}$ is contractible, and hence acyclic. Therefore $F \otimes F'$ is aspherical. □

Chapter 16

Conclusion

We have now come to the end of our description of this intricate structure. We hope to have shown how this fits together and allows a new approach to algebraic topology, in which some nonabelian information is successfully taken into account. We also wanted to convey how the good modelling of the geometry by the algebra, the way the algebra gives power and reality to some basic intuitions, is a key to the success.

We have presented the material in a way which we hope will convince you that the intricacy of the justification of the theory does not detract from the fact that crossed complexes theory are usable as a tool without knowing exactly why they works. That is, we have given a pedagogical order rather than a logical and structural order. It should be emphasised that the order of discovery followed the logical order! The conjectures were made and verified in terms of ω-groupoids, and we were amazed that the theory of crossed complexes, which was in essence already available, fitted with this so nicely.

It is also surprising that this corpus of work followed from a simple aesthetic question posed in 1965, to find a determination of the fundamental group of the circle which avoided the detour of setting up covering space theory. This led to nonabelian cohomology, [Bro65], and then to groupoids, [Bro67]. The latter suggested the programme of rewriting homotopy theory replacing the word 'group' by 'groupoid' and seeing if the result was an improvement!

What more is there to do? We explain some potential areas of work in the next section.

16.1 Problems

There are a number of standard methods and results in algebraic topology to which the techniques of crossed complexes given here have not been applied. So we leave these open for work to be done, and to decide if the uses of crossed complexes in these areas can advance the subject of algebraic topology. Where we do not give references, then we expect the reader to rely on Wikipedia for further details.

Problem 16.1.1 Investigate Mayer-Vietoris type exact sequences for a pullback of a fibration of crossed complexes, analogous to that given for a pullback of a covering morphism of groupoids in [Bro06, Section 10.7]. See also [BHK83].

Problem 16.1.2 Can one use crossed complexes to give a finer form of Poincaré Duality? This means developing cup and cap products, which should be no problem, and also coefficients in an object with an analogue of a 'ring structure'. These could be the crossed differential algebras (i.e. monoid objects in the monoidal category Crs) considered in [BT97], and the braided regular crossed modules of [BG89a], further developed in [AU07].

Problem 16.1.3 Another standard area is fixed point theory, which includes the Lefschetz theory, involving homology, and also Nielsen theory, involving the fundamental group.

Problem 16.1.4 Are there results on the fundamental crossed complex of an orbit space of a filtered space analogous to those for the fundamental groupoid of an orbit space given in [Bro06, Chapter 11]? Some related work is in [HT82].

Problem 16.1.5 Are there applications of crossed complexes to the nonabelian cohomology of fibre spaces? Could the well developed acyclic model theory and fibre spaces of [GM57] be suitably modified and used? The spectral sequence of filtered crossed complexes has been developed by Baues in [Bau89], but surely more work needs to be done.

Problem 16.1.6 Is there a non-Abelian *homological perturbation theory* for constructing non-Abelian twisted tensor products from fibrations? Or for constructing small free crossed resolutions of groups? References for the standard theory my be found by a web search.

Problem 16.1.7 The standard theory of chain complexes makes much use of double chain complexes. Double crossed complexes have been defined in [Ton94] but presumably there is much more to be done here.

Problem 16.1.8 The theory of equivariant crossed complexes has already been developed in [BGPT97, BGPT01]. However notions such as fibrations of crossed complexes have not been applied here.

Problem 16.1.9 Can one make progress with nonabelian cohomology operations? The tensor product of crossed complexes is symmetric, as proved in Chapter 13. So if K is a simplicial set, then we can consider the non-commutativity of the diagonal map $\Delta : \pi|K| \to \pi|K| \otimes |K|$. If T is the twisting map $A \otimes B \to B \otimes A$, then there is a natural homotopy $T\Delta \simeq \Delta$, by the usual acyclic models argument. This look like the beginnings of a theory of non-Abelian Steenrod cohomology operations. Does such a theory exist and does it hold any surprises? By contrast, [Bau89] gives an obstruction to the existence of a Pontrjagin square with local coefficients.

Problem 16.1.10 One use of chain complexes is in defining Kolmogorov-Steenrod homology. One takes the usual net of polyhedra defined as the nerves of open covers of a space X, with maps between them induced by choices of refinements. The result is a homotopy coherent diagram of polyhedra. It is shown in [Cor87] that a strong homology theory results by taking the chain complexes of this net, and forming the chain complex which is the homotopy inverse limit. What sort of strong homology theory results from using the fundamental crossed complexes of the nerves instead of the chain complexes? Is there a kind of "strong fundamental groupoid", and could this be related to defining universal covers of spaces which are not locally 'nice'?

Problem 16.1.11 There are a number of areas of algebraic topology where chain complexes with a group of operators are used, for example [RW90]. Is it helpful to reformulate this work in terms of crossed complexes?

Problem 16.1.12 Another example for the last problem is the work of Dyer and Vasquez in [DV73] on CW-models for one-relator groups.

Problem 16.1.13 Find applications of these non-Abelian constructions to configuration space theory and mapping space theory, particularly the theory of spaces of rational maps.

Problem 16.1.14 A further aim is to use these methods in the theory of stacks and gerbes, and more generally in differential topology and geometry.

Problem 16.1.15 Investigate the relation between the cocycle approach to Postnikov invariants and that given using triple cohomology in [BFGM05].

Other problems in crossed complexes and related areas are given in [Bro90].

Appendices

Appendix A

A resumé of some category theory.

A.1 Introduction

A categorical approach is basic to this book, and we use freely notions of category, functor, natural transformation, pushout, product category covered in the book 'Topology and Groupoids', [Bro06].

Some of our key proofs, for example of the HHvKT, follow the pattern of: we verify the universal property. One importance of this is that we verify in this way that for example a particular fundamental groupoid $\pi_1(X, X_0)$ is given as a pushout of groupoids; but the proof makes no claim as to the general existence of pushouts of groupoids, nor does it show how to construct pushouts of groupoids in general. So the theorem raises questions as to how to prove existence of pushouts of groupoids, and how to construct them in practical ways.

In addition to the above topics, we need at various stages limits and colimits, equalisers and coequalisers, adjoint functors, ends and coends, cartesian closed categories, monoidal closed categories, and for all of these there are excellent texts available (for example Mac Lane, [ML71], Adamek-Herrlich-Strecker, [AHS06] (downloadable), and many others). Readers will also profit from accounts of these topics on Wikipedia and on Planet Math.

We find it difficult to give an adequate and complete coverage of what we need here, since that would be too large a task. Further, there is a considerable amount of information freely available online, including downloadable texts, or partial texts, and also web encyclopedia. Therefore the aim of this Appendix is to indicate the necessary background and to supply more detail only when we can present or highlight a particular viewpoint or the material is not so accessible in the format we need. So this Appendix should be supplemented with downloadable material.

A.2 Representable functors

As a start, we give an introduction to the notion of representable functor, since this is simple but includes a pattern of argument which may not be so familiar.

Let C be a category. Then for each $d \in C$ there is a functor $C_d : C^{op} \to \mathsf{Sets}$ given by $C_d(c) = C(c, d)$. An important property of such functors is the following. If $h : d \to e$ is a morphism in C then h induces a natural transformation

$$C_h : C_d \to C_e,$$

given by $C_h(c) = C(c, h)$. Thus if $f : c \to b$ in C, we need to verify the commutativity of the diagram

$$\begin{tikzcd} C(c,d) \arrow[r, "h_*"] & C(c,e) \\ C(b,d) \arrow[u, "f^*"] \arrow[r, "h_*"'] & C(b,e) \arrow[u, "f^*"'] \end{tikzcd}$$

Indeed for any $g : b \to d$ the evaluation of both ways round the diagram yields hgf, so that the proof of naturality follows from associativity of the composition in C.

The converse of this result is easy to prove but turns out to be significant.[1]

Proposition A.2.1 *If* $d, e \in C$ *then there is a natural bijection*

$$\mathrm{Nat}(C_d, C_e) \to C(d, e).$$

Proof Suppose $\eta : C_d \to C_e$ is a natural transformation, yielding for each $c \in C$ a function $\eta(c) : C(c, d) \to C(c, e)$. The naturality condition states that for each $f : c \to b$ in C the first of the following diagrams is commutative:

$$\begin{tikzcd} C(c,d) \arrow[r, "\eta_c"] & C(c,e) \\ C(b,d) \arrow[u, "f^*"] \arrow[r, "\eta_b"'] & C(b,e) \arrow[u, "f^*"'] \end{tikzcd} \qquad \begin{tikzcd} C(c,d) \arrow[r, "\eta_c"] & C(c,e) \\ C(d,d) \arrow[u, "f^*"] \arrow[r, "\eta_d"'] & C(d,e) \arrow[u, "f^*"'] \end{tikzcd} \tag{A.2.1}$$

Now choose b to be d, and set $g = \eta_d(1_d) : d \to e$. In order to evaluate $\eta_c(f)$ where $f : c \to d$ we use the second commutative diagram. Then $\eta_c(f^*(1_d)) = \eta_c(f)$, while $f^*\eta_d(1_d) = f^*(g) = gf$. □

The idea can be extended.

Definition A.2.2 A functor $T : C^{op} \to \mathsf{Sets}$ is called *representable* if it is naturally equivalent to a functor C_c for some object c of C. Then c is called a *representing object* for T, or we say T is *represented* by c.

Proposition A.2.3 *If functors* $T, U : C^{op} \to \mathsf{Sets}$ *are represented by objects* d, e *of* C*, then there is a bijection*

$$\mathrm{Nat}(T, U) \cong C(d, e).$$

In particular, a natural equivalence $T \cong U$ *is determined completely by an isomorphism* $d \cong e$*.*

The proof is easy from proposition A.2.1.

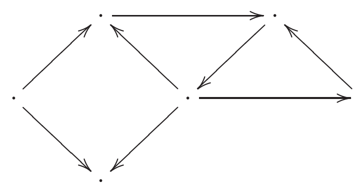
A.3 Colimits and limits

We concentrate on the notion of colimit since this is a general concept closely related to the formulation of local-to-global properties. The idea is to give a general formulation of 'gluing', of putting together a complex object from smaller pieces, and rules for the gluing, to give what is called a *colimit*.

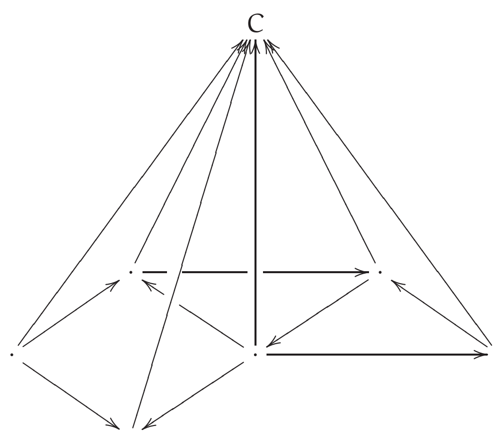
[1] J.H.C. Whitehead once remarked: It is the snobbery of the young to suppose that a theorem is trivial because the proof is trivial!

The ‘input data’ for a colimit is a diagram D, that is a collection of some objects in a category C and some morphisms between them. The output will be an object $\mathrm{Colim} D$ in C. A (co)cone with base the diagram D and vertex C say consists of arrows from the objects of the diagram to the vertex C satisfying a ‘commutativity’ condition: any path from a given object to C composes to give the same composite arrow. Any such cocone factors through the colimiting cocone:

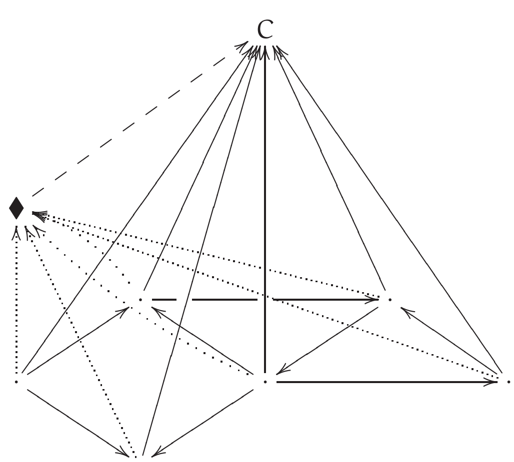
D =



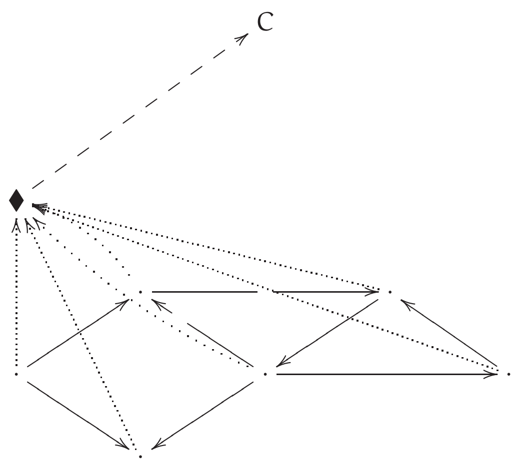
Cocone with base D and vertex C:



The next step is where the colimit sits in this picture (⧫ $=$ $\mathrm{colimit}$ D and the dotted arrows represent new morphisms):



and stripping away the 'old' cocone gives the factorisation of the cocone via the colimit:



INTUITIONS:

From beyond (or above in our diagrams) D, an object 'sees' the diagram D 'mediated' through its colimit, i.e. if it tries to interact with the whole of D, it has to do it via $\operatorname{colim} D$. Conversely, any interaction of $\operatorname{colim} D$ with other objects comes from the whole of the diagram D.

Example A.3.1 The lcm of two positive integers a, b can be seen as the colimit of the diagram

$$\begin{tikzcd} a & & b \\ & \gcd(a,b) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

The gcd, from a lower level of the hierarchy, 'measures' the interaction of a and b.

Some people have viewed biological organs as colimits of the diagrams of interacting cells within them.

Remark A.3.2 WARNING. Often colimits do not exist (in C) for some diagrams. However, one can add colimits in a completion process, i.e. freely for a class of diagrams, and then compare these 'virtual colimits' with any that happen to exist.

It is important to note that a colimit has more structure than merely the disjoint union of its individual parts, since it depends on the arrows of the diagram D as well as the objects. Thus the specification for a colimit object of the arrows which define it can be thought of as a 'subdivision' of the colimit object. This is why the notion is of importance in local-to-global questions.

We now give a more formal definition. First note that it is convenient to consider not a diagram D but a small category, say S. This category can be obtained from D as the free category on the graph D factored out by relations given by the commutative triangles of D. So we consider a colimit in C as defined by a functor $T : S \to C$. The colimit of T, if it exists, is an object of C but it is convenient to think of this as a constant functor $L : S \to C$. The relation between T and L, the *cocone*, is defined to be a natural transformation $\eta : T \Rightarrow L$. Thus η gives for each arrow $s : x \to y$ of

S a commutative diagram

$$\begin{tikzcd} Tx \arrow[rr, "Ts"] \arrow[dr, "\eta x"'] & & Ty \arrow[dl, "\eta y"] \\ & Lx = Ly & \end{tikzcd} \tag{A.3.1}$$

So we have our definition:

Definition A.3.3 A *colimit* $\operatorname{colim} T$ of a functor $T : S \to C$ is a natural transformation $\eta : T \Rightarrow L$ to a constant functor, which is universal for natural transformations to constant functors: that is, if $\xi : T \to L'$ is a natural transformation to a constant functor L', then there is a unique natural transformation $\phi : L \to L'$ such that $\phi \circ \eta = \chi$. (Note that a natural transformation between constant functors to C reduces to a morphism of C between their values.) If a colimit of T exists then it is unique up to natural equivalence, and is written $\operatorname{colim} T$; it is thought of either as a constant functor to C or as an object of C; it always comes with its universal cocone $T \Rightarrow \operatorname{colim} T$. Sometimes the colimit is written as $\operatorname{colim}_x T(x)$ where x ranges over the objects of S; this is an abuse of language since the morphisms of S are crucial to the definition. □

Example A.3.4 (i) A special case is the coproduct. In this case, S is the discrete category on a set of objects.

(ii) Another example is the pushout: here the diagram D has three objects, say 0, 1 and 2, and two arrows from 0, namely $0 \to 1, 0 \to 2$.

(iii) Another example is the coequaliser: here the diagram D has two objects say 1 and 2 and two arrows $1 \rightrightarrows 2$. □

The next proposition shows that colimits may be constructed from coproducts and coequalisers.

Proposition A.3.5 *If functors* $S \to C$ *admit coproducts and coequalisers, then they admit colimits.*

In a similar spirit, we define limits of a functor.

Definition A.3.6 Let S be a small category, and $T : S \to C$ a functor. A *limit* of T is a constant functor $L : S \to C$ and a natural transformation $\epsilon : L \Rightarrow T$ (called the *cone* on T) with the universal property: for any natural transformation $\xi : L' \Rightarrow T$ from a constant functor L' to T, there exists a unique natural transformation $\phi : L' \to L$ such that $\epsilon \circ \phi = \xi$. Then L is also written $L = \lim T = \lim_x T(x)$.

A.4 Generating objects and dense subcategories

In the category of groups the infinite cyclic group C_∞ plays a key role. This leads to the following definition.

Definition A.4.1 A set S of objects in a category C is said to be *generating* C if for all pairs of morphisms $f, g : c \to d$ on objects of C, $f = g$ if and only if $fh = gh$ for all $s \in S$ and morphisms $h : s \to c$ in C. □

Example A.4.2 In the category Sets of sets, any singleton is a generator. In the category Groups of groups the infinite cyclic group C_∞ is a generator. In the category Gpds of groupoids the unit interval groupoid $\mathcal{I}$ is a generator. Note that in these examples the generator identifies the elements of a in each case a group or groupoid, but gives no further structural information. This leads to our next definition. □

Definition A.4.3 An inclusion $K : D \to C$ of a subcategory of a category is called *dense in* C if D is small and for all objects d, e of C the canonical function

$$\mathrm{Nat}(C(K(-), d), C(K(-), e) \to C(d, e) \tag{A.4.1}$$

is a bijection. □

Remark A.4.4 The meaning of this is that we can recover the morphisms $d \to e$ in C from information on the way the dense subcategory maps to d and e. Note that a universal property is defined by relating to all objects of a category. The advantage of a dense subcategory is that in principle, and for some purposes, we need look only at the objects of that dense subcategory. □

Example A.4.5 The full subcategory of Groups on the object $F\{x, y\}$, the free group on the elements x, y, is dense in the category of groups. The essential part of the argument is to show that if G, H are groups, then a function $f : G \to H$ is a morphism of groups if and only if $fg : F\{x, y\} \to H$ is a morphism for every morphism $g : F\{x, y\} \to G$ of groups. The proof of this is a nice little exercise, as is working out the analogous example for groupoids. □

A.5 Adjoint functors

One of the concepts that took a bit more to unearth was that of a pair of adjoint functors. Nevertheless, is almost ubiquitous and most fruitful.

To define this concept we consider two categories C and D, two functors between them $f : C \to \mathcal{D}$ and $g : D \to C$. We say that f is left adjoint of g (or that g is right adjoint of f) if there is an adjunction between them, i.e. a natural equivalence between the "functors"

$$\phi : C(-, g(-)) \cong D(f(-), -).$$

Since these are really bifunctors (moreover covariant in one variable and contravariant in the other one) and we do not want to digress in this direction, perhaps it is better to indicate how ϕ is given and which properties it has to satisfy.

It is a map

$$\phi : \mathrm{Ob}\ C \times \mathrm{Ob}\ D \to \mathsf{Sets}$$

such that for any $x \in \mathrm{Ob}\ C$ and $y \in \mathrm{Ob}\ \mathcal{D}$, the map

$$\phi(x, y) : C(x, g(y)) \to D(f(x), y)$$

is a bijection and it is natural with respect to both x and y, i.e. for any $a \in C(x, x')$ the diagram

$$\begin{tikzcd}
C(x, g(y)) \arrow[r, "\phi(x,y)"] & D(f(x), y) \\
C(x', g(y)) \arrow[u, "a_*"] \arrow[r, "\phi(x',y)"'] & \mathcal{D}(f(x'), y) \arrow[u, "f(a)_*"']
\end{tikzcd}$$

commutes, where a_* is given by composition with a, and for any $b \in D(y, y')$ the diagram

$$\begin{tikzcd} C(x, g(y)) \arrow[r, "\phi(x,y)"] \arrow[d, "g(b)^*"'] & D(f(x), y) \arrow[d, "b^*"] \\ C(x, g(y')) \arrow[r, "\phi(x,y')"'] & D(f(x), y') \end{tikzcd}$$

also commutes, where b^* is given by composition with b

Example A.5.1 There are many examples of adjoint pairs coming from algebra and topology. Let us mention

(i) All free constructions are adjoint of the corresponding forgetful functors (free group, free R-module, free category over a directed graph, etc);

(ii) the field of quotients of an integral domain is adjoint to the inclusion of the category of fields in that of integral domains;

(iii) the completion of a metric space is adjoint to the inclusion of the category of complete metric spaces in such of metric spaces;

(iv) the abelianisation of a group is left adjoint to the inclusion of the category of abelian groups in that of groups. □

Let us consider some functors that are associated to any adjunction and, under some conditions, determine it. The first construction is the unit, a natural transformation

$$\eta : 1_C \to gf$$

For any $x \in \mathrm{Ob}\ C$, $\eta(x) : x \to gf(x)$ is $\phi^{-1}_{f(x)}$. It is easy to prove naturality. Moreover the unit is universal in the following sense

Proposition A.5.2 *For any* $x \in \mathrm{Ob}\ C$, $\eta(x)$ *is universal with respect to* g, *i.e. for any morphism* $h : x \to g(y)$ *there is a unique morphism* $h' : f(x) \to y$ *so that the diagram*

$$\begin{tikzcd} x \arrow[r, "\eta(x)"] \arrow[d, "h"'] & gf(x) \arrow[dl, "g(h')"] \\ g(y) & \end{tikzcd}$$

commutes

It is easy to see that we can recover the adjunction ϕ from η since, for any $x \in \mathrm{Ob}\ C$, $y \in \mathrm{Ob}\ D$, $a \in D(f(x), y)$, $\phi^{-1}(x, y)(a)$ is $\eta(x)g(a)$.

The same result can be got dually. The counit is a natural transformation

$$\varepsilon : fg \to 1_D$$

For any $xy \in \mathrm{Ob}\ D$, $\varepsilon(y) : fg(y) \to y$ is $\phi 1_{g(y)}$. It is easy to prove naturality. Moreover the counit is universal in the following sense

Proposition A.5.3 *For any* $y \in \mathrm{Ob}\ \mathsf{C}$, $\varepsilon(y)$ *is universal with respect to* f, *i.e. for any morphism* $h : f(x) \to y$ *there is a unique morphism* $h' : g(y) \to x$ *so that the diagram*

$$\begin{tikzcd} fg(y) \arrow[r, "f(h')"] \arrow[d, "\varepsilon(y)"'] & f(x) \arrow[dl, "h"'] \\ y & \end{tikzcd}$$

commutes

It is easy to see that we can recover the adjunction ϕ from ε since, for any $x \in \mathrm{Ob}\ \mathsf{C}$, $y \in \mathrm{Ob}\ \mathsf{D}$ and $a \in \mathsf{C}(x, g(y)$ $\phi(x, y)(a)$ is $\varepsilon(y)f(a)$.

A.6 Adjoint functors, limits and colimits

[There lots of excellent accounts of adjoint functors, e.g. wikipedia, downloadable texts such as a TAC Reprint, and here we can say just that readers need to know: definition, unit and counit, including notation, and preservation of limits and colimits.]

One of the most useful results about adjoint functors is that on preservation of limits and colimits as follows.

Proposition A.6.1 *Let* $\phi : \mathsf{C}(-, g(-)) \cong \mathsf{D}(f(-), -)$ *be an adjunction between the functors* $f : \mathsf{C} \to \mathsf{D}$, $g : \mathsf{D} \to \mathsf{C}$. *Then* f *preserves colimits, and* g *preserves limits.*

Proof We first prove f preserves colimits. Let X be a small category and $T : X \to \mathsf{C}$ a functor. We use the following set of natural equivalences for $c \in \mathsf{C}$:

$$\begin{aligned} \mathsf{C}(f \operatorname{colim} T, c) &\cong \mathsf{D}(\operatorname{colim} T, gc) && \text{by adjointness} \\ &\cong \lim \mathsf{D}(T, gc) && \text{this needs an earlier justification} \\ &\cong \lim \mathsf{C}(fT, c) && \text{by adjointness} \\ &\cong \mathsf{C}(\operatorname{colim} fT, c). && \text{by an earler result!} \end{aligned}$$

By the representability proposition A.2.3, there is a natural equivalence $f \operatorname{colim} T \cong \operatorname{colim} fT$.

A similar argument, using $\mathsf{D}(d, \lim S) \cong \lim \mathsf{D}(d, S)$, proves that g preserves limits. □

Remark A.6.2 This result is useful in quite basic constructions in topology and algebra. For example, it is standard that the forgetful functor $U : \mathsf{Top} \to \mathsf{Sets}$ giving the underlying set of a topological space has left and right adjoints, given respectively by the discrete, and the indiscrete topologies on a set. Hence the underlying set of the product of topological spaces is the product of the underlying sets. The property we want of the product of spaces is the universal property, since this enables one to construct continuous functions.

A.7 Fibrations of categories

We recall the definition of fibration of categories.

Definition A.7.1 Let $\Phi : \mathsf{X} \to \mathsf{B}$ be a functor. A morphism $\phi : Y \to X$ in X over $u := \Phi(\phi)$ is called *cartesian* if and only if for all $\nu : K \to J$ in B and $\theta : Z \to X$ with $\Phi(\theta) = u\nu$ there is a unique morphism $\psi : Z \to Y$ with $\Phi(\psi) = \nu$ and $\theta = \phi\psi$.

This is illustrated by the following diagram:

$$\begin{tikzcd}
Z \arrow[r, dotted, "\psi"'] \arrow[rr, bend left, "\theta"] & Y \arrow[r, "\phi"'] & X & {} \arrow[d, "\Phi"] \\
K \arrow[r, "\nu"'] \arrow[rr, bend left, "u\nu"] & J \arrow[r, "u"'] & I & {}
\end{tikzcd}$$

□

It is straightforward to check that cartesian morphisms are closed under composition, and that ϕ is an isomorphism if and only if ϕ is a cartesian morphism over an isomorphism.

A morphism $\alpha : Z \to Y$ is called *vertical* (with respect to Φ) if and only if $\Phi(\alpha)$ is an identity morphism in B. In particular, for $I \in \mathsf{B}$ we write X_I, called the *fibre over* I, for the subcategory of X consisting of those morphisms α with $\Phi(\alpha) = \mathrm{id}_I$.

Definition A.7.2 The functor $\Phi : \mathsf{X} \to \mathsf{B}$ is a *fibration* or *category fibred over* B if and only if for all $u : J \to I$ in B and $X \in \mathsf{X}_I$ there is a cartesian morphism $\phi : Y \to X$ over u: such a ϕ is called a *cartesian lifting* of X along u. □

Notice that cartesian liftings of $X \in \mathsf{X}_I$ along $u : J \to I$ are unique up to vertical isomorphism: if $\phi : Y \to X$ and $\psi : Z \to X$ are cartesian over u, then there exist vertical arrows $\alpha : Z \to Y$ and $\beta : Y \to Z$ with $\phi\alpha = \psi$ and $\psi\beta = \phi$ respectively, from which it follows by cartesianness of ϕ and ψ that $\alpha\beta = \mathrm{id}_Y$ and $\beta\alpha = \mathrm{id}_Z$ as $\psi\beta\alpha = \phi\alpha = \psi = \psi\,\mathrm{id}_Y$ and similarly $\phi\beta\alpha = \phi\,\mathrm{id}_Y$.

Example A.7.3 The forgetful functor, $\mathrm{Ob} : \mathsf{Gpds} \to \mathsf{Sets}$, from the category of groupoids to the category of sets is a fibration. We can for a groupoid G over I and function $u : J \to I$ define the cartesian lifting $\phi : H \to G$ as follows: for $j, j' \in J$ set

$$H(j, j') = \{(j, g, j') \mid g \in G(uj, uj')\}$$

with composition

$$(j_1, g_1, j_1')(j, g, j') = (j_1, g_1 g, j'),$$

with ϕ given by $\phi(j, g, j') = g$. The universal property is easily verified. The groupoid H is usually called the *pullback* of G by u. This is a well known construction (see for example [Mac05, §2.3], where pullback by u is written $u^{\downarrow\downarrow}$). Q.E.D.

Definition A.7.4 If $\Phi : \mathsf{X} \to \mathsf{B}$ is a fibration, then using the axiom of choice for classes we may select for every $u : J \to I$ in B and $X \in \mathsf{X}_I$ a cartesian lifting of X along u

$$u^X : u^*X \to X.$$

Such a choice of cartesian liftings is called a *cleavage* or *splitting* of Φ.

If we fix the morphism $u : J \to I$ in B, the splitting gives a so-called *reindexing functor*

$$u^* : \mathsf{X}_I \to \mathsf{X}_J$$

defined on objects by $X \mapsto u^*X$ and the image of a morphism $\alpha : X \to Y$ is $u^*\alpha$ the unique vertical arrow commuting the diagram:

$$\begin{tikzcd} u^*X \arrow[r, "u^X"] \arrow[d, dashed, "u^*\alpha"'] & X \arrow[d, "\alpha"] \\ u^*Y \arrow[r, "u^Y"'] & Y \end{tikzcd}$$

□

We can use this reindexing functor to get an adjoint situation for each $u : J \to I$ in B.

Proposition A.7.5 *Suppose* $\Phi : X \to B$ *is a fibration of categories,* $u : J \to I$ *in* B*, and a reindexing functor* $u^* : X_I \to X_J$ *is chosen. Then there is a bijection*

$$X_J(Y, u^*X) \cong X_u(Y, X)$$

natural in $Y \in X_J$, $X \in X_I$ *where* $X_u(Y, X)$ *consists of those morphisms* $\alpha \in X(Y, X)$ *with* $\Phi(\alpha) = u$.

Proof This is just a restatement of the universal properties concerned. □

In general for composable maps $u : J \to I$ and $v : K \to J$ in B it does not hold that

$$v^*u^* = (uv)^*$$

as may be seen with the fibration of Example A.7.3. Nevertheless there is a natural equivalence $c_{u,v} : v^*u^* \simeq (uv)^*$ as shown in the following diagram in which the full arrows are cartesian and where $(c_{u,v})_X$ is the unique vertical arrow making the diagram commute:

$$\begin{tikzcd} v^*u^*X \arrow[r, "v^{u^*X}"] \arrow[d, dashed, "(c_{u,v})_X"', "\cong"] & u^*X \arrow[d, "u^X"] \\ (uv)^*X \arrow[r, "(uv)^X"'] & X \end{tikzcd}$$

Let us consider this phenomenon for our main examples:

Example A.7.6 1.- Typically, for $\Phi_B = \partial_1 : B^2 \to B$, the fundamental fibration for a category with pullbacks, we do not know how to choose pullbacks in a functorial way.

2.- In considering groupoids as a fibration over sets, if $u : J \to I$ is a map, we have a reindexing functor, also called *pullback*, $u^* : \mathsf{Gpds}_I \to \mathsf{Gpds}_J$. We notice that v^*u^*Q is naturally isomorphic to, but not identical to $(uv)^*Q$. Q.E.D.

A result which aids understanding of our calculation of pushouts and some other colimits of groupoids, modules, crossed complexes and higher categories is the following. Recall that a category C is *connected* if for any $c, c' \in C$ there is a sequence of objects $c_0 = c, c_1, \ldots, c_{n-1}, c_n = c'$ such that for each $i = 0, \ldots, n-1$ there is a morphism $c_i \to c_{i+1}$ or $c_{i+1} \to c_i$ in C. The sequence of morphisms arising in this way is called a *zig-zag* from c to c' of length n.

Theorem A.7.7 *Let* $\Phi : X \to B$ *be a fibration, and let* $J \in B$*. Then the inclusion* $i_J : X_J \to X$ *preserves colimits of connected diagrams.*

Proof We will need the following diagrams:

$$\begin{tikzcd}
T(c) \arrow[d, "\psi(c)"] \arrow[dr, "\theta(c)"] \arrow[dd, bend right=60, "T(f)"'] & \\
Y \arrow[r, "\phi"] & X \\
T(c') \arrow[u, "\psi(c')"'] \arrow[ur, "\theta(c')"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
T(c) \arrow[d, "\psi(c)"] \arrow[dr, "\theta(c)"] \arrow[dd, bend right=60, "\gamma(c)"'] & \\
Y \arrow[r, "\phi"] & X \\
L \arrow[u, dotted, "\psi'"'] \arrow[ur, dotted, "\theta'"'] &
\end{tikzcd}$$

$$\Big\downarrow \Phi \qquad\qquad\qquad \Big\downarrow \Phi$$

$$\begin{tikzcd} J \arrow[r, dotted, "u"'] & I \end{tikzcd} \qquad\qquad \begin{tikzcd} J \arrow[r, dotted, "u"'] & I \end{tikzcd}$$

(a) (b)

Let $T : C \to X_J$ be a functor from a small connected category C and suppose T has a colimit $L \in X_J$. So we regard L as a constant functor $L : C \to X_J$ which comes with a universal cocone $\gamma : T \Rightarrow L$ in X_J. Let $i_J : X_J \to X$ be the inclusion. We prove that the natural transformation $i_J\gamma : i_J T \Rightarrow i_J L$ is a colimit cocone in X.

We use the following lemma.

Lemma A.7.8 *Let* $X \in X$*, with* $\Phi(X) = I$*, be regarded as a constant functor* $X : C \to X$ *and let* $\theta : i_J T \Rightarrow X$ *be a natural transformation, i.e. a cocone. Then*
(i) for all $c \in C$*,* $u = \Phi(\theta(c)) : J \to I$ *in* B *is independent of* c*, and*
(ii) the cartesian lifting $Y \to X$ *of* u *determines a cocone* $\psi : T \Rightarrow Y$.

Proof The natural transformation θ gives for each object the morphism $\theta(c) : T(c) \to X$ in X. Since C is connected, induction on the length of a ziz-zag shows it is sufficient to prove (i) when there is a morphism $f : c \to c'$ in C. By naturality of θ, $\theta(c')T(f) = \theta(c)$. But $\Phi T(f)$ is the identity, since T has values in X_J, and so $\Phi(\theta(c)) = \Phi(\theta(c'))$. Write $u = \Phi(\theta(c))$.

Since Φ is a fibration, there is a $Y \in X_J$ and a cartesian lifting $\phi : Y \to X$ of u. Hence for each $c \in C$ there is a unique vertical morphism $\psi(c) : T(c) \to Y$ in X_J such that $\phi\psi(c) = \theta(c)$. We now prove that ψ is a natural transformation $T \Rightarrow Y$ in X_J, where Y is regarded as a constant functor.

To this end, let $f : c \to c'$ be a morphism in X_J. We need to prove $\psi(c) = \psi(c')T(f)$.

The outer part of diagram (a) commutes, since θ is a natural transformation. The upper and lower triangles commute, by construction of ϕ. Hence

$$\phi\psi(c) = \theta(c) = \theta(c')T(f) = \phi\psi(c')T(f).$$

Now $T(f)$, $\psi(c)$ and $\psi(c')$ are all vertical. By the universal property of ϕ, $\psi(c) = \psi(c')T(f)$, i.e. the left hand cell commutes. That is, ψ is a natural transformation $T \Rightarrow Y$ in X_J. □

To return to the theorem, since L is a colimit in X_J, there is a unique vertical morphism $\psi' : L \to Y$ in the right hand diagram (b) such that for all $c \in C$, $\psi'\gamma(c) = \psi(c)$. Let $\theta' = \phi\psi' : L \to X$. This gives a morphism $\theta' : L \to X$ such that $\theta'\gamma(c) = \theta(c)$ for all c, and, again using universality of ϕ, this morphism is unique. □

Remark A.7.9 The connectedness assumption is essential in the Theorem. Any small category C is the disjoint union of its connected components. If $T : C \to X$ is a functor, and X has colimits, then $\operatorname{colim} T$ is the coproduct (in X) of the $\operatorname{colim} T_i$ where T_i is the restriction of T to a component C_i. But given two objects in the same fibre of $\Phi : X \to B$, their coproduct in that fibre is in general not the same as their coproduct in X. For example, the coproduct of two groups in the category of groups is the free product of groups, while their coproduct as groupoids is their disjoint union. □

Remark A.7.10 A common application of the theorem is that the inclusion $X_J \to X$ preserves pushouts. This is relevant to our application of pushouts in section A.9. Pushouts are used to construct free crossed modules as a special case of induced crossed modules, [BH78], and to construct free crossed complexes as explained in [BH91, BG89b]. Q.E.D.

Remark A.7.11 George Janelidze has pointed out a short proof of Theorem A.7.7 in the case Φ has a right adjoint, and so preserves colimits, which applies to our main examples here. If the image of T is inside $\Phi(b)$, then ΦT is the constant diagram whose value is $\{b, 1_b\}$, and if C is connected this implies that $\operatorname{colim} \Phi T = b$. But if $\Phi(\operatorname{colim} T) = \operatorname{colim} \Phi T = b$, then $\operatorname{colim} T$ is inside $\Phi(b)$. Q.E.D.

A.8 Cofibrations of categories

We now give the duals of the above results.

Definition A.8.1 Let $\Phi : X \to B$ be a functor. A morphism $\psi : Z \to Y$ in X over $v := \Phi(\psi)$ is called *cocartesian* if and only if for all $u : J \to I$ in B and $\theta : Z \to X$ with $\Phi(\theta) = uv$ there is a unique morphism $\phi : Y \to X$ with $\Phi(\phi) = u$ and $\theta = \phi\psi$.

This is illustrated by the following diagram:

$$\begin{tikzcd}
Z \arrow[r, "\psi"'] \arrow[rr, bend left, "\theta"] & Y \arrow[r, dotted, "\phi"'] & X & \arrow[dd, "\Phi"] \\
 & & & \\
K \arrow[r, "v"'] \arrow[rr, bend left, "uv"] & J \arrow[r, "u"'] & I &
\end{tikzcd}$$

□

It is straightforward to check that cocartesian morphisms are closed under composition, and that ψ is an isomorphism if and only if ψ is a cocartesian morphism over an isomorphism.

Definition A.8.2 The functor $\Phi : X \to B$ is a *cofibration* or *category cofibred over* B if and only if for all $v : K \to J$ in B and $Z \in X_K$ there is a cartesian morphism $\psi : Z \to Z'$ over v: such a ψ is called a *cocartesian lifting* of Z along v. □

The cocartesian liftings of $Z \in X_K$ along $v : K \to J$ are also unique up to vertical isomorphism.

Remark A.8.3 As in Definition A.7.4, if $\Phi : X \to B$ is a cofibration, then using the axiom of choice for classes we may select for every $v : K \to J$ in B and $Z \in X_K$ a cocartesian lifting of Z along v

$$v_Z : Z \to v_* Z.$$

Under these conditions, the functor ν_* is commonly said to give the objects *induced* by ν. Examples of induced crossed modules of groups are developed in [BW03], following on from the first definition of these in [BH78]. □

We now have the dual of Proposition A.7.5.

Proposition A.8.4 *For a cofibration $\Phi : X \to B$, a choice of cocartesian liftings of $\nu : K \to J$ in B yields a functor $\nu_* : X_K \to X_J$, and an adjointness*

$$X_J(\nu_* Z, Y) \cong X_\nu(Z, Y)$$

for all $Y \in X_J$, $Z \in X_K$.

We now state the dual of Theorem A.7.7.

Theorem A.8.5 *Let $\Phi : X \to B$ be a category cofibred over B. Then the inclusion of each fibre of Φ into X preserves limits of connected diagrams.*

Many of the examples we are interested in are both fibred and cofibred. For them we have an adjoint situation.

Proposition A.8.6 *For a functor $\Phi : X \to B$ which is both a fibration and cofibration, and a morphism $u : J \to I$ in B, a choice of cartesian and cocartesian liftings of u gives an adjointness*

$$X_J(Y, u^* X) \cong X_I(u_* Y, X)$$

for $Y \in X_J$, $X \in X_I$.

It is interesting to get a characterisation of the cofibration property for a functor that already is a fibration. The following is a useful weakening of the condition for cocartesian in the case of a fibration of categories.

Proposition A.8.7 *Let $\Phi : X \to B$ be a fibration of categories. Then $\psi : Z \to Y$ in X over $\nu : K \to J$ in B is cocartesian if only if for all $\theta' : Z \to X'$ over ν there is a unique morphism $\psi' : Y \to X'$ in X_J with $\theta' = \psi'\psi$.*

Proof The 'only if' part is trivial. So to prove 'if' we have to prove that for any $u : J \to I$ and $\theta : Z \to X$ such that $\Phi(\theta) = u\nu$, there exists a unique $\phi : Y \to X$ over u completing the diagram

$$\begin{tikzcd} Z \arrow[r, "\psi"'] \arrow[rr, bend left, "\theta"] & Y \arrow[r, dotted, "\phi"'] & X & \\ & & & {} \arrow[d, "\Phi"] \\ & & & {} \\ K \arrow[r, "\nu"'] & J \arrow[r, "u"'] & I. & \end{tikzcd}$$

Since Φ is a fibration there is a cartesian morphism $\kappa : X' \to X$ over u. By the cartesian property, there is a unique morphism $\theta' : Z \to X'$ over ν such that $\kappa\theta' = \theta$, as in the diagram

$$\begin{tikzcd} Z \arrow[r, dashed, "\theta'"'] \arrow[rr, bend left, "\theta"] & X' \arrow[r, "\kappa"'] & X. \end{tikzcd}$$

Now, suppose $\phi : Y \to X$ over $u : J \to I$ satisfies $\phi\psi = \theta$, as in the diagram:

$$\begin{tikzcd}
Z \arrow[r, "\psi"'] \arrow[rr, bend left, "\theta"] \arrow[dr, dashed, "\theta'"'] & Y \arrow[r, dotted, "\phi"'] & X \\
& X' \arrow[ur, "\kappa"'] &
\end{tikzcd}$$

By the given property of ψ there is a unique morphism $\psi' : Y \to X'$ in X_J such that $\psi'\psi = \theta'$. By the cartesian property of κ, there is a unique morphism ϕ' in X_J such that $\kappa\phi' = \phi$. Then

$$\kappa\psi'\psi = \kappa\theta' = \theta = \phi\psi = \kappa\phi'\psi.$$

By the cartesian property of κ, and since $\psi'\psi,\ \phi'\psi$ are over uv, we have $\psi'\psi = \phi'\psi$. By the given property of ψ, and since $\phi',\ \psi'$ are in X_J, we have $\phi' = \psi'$. So $\phi = \kappa\psi'$, and this proves uniqueness.

But we have already checked that $\kappa\psi'\psi = \theta$, so we are done. □

The following Proposition allows us to prove that a fibration is also a cofibration by constructing the adjoints u_* of u^* for every u.

Proposition A.8.8 *Let $\Phi : X \to B$ be a fibration of categories. Let $u : J \to I$ have reindexing functor $u^* : X_I \to X_J$. Then the following are equivalent:*

(i) *For all $Y \in X_J$, there is a morphism $u_Y : Y \to u_*Y$ which is cocartesian over u;*

(ii) *there is a functor $u_* : X_J \to X_I$ which is left adjoint to u^*.*

Proof That (ii) implies (i) is clear, using Proposition A.8.7, since the adjointness gives the bijection required for the cocartesian property.

To prove that (i) implies (ii) we have to check that the allocation $Y \mapsto u_*(Y)$ gives a functor that is adjoint to u^*. As before the adjointness comes from the cocartesian property.

We leave to the reader the check the details of the functoriality of u_*. □

To end this section, we give a useful result on compositions.

Proposition A.8.9 *The composition of fibrations, (cofibrations), is also a fibration (cofibration).*

Proof We leave this as an exercise. □

A.9 Pushouts and cocartesian morphisms

Here is a small result which we use in this section and section **??**, as it applies to many examples, such as the fibration $\mathrm{Ob} : \mathsf{Gpds} \to \mathsf{Set}$.

Proposition A.9.1 *Let $\Phi : X \to B$ be a functor that has a left adjoint D. Then for each $K \in \mathrm{Ob}\, B$, $D(K)$ is initial in X_K. In fact if $u : K \to J$ in B, then for any $X \in X_J$ there is a unique morphism $\epsilon_K : DK \to X$ over u.*

Proof This follows immediately from the adjoint relation $X_u(DK, X) \cong B(K, \Phi X)$ for all $X \in \mathrm{Ob}\, X_J$. □

Special cases of cocartesian morphisms are used in [Bro06, BH78, BH81a], and we review these in section **??**. A construction which arises naturally from the various Higher Homotopy van Kampen theorems is given a general setting as follows:

Theorem A.9.2 *Let* $\Phi : X \to B$ *be a fibration of categories which has a left adjoint* D*. Suppose that* X *admits pushouts. Let* $\nu : K \to J$ *be a morphism in* B*, and let* $Z \in X_K$*. Then a cocartesian lifting* $\psi : Z \to Y$ *of* ν *is given precisely by the pushout in* X*:*

$$\begin{tikzcd} D(K) \arrow[r, "D(\nu)"] \arrow[d, "\epsilon_K"'] & D(J) \arrow[d, "\epsilon_J"] \\ Z \arrow[r, "\psi"'] & Y \end{tikzcd} \tag{*}$$

Proof Suppose first that diagram (*) is a pushout in X. Let $u : J \to I$ in B and let $\theta : Z \to X$ satisfy $\Phi(\theta) = u\nu$, so that $\Phi(X) = I$. Let $f : D(J) \to X$ be the adjoint of $u : J \to \Phi(X)$.

$$\begin{tikzcd} D(K) \arrow[r, "D(\nu)"] \arrow[d, "\epsilon_K"'] & D(J) \arrow[d, "\epsilon_J"'] \arrow[ddr, "f"] & \\ Z \arrow[r, "\psi"] \arrow[drr, "\theta"'] & Y \arrow[dr, dotted, "\phi"] & \\ & & X \end{tikzcd} \tag{**}$$

$$\begin{tikzcd} K \arrow[r, "\nu"'] & J \arrow[r, "u"'] & I \end{tikzcd}$$

Then $\Phi(fD(\nu)) = u\nu = \Phi(\theta\epsilon_K)$ and so by Proposition A.9.1, $fD(\nu) = \theta\epsilon_K$. The pushout property implies there is a unique $\phi : Y \to X$ such that $\phi\psi = \theta$ and $\phi\epsilon_J = f$. This last condition gives $\Phi(\phi) = u$ since $u = \Phi(f) = \Phi(\phi\epsilon_J) = \Phi(\phi)\,\mathrm{id}_J = \Phi(\phi)$.

For the converse, we suppose given $f : D(J) \to X$ and $\theta : Z \to X$ such that $\theta\epsilon_K = fD(\nu)$. Then $\Phi(\theta) = u\nu$ and so there is a cocartesian lifting $\phi : Y \to X$ of u. The additional condition $\phi\epsilon_J = f$ is immediate by Proposition A.9.1. □

Corollary A.9.3 *Let* $\Phi : X \to B$ *be a fibration which has a left adjoint and suppose that* X *admits pushouts. Then* Φ *is also a cofibration.*

In view of the construction of hierarchical homotopical invariants as colimits from the HHvKT in Chapter 8 [28], the following is worth recording, as a consequence of Theorem A.7.7.

Theorem A.9.4 *Let* $\Phi : X \to B$ *be fibred and cofibred. Assume* B *and all fibres* X_I *are cocomplete. Let* $T : C \to X$ *be a functor from a small connected category. Then* $\operatorname{colim} T$ *exists and may be calculated as follows:*

(i) *First calculate* $I = \operatorname{colim}(\Phi T)$*, with cocone* $\gamma : \Phi T \Rightarrow I$*;*

(ii) *for each* $c \in C$ *choose cocartesian morphisms* $\gamma'(c) : T(c) \to X(c)$*, over* $\gamma(c)$ *where* $X(c) \in X_I$*;*

(iii) *make* $c \mapsto X(c)$ *into a functor* $X : C \to X_I$*, so that* γ' *becomes a natural transformation* $\gamma' : T \Rightarrow X$*;*

(iv) *form* $Y = \operatorname{colim} X \in X_I$ *with cocone* $\mu : X \Rightarrow Y$.

Then Y *with* $\mu\gamma' : T \Rightarrow Y$ *is* $\operatorname{colim} T$.

Proof We first explain how to make X into a functor.

We will in stages build up the following diagram:

$$\begin{tikzcd}
 & T(c) \arrow[rr, "\gamma'(c)"'] \arrow[dr, "T(f)"'] \arrow[rrrr, bend left, "\eta"] & & X(c) \arrow[r, "\mu(c)"] \arrow[d, dashed, "X(f)"] & Y \arrow[r, dotted, "\tau"'] \arrow[d, "1"] & Z \\
 {} \arrow[dd, "\Phi"] & & T(c') \arrow[r, "\gamma'(c')"'] & X(c') \arrow[r, "\mu(c')"'] & Y \arrow[r, dotted, "\tau'"'] & Z \\
 & & & & & \\
 {} & K \arrow[r, "\Phi T(f)"'] & J \arrow[r, "\gamma(c')"'] & I \arrow[r, "1"'] & I \arrow[r, "w"'] & H
\end{tikzcd} \tag{A.9.1}$$

Let $f : c \to c'$ be a morphism in C, $K = \Phi T(c), J = \Phi T(c')$. By cocartesianness of $\gamma'(c)$, there is a unique vertical morphism $X(f) : X(c) \to X(c')$ such that $X(f)\gamma'(c) = \gamma'(c')T(f)$. It is easy to check, again using cocartesianness, that if further $g : c' \to c''$, then $X(gf) = X(g)X(f)$, and $X(1) = 1$. So X is a functor and the above diagram shows that γ' becomes a natural transformation $T \Rightarrow X$.

Let $\eta : T \Rightarrow Z$ be a natural transformation to a constant functor Z, and let $\Phi(Z) = H$. Since $I = \operatorname{colim}(\Phi T)$, there is a unique morphism $w : I \to H$ such that $w\gamma = \Phi(\eta)$.

By the cocartesian property of γ', there is a natural transformation $\eta' : X \Rightarrow Z$ such that $\eta'\gamma' = \eta$.

Since Y is also a colimit in X of X, we obtain a morphism $\tau : Y \to Z$ in X such that $\tau\mu = \eta'$. Then $\tau\mu\gamma' = \eta'\gamma' = \eta$.

Let $\tau' : Y \to Z$ be another morphism such that $\tau'\mu\gamma' = \eta$. Then $\Phi(\tau) = \Phi(\tau') = w$, since I is a colimit. Again by cocartesianness, $\tau'\mu = \tau\mu$. By the colimit property of Y, $\tau = \tau'$. □

This with Theorem A.9.4 shows how to compute colimits of connected diagrams in the examples we discuss in sections A.10 to **??**, and in all of which a van Kampen type theorem is available giving colimits of algebraic data for some glued topological data.

Corollary A.9.5 *Let* $\Phi : X \to B$ *be a functor satisfying the assumptions of theorem A.9.4. Then* X *is connected complete, i.e. admits colimits of all connected diagrams.*

A.10 Groupoids bifibred over sets

We have already seen in Example A.7.3 that the functor $\mathrm{Ob} : \mathsf{Gpds} \to \mathsf{Sets}$ is a fibration. It also has a left adjoint D assigning to a set I the discrete groupoid on I, and a right adjoint assigning to a set I the codiscrete groupoid on I.

It follows from general theorems on algebraic theories that the category Gpds is cocomplete, and in particular admits pushouts, and so it follows from previous results that $\mathrm{Ob} : \mathsf{Gpds} \to \mathsf{Sets}$ is also a cofibration. A construction of the cocartesian liftings of $u : I \to J$ for G a groupoid over I is given in terms of words, generalising the construction of free groups and free products of groups, in [Hig71, Bro06]. In these references the cocartesian lifting of u to G is called a *universal morphism*, and is written $u_* : G \to U_u(G)$. This construction is of interest as it yields a normal form for the elements of $U_u(G)$, and hence u_* is injective on the set of non-identity elements of G.

A homotopical application of this cocartesian lifting is the following theorem on the fundamental groupoid. It shows how identification of points of a discrete subset of a space can lead to 'identifications of the objects' of the fundamental groupoid:

Theorem A.10.1 *Let (X, A) be a pair of spaces such that A is discrete and the inclusion $A \to X$ is a closed cofibration. Let $f : A \to B$ be a function to a discrete space B. Then the induced morphism*

$$\pi_1(X, A) \to \pi_1(B \cup_f X, B)$$

is the cocartesian lifting of f.

This theorem immediately gives the fundamental group of the circle S^1 as the infinite cyclic group C, since S^1 is obtained from the unit interval $[0, 1]$ by identifying 0 and 1, as shown in the Introduction in diagram (1.7.3). The theorem is a translation of [Bro06, 9.2.1], where the words 'universal morphism' are used instead of 'cocartesian lifting'. Section 8.2 of [Bro06] shows how free groupoids on directed graphs are obtained by a generalisation of this example.

The calculation of colimits in a fibre Gpds_I is similar to that in the category of groups, since both categories are protomodular, [BB04]. Thus a colimit is calculated as a quotient of a coproduct, where quotients are themselves obtained by factoring by a normal subgroupoid. Quotients are discussed in [Hig71, Bro06].

Theorem A.9.4 now shows how to compute general colimits of groupoids.

We refer again to [Hig71, Bro06] for further developments and applications of the algebra of groupoids. We generalise some aspects of the above to modules, crossed modules and crossed complexes in Chapter 7.

The following subsections cover some aspects of groupoid theory needed earlier.

A.10.1 Covering morphisms of groupoids

For the convenience of readers, and to fix the notation, we recall here the basic facts on covering morphisms of groupoids. Proofs can be found in the books [Bro06, **?**]. However we find it convenient to adopt different conventions, focussing on costars rather than stars, which ensure that some of our formulae in Subsection 11.2.4 work out in a nice way, see equation (A.10.3).

Let G be a groupoid. For each object a of G the *Costar* of a_0 in G, denoted by $\mathrm{Cost}_G\, a_0$, is the union of the sets $G(a, a_0)$ for all objects a of G, i.e. $\mathrm{Cot}_G a_0 = \{g \in G \mid tg = a_0\}$. A morphism $p : \widetilde{G} \to G$ of groupoids is a *covering morphism* if for each object $\widetilde{a}$ of $\widetilde{G}$ the restriction of p

$$\mathrm{Cost}_{\widetilde{G}}\, \widetilde{a} \to \mathrm{Cost}_G\, p\widetilde{a}$$

is bijective. In this case $\widetilde{G}$ is called a *covering groupoid of G*.

A basic result for covering groupoids is *unique path lifting*. That is, let $p : \widetilde{G} \to G$ be a covering morphism of groupoids, and let $(g_1, g_2, \ldots, g_n)$ be a sequence of composable elements of G. Let $\tilde{a} \in \mathrm{Ob}(\widetilde{G})$ be such that $p\tilde{a}$ is the target of g_n. Then there is a unique composable sequence $(\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_n)$ of elements of $\widetilde{G}$ such that $\tilde{g}_n$ ends at $\tilde{a}$ and $p\tilde{g}_i = g_i, i = 1, \ldots, n$.

If G is a groupoid, the slice category $\mathsf{GpdsCov}/G$ of coverings of G has as objects the covering morphisms $p : H \to G$ and has as arrows (morphisms) the commutative diagrams of morphisms of groupoids, where p and q are covering morphisms,

$$\begin{tikzcd} H \arrow[rr, "f"] \arrow[dr, "p"'] & & K \arrow[dl, "q"] \\ & G & \end{tikzcd}$$

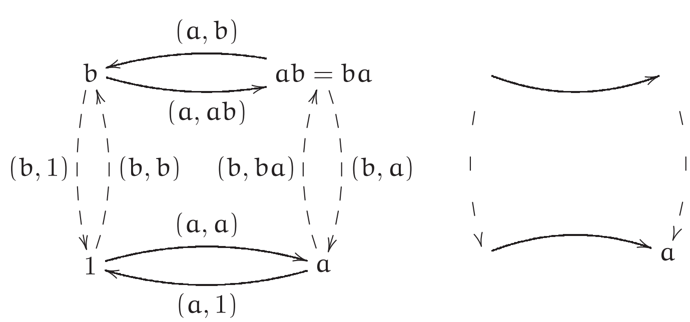
By a standard result on compositions and covering morphisms ([Bro06, 10.2.3]), f also is a covering morphism. It is convenient to write such a diagram as a triple (f, p, q). The composition in $\mathsf{GpdsCov}/G$ is then given as usual by

$$(g, q, r)(f, p, q) = (gf, p, r).$$

It is a standard result (see for example [Hig71, Bro70]) that the category $\mathsf{GpdsCov}/G$ is equivalent to the functor category $\mathsf{Sets}^{G^{op}}$. Thus if $X : G^{op} \to \mathsf{Sets}$ is a functor, then $\widetilde{G} = G \ltimes X$ has object set the disjoint union of the sets $X(a)$ for $a \in Ob(G)$ and arrows $y \to x$ the pairs (g, x) such that $x \in X(tg)$ and $y = X(g)x$; in operator notation: $(g, x) : gx \to x$. The composition is then $(g', gx)(g, x) = (g'g, x)$. The projection morphism $G \ltimes X \to G$, $(g, x) \mapsto g$, is then a covering morphism.

This 'semidirect product' or 'Grothendieck construction' [29] is fundamental for constructing covering morphisms to the groupoid G. For example, if a_0 is an object of the transitive groupoid G, and A is a subgroup of the object group $G(a_0)$ then the groupoid G operates on the family of cosets $\{gA \mid g \in \mathrm{Cost}_G\ a_0\}$, by $g'.(gA) = g'gA$ whenever $g'g$ is defined, and the associated covering morphism $\widetilde{G} \to G$ defines the covering groupoid $\widetilde{G}$ of the groupoid G determined by the subgroup A. When A is trivial this gives the *universal cover* at a_0 of the groupoid G. In particular, this gives the universal covering groupoid of a group, whose objects are the elements of G and arrows are pairs $(g, h) : gh \to h$, $g, h \in G$. Then G operates on the right of the universal cover by $(g, h)^k = (g, hk)$. This operation preserves the map p and is called a *covering transformation*.

Example A.10.2 Here is a simple example: the universal covering groupoid $\widetilde{K}$ of the Klein 4-group $K = C_2 \times C_2$ with elements say $1, a, b, ab$. The group is generated by a, b with the relations $a^2, b^2, aba^{-1}b^{-1}$, which we write respectively r, s, t. Then $\widetilde{K}$ has the elements of K as vertices and an arrow $(g, x) : gx \to x$ for each $g, x \in K$. The covering morphism $p : \widetilde{K} \to K$ is $(g, x) \mapsto g$. In terms of the generators a, b we obtain a diagram of $\widetilde{K}$ as the left hand diagram in the following picture:



Note that for example $(a, ab) : b \to ab$ because $a^2 = 1$. The right hand diagram illustrates a lift of the path $b^{-1}a^{-1}ba$ in K to a path starting and ending at a in the diagram of $\widetilde{K}$. You should draw the similar loops starting in turn at $1, b, ab$. We show in Section 11.2.4 that in the context of covering morphisms of crossed complexes these four loops form boundaries of four 'lifts' of the relation t. $\square$

Example A.10.3 Given a morphism $\phi : F \to G$ of groups, let $q : \widehat{F} \to F$ be the pullback by ϕ of the universal covering morphism $p : \widetilde{G} \to G$ giving a commutative diagram

$$\begin{tikzcd} \widehat{F} \arrow[r, "\bar{\phi}"] \arrow[d, "q"'] & \widetilde{G} \arrow[d, "p"] \\ F \arrow[r, "\phi"'] & G \end{tikzcd} \tag{A.10.1}$$

Note that an arrow in $\widehat{F}$ is a pair $(u, (\phi u, g)) : (\phi u)g \to g$, $u \in F, g \in G$. Since u determines ϕu, we can write an arrow of $\widehat{F}$ as $(u, g) : (\phi u)g \to g$. Again, G operates on the right of $\widehat{F}$ by $(u, g)^k = (u, gk)$, $k \in G$.

If X is a set of generators of the group G, we have an epimorphism $\phi : F \to G$ where F is the free group on the set X. Let $\widehat{X}$ be the graph $q^{-1}(X)$ in $\widehat{F}$. This is called the *Cayley graph* of the set of generators X of G. Its vertices are the elements of G and the arrows are pairs $(x, g) : (\phi x)g \to g$. For our particular example with generators of the Klein group K this Cayley graph is often drawn in an abbreviated form as:

$$\begin{tikzcd} b \arrow[d, "b"'] & ab \arrow[l, "a"'] \arrow[d, "b"] \\ a & a \arrow[l, "a"] \end{tikzcd} \tag{A.10.2}$$

□

Exercise A.10.4 Carry out a similar analysis to the above for the universal cover of the symmetric group S_3, whose Cayley graph is drawn in Example 3.1.6.

The following is a key result.

Proposition A.10.5 *Given the epimorphism $\phi : F \to G$ where $F = F(X)$ is the free group on the set X of generators of G, then $\widehat{F}$ is the free groupoid on the graph $\widehat{X}$, whose arrows can be written $(x, g) : (\phi x)g \to g$.*

Proof This is [Bro06, 8.2.1 Corollary 1]. See also [Hig71, Theorem 8, p.112]. The proofs use the solution of the word problem. □

This construction is used in Section 11.2.4 for computing resolutions, and is also relevant to Section 8.4.

Remark A.10.6 The main reason for our choice of conventions on covering morphisms is the following. Let G be a group and $p : \widetilde{G} \to G$ its universal covering morphism. Then G operates on the right of the groupoid $\widetilde{G}$ by $(g, h)^k = (g, hk)$, $(g, h) \in \widetilde{G}, k \in G$. Let $e : G \to \widetilde{G}$ be the function $g \mapsto (g, 1) : g \to 1$. Then one easily checks that

$$e(gh) = e(g)^h \, e(h)$$

Thus e is a (nonabelian) derivation. Also if $\phi : F \to G$ is a morphism of groups and $q : \widehat{F} \to F$ is the pullback of p by ϕ, then G again operates on the groupoid $\widehat{F}$ and $d : F \to \widehat{F}$ given by $u \mapsto (u, 1)$ satisfies

$$d(uv) = d(u)^{\phi v} \, d(v), \tag{A.10.3}$$

i.e. d is a (nonabelian) ϕ-derivation. □

Remark A.10.7 It is also useful to note that in the situation of the last remark, if

$$(x_1, g_1), (x_2, g_2), \dots, (x_n, g_n)$$

is a sequence of composable morphisms of $\widehat{F}$, so that $(\phi x_i)g_i = g_{i-1}, 1 < i \leqslant n$, where the x_i or their inverse belongs to X, then their composite is $(x_1 x_2 \dots x_n, t g_n)$. □

A.10.2 Abelianisation of groupoids

We will need in sections 7.5.3 and 14.7 the notion of abelianisation of a groupoid.

Let Ab, Groups, Gpds denote respectively the categories of abelian groups, groups, and groupoids. Each of the inclusions

$$\mathsf{Ab} \to \mathsf{Groups} \to \mathsf{Gpds} \tag{A.10.4}$$

has a left adjoint. That from groupoids to groups is called the *universal group* UG of a groupoid G and is described in detail in [Bro06, Chapter 8] and [Hig71]. In particular, the universal group of a groupoid G is the free product of the universal groups of the transitive components of G.

It follows that we have what we call the *universal abelianisation* G^{totab} of a groupoid, namely the usual abelianisation of the group UG. It is isomorphic to the direct sum of the G_i^{totab} over all components G_i of G. Any transitive groupoid G may be written in a non canonical way as the free product $G(a_0) * T$ of a vertex group $G(a_0)$ and an indiscrete or tree groupoid T (This result has been used to suggest that 'groupoids reduce to groups'; but this is analogous to suggesting that vector spaces reduce to numbers!). Then

$$UG \cong G(a_0) * UT$$

and UT is the free group on the elements $x : a_0 \to a$ in T for all $a \in \mathrm{Ob}(T)$, $a \neq a_0$. So for a transitive groupoid G with $a_0 \in \mathrm{Ob}\, G$

$$G^{\mathrm{totab}} \cong G(a_0)^{\mathrm{ab}} \oplus F$$

where F is the free abelian group on the elements $x : a_0 \to a$ in T for all $a \in \mathrm{Ob}(T)$, $a \neq a_0$, for T a wide tree subgroupoid of G.

However we also need a more restrictive abelianisation of a groupoid G with object set I, which we write G^{ab}. Here the abelianisation takes place in the category of groupoids with object set I, and an abelian groupoid over I is one in which all vertex groups are abelian. It is this construction which we apply to C_2 as part of the abelianisation ∇C of a crossed complex C, giving a chain complex with $\pi_1 C$ as groupoid of operators, in section 7.5.3.

Exercise A.10.8 A groupoid G is *abelian* if all its vertex groups are abelian. Show that the abelian groupoids form a reflexive subcategory of the category of all groupoids. □

Notes

[28]p. 449 See also [BL87b, BL87a, ES87, EM08].

[29]p. 452 This has also been developed by C. Ehresmann in [Ehr57], in which he defines both an action of a category and the associated "category of hypermorphisms" (and also what he calls the complete enlargement of a species of structures in the case of local groupoids).

Appendix B

Closed categories.

In section 9.1 we have given an account of various exponential laws. Here we give a sketch of some of the underlying categorical ideas.

In specialising to the category of groupoids, we get some indication of possible notion of 'higher order symmetry.

B.1 Products of categories and coherence

Let Cat be the category of all small categories with morphisms being the functors. This category is known to be complete and cocomplete. The product of categories is constructed in for example [Bro06, section 6.4], and has the universal property of a product in a category.

Let C, D be categories. The *product* $C \times D$ is defined to have objects all pairs (x, y) for x in $\mathrm{Ob}(C)$, y in $\mathrm{Ob}(D)$ and to have as arrows the pairs (c, d), for c in C, d in D–thus the set $C \times D$ is just the cartesian product of the two sets. Also, if $c : x \to x'$ in C, $d : y \to y'$ in D, then we take in $C \times D$

$$(c, d) : (x, y) \to (x', y').$$

The composition is defined as one would expect by

$$(c', d')(c, d) = (c'c, d'd)$$

whenever $c'c, d'd$ are defined. It is very easy to show that $C \times D$ is a category.

Notice also that if c, d have inverses c^{-1}, d^{-1} then (c, d) has inverse (c^{-1}, d^{-1}). It follows that if C, D are both groupoids then so also is $C \times D$.

Let $p_1 : C \times D \to C, p_2 : C \times D \to D$ be the obvious projection functors. Then we have the universal property: *if* $f : E \to C, g : E \to D$ *are functors then there is a unique functor* $(f, g) : E \to C \times D$ *such that* $p_1(f, g) = f, p_2(f, g) = g$. The proof is easy and is left to the reader. As usual, this property characterises the product up to isomorphism.

Note that this is how product is defined in elementary category theory. So, in an interesting kind of self reference, we use category theory to discuss category theory itself. This is partly because of the dual role of categories and groupoids in mathematics – on the one hand for metamathematical considerations, and on the other as algebraic objects in their own right.

Let $f : C \times D \to E$ be a functor, where C, D, E are categories. If 1_x is the identity at x in C, then let us write $f(x, d)$ for $f(1_x, d)$ where d is any arrow in D. Similarly, let us write $f(c, y)$ for $f(c, 1_y)$

for any object y of D and a arrow c of C. Then, as is easily verified, $f(x, \sim)$ is a functor $D \to E$ (called the x-*section* of f) and $f(\sim, y)$ is a functor $C \to E$ (called the y-*section* of f). These two families of functors determine f. If $c : x \to x', d : y \to y'$ are arrows in C, D respectively then we have a commutative diagram

$$\begin{tikzcd}
f(x,y) \arrow[r, "f(c,y)"] \arrow[d, "f(x,d)"'] \arrow[dr, "f(c,d)"] & f(x',y) \arrow[d, "f(x',d)"] \\
f(x,y') \arrow[r, "f(c,y')"'] & f(x',y')
\end{tikzcd}$$

since $f(1_{x'}c, d1_y) = f(c, d) = f(c1_x, 1_{y'}d)$.

Proposition B.1.1 *Suppose for each x in $\mathrm{Ob}(C)$ and y in $\mathrm{Ob}(D)$ we are given functors*

$$f(x, \sim) : D \to E, f(\sim, y) : C \to E$$

such that $f(x, y)$ is a unique object of E. Suppose for each $c : x \to x'$ in C and $d : y \to y'$ in D the outer square of B.1 commutes. Then the diagonal composite $f(c, d)$ makes f a functor $C \times D \to E$. All functors $C \times D \to E$ arise in this way.

Proof The verification of the preservation of the identity for f is easy since

$$\begin{aligned}
f(1_x, 1_y) &= f(1_x, y)f(x, 1_y) \\
&= 1_{f(x,y)}1_{f(x,y)} \\
&= 1_{f(x,y)}.
\end{aligned}$$

The verification of the composition rule involves a diagram of four commutative squares:

$$\begin{tikzcd}
f(x,y) \arrow[r, "f(c,y)"] \arrow[d, "f(x,d)"'] \arrow[dr, "f(c,d)"] & f(x',y) \arrow[r, "f(c',y)"] \arrow[d, "f(x',d)"] & f(x'',y) \arrow[d, "f(x'',d)"] \\
f(x,y') \arrow[r, "f(c',y')"] \arrow[d, "f(x,d')"'] & f(x',y') \arrow[r, "f(c',y')"] \arrow[d, "f(x',d')"'] \arrow[dr, "f(c',d')"] & f(x'',y') \arrow[d, "f(x'',d')"] \\
f(x,y'') \arrow[r, "f(c,y'')"] & f(x',y'') \arrow[r, "f(c',y'')"] & f(x'',y'')
\end{tikzcd}$$

The last statement is clear from the discussion preceding B.1.1. □

It is clear that if G, H are groupoids, regarded as a special case of categories, then their product $G \times H$ as categories is also a groupoid. This defines the product of groupoids.

B.2 Cartesian closed categories

We have already given in section 9.1 some background to the fundamental notion of an 'exponential law'. Here we will sketch the ideas for one aspect of that, and how the category Cat of small categories comes into this framework with an exponential law of the form of a natural bijection

$$\mathsf{Cat}(C \times D, E) \cong \mathsf{Cat}(C, \mathsf{CAT}(D, E)) \tag{B.2.1}$$

for all small categories C, D, E. The small category $\mathsf{CAT}(D, E)$ has objects the functors $D \to E$ and morphisms the natural transformations.

We will not give a proof of this, but sketch some of the ideas in a way related to previous work.

In section 6.1 we have defined the notion of double category and given the example of the double category $\square E$ of commuting squares in a category E. This double category has two compositions which were there written $+_1, +_2$ and here we will write $\circ_1, \circ_2$. This gives rise to two categories $\square_1 E$, $\square_2 E$ in which the morphisms are the commutative squares in E but in which the compositions are respectively $\circ_1, \circ_2$.

Proposition B.2.1 *The natural transformations of functors* $D \to E$ *are bijective with the elements of* $\mathsf{Cat}(D, \square_2 E)$.

That is, instead of saying that a natural transformation $\phi : F \to G$ assigns to each object d of D a morphism $\phi(d) : F(d) \to G(d)$ in E such that for every morphism $f : d \to d'$ in D a certain square diagram in E commutes, we say that a natural transformation ϕ is a functor $D \to \square_2 E$, and the composition of natural transformations is determined by the composition $\circ_1$ in $\square E$. This approach has been used in [BN79], where it has the advantage of applying to the topological case.

B.3 The internal hom for categories and groupoids

Let us prove that the category of small categories (and that of groupoids) is closed. Thus, for any couple of small categories (groupoids) C, D, we need to construct the small category (groupoid) of internal morphisms from C to D that we are going to denote as $\mathsf{CAT}(C, D)$ ($\mathsf{GPDS}(C, D)$).

The objects of $\mathsf{CAT}(C, D)$ are $\mathsf{Cat}(C, D)$, all functors (morphisms) between the given categories.

Its arrows are all the natural transformations between such functors. Recall that a natural transformation $\phi : f \Rightarrow f'$ between two functors $f, f' : C \to D$ is a family of arrows $\{\phi : f(x) \to f'(x) \mid x \in \mathrm{Ob}(C)\}$ such that for any arrow in C, $c : x \to x'$, the diagram

$$\begin{tikzcd} f(x) \arrow[r, "f(a)"] \arrow[d, "\phi(x)"'] & f(x') \arrow[d, "\phi(x')"] \\ f'(x) \arrow[r, "f'(a)"'] & f'(x') \end{tikzcd}$$

commutes. We are going to denote this diagram as $\phi(a)$.

The source, target and identity of $\mathsf{CAT}(C, D)$ are the obvious one. For any two natural transformations $\phi : f \Rightarrow f'$ and $\phi' : f' \Rightarrow f''$, we define the composition $\phi'\phi$ by $\phi'\phi(x) = \phi'(\phi(x))$. It is clear that this composition completes the structure of category over $\mathsf{CAT}(C, D)$.

It is immediate to see that when C and D are groupoids, any natural transformation $\phi : f \Rightarrow f'$ has inverse ϕ^{-1} defined by $\phi^{-1}(x) = (\phi(x))^{-1}$. Thus $\mathsf{CAT}(C, D)$ is a groupoid that we denote by $\mathsf{GPDS}(C, D)$.

The construction of internal morphisms $\mathsf{CAT}(C, D)$ is natural in C and D. Let us check that it is the adjoint of the cartesian product using essentially the same procedure as in Sets.

Theorem B.3.1 *If* C, D, E *are small categories, there is a natural bijection of sets*

$$\theta : \mathsf{Cat}(C \times D, E) \cong \mathsf{Cat}(C, \mathsf{CAT}(D, E))$$

Proof To define θ, let us start with any functor $f : C \times D \to E$ and we are going to construct the functor $\theta(f) = \hat{f} : C \to \mathsf{CAT}(D, E)$.

For an object $x \in \mathrm{Ob}(C)$, its image is the functor $\hat{f}(x) : D \to E$ given by the x-section of f, i.e. $\hat{f}(x) = f(x, \sim)$

Now, let $c : x \to x'$ be an arrow in C. The natural transformation $\hat{f}(c) : f(x, \sim) \to f(x', \sim)$ is defined by assigning to each object y in D an arrow $\hat{f}(c)(y) = f(c, y)$. It is clear that for any arrow $d : y \to y'$ in D the square

$$\begin{tikzcd} f(x,y) \arrow[r, "f(c,y)"] \arrow[d, "f(x,d)"'] & f(x',y) \arrow[d, "f(x',d)"] \\ f(x,y') \arrow[r, "f(c,y')"'] & f(x',y') \end{tikzcd}$$

commutes.

To prove bijectivity, we construct $\phi = \theta^{-1}$. Thus, for any functor $g : C \to \mathsf{CAT}(D, E)$ we define a functor $\phi(g) = \hat{g} : C \times D \to E$ using B.1.1 by giving its sections $\hat{g}(x, \sim) : D \to E$, and $\hat{g}(\sim, y) : C \to E$, and verifying the commutativity of the appropriate diagram.

For any x object in C, we define the x-section $\hat{g}(x, \sim) = g(x) : D \to E$. Then, on objects $\hat{g}(x, y) = g(x)(y)$.

For any y object in D, the functor $\hat{g}(\sim, y)$ is clear on objects. Let $c : x \to x'$ be an arrow of C. The natural transformation $g(c) : g(x) \Rightarrow g(y)$ is given by $g(c)(y) : g(x, y) \to g(x', y)$. We take $\hat{g}(c, y) = g(c)(y)$.

These sections give a functor $C \to E$ because the commutativity of the square is a direct consequence of naturality. $\square$

Corollary B.3.2 *There is a natural isomorphism of categories*

$$\Theta : \mathsf{CAT}(C \times D, E) \cong \mathsf{CAT}(C, \mathsf{CAT}(D, E))$$

that on objects is θ.

Our interest lays not only in general small categories but mainly in groupoids. It is clear that if G and H are groupoids, the category $\mathsf{CAT}(G, H)$ is also a groupoid that we represent by $\mathsf{GPDS}(G, H)$. The same bijection above proves that this internal morphisms make Gpds a cartesian closed category.

Corollary B.3.3 *If* G, H, K *are groupoids, there is a natural bijection of sets*

$$\mathsf{Gpds}(G \times H, K) \cong \mathsf{Gpds}(G, \mathsf{GPDS}(H, K))$$

and hence a natural isomorphism of groupoids

$$\mathsf{GPDS}(G \times H, K) \cong \mathsf{GPDS}(G, \mathsf{GPDS}(H, K)).$$

The reader will have noticed that since groups are special cases of groupoids, this corollary applies to the case when G, H, K are all groups and then yields a bijection of sets

$$\mathsf{Groups}(G \times H, K) \cong \mathsf{Gpds}(G, \mathsf{GPDS}(H, K))$$

natural with respect to morphisms of G, H, K. Thus to obtain an adjoint to the cartesian product of groups, we have to go outside the category of groups since $\mathsf{GPDS}(H, K)$ has, in general, more than one object. We shall go back to this case in section B.6

The applications of this exponential law confirm again that the sensible approach is to study the algebraic objects which arise in say a given geometric situation, and to examine their uses in order to see how their algebraic properties match up to the formal requirements of the geometric situation. An important aspect of the properties of the algebraic objects is the properties of the category of these objects. As we see above, the category of groups has limitations, in that it is not cartesian closed. On the other hand, the category of groupoids is cartesian closed. We will obtain an application of this in the next section.

The above result is a special case of the result that the category of categories or of groupoids internal to a cartesian closed category is also cartesian closed. (Ehresmann and Ehresmann.)

In order to use the preceding results we have to make some deductions from them and get familiar with the deductions of some standard operations. Some of these arguments work in a general cartesian closed category, but it is important to become familiar with a particular example other than the standard category of sets, in which it is possible to proceed in an *ad hoc* basis.

B.4 The monoid of endomorphisms in the case of groupoids

It is well known that in the case of a cartesian closed category C, for any object E the internal endomorphisms E^E may be given a monoid structure. We are going to study the case of the category of groupoids. For the general case see [Kel36].

As we have seen for any groupoids, G, H, and K there are natural bijections

$$\mathsf{Gpds}(G \times H, K) \cong \mathsf{Gpds}(G, \mathsf{GPDS}(H, K)).$$

In particular, for any groupoids G and H there is a bijection

$$\phi : \mathsf{Gpds}(\mathsf{GPDS}(G, H), \mathsf{GPDS}(G, H)) \to \mathsf{Gpds}(\mathsf{GPDS}(G, H) \times G, H).$$

We are going to study the *evaluation*,

$$\varepsilon_{GH} = \phi(1_{\mathsf{GPDS}(G,H)}) : \mathsf{GPDS}(G, H) \times G \to H$$

i.e.the functor corresponding to the identity $1_{\mathsf{GPDS}(G,H)}$ under the above bijection.

Remark B.4.1 Let us see the action of the evaluation recalling the definition of ϕ. So, to define ε_{GH}, we give its sections.

For any functor $f : G \to H$, the section $\varepsilon_{GH}(f, \sim) : G \to H$ is defined to be f. Then, on objects, we have $\varepsilon_{GH}(f, x) = f(x)$, for any functor $f : G \to H$ and object $x \in G$.

For any object x in G, the section $\varepsilon_{GH}(\sim, x) : \mathsf{GPDS}(G, H) \to H$ is defined on objects as before, and for any natural transformation $\phi : f \Rightarrow f'$, $\varepsilon_{GH}(\sim, x)(\phi) = \phi(x)$.

Then, for any natural transformation $\phi : f \Rightarrow f'$ and arrow $a : x \to y$, $\varepsilon_{GH}(\phi, a)$ is the common composition of the commutative square

$$\begin{tikzcd}
f(x) \arrow[r, "f(a)"] \arrow[d, "\phi(x)"'] \arrow[dr, "\phi(a)"] & f(x') \arrow[d, "\phi(x')"] \\
f'(x) \arrow[r, "f'(a)"'] & f'(x')
\end{tikzcd}$$

that we call $\phi(a)$. notice that for $a = 1_x$, we have $\phi(1_x) = \phi(x)$. $\square$

Using the evaluation maps ε_{GH} we can define the map

$$\alpha : \mathsf{GPDS}(H,K) \times \mathsf{GPDS}(G,H) \times G \xrightarrow{1\times\varepsilon_{GH}} \mathsf{GPDS}(H,K) \times H \xrightarrow{\varepsilon_{HK}} K$$

Now, using the bijection above, we get a functor

$$* = \theta(\alpha) : \mathsf{GPDS}(H,K) \times \mathsf{GPDS}(G,H) \to \mathsf{GPDS}(G,K)$$

that we call the *product* of internal morphisms.

Remark B.4.2 To study the product functor, it is better to have a more explicit description of α. On objects, for any two morphisms of groupoids $g : H \to K$, $f : G \to H$ and an object x in G, we have $\alpha(g, f, x) = g(f(x))$. On morphisms, given two natural transformations $\psi : g \Rightarrow g'$, $\phi : f \Rightarrow f'$ and an arrow $a : x \to x'$, $\alpha(\psi, \phi, a) = \psi\phi(a)$

Now, we construct $\theta(\alpha)$ following B.3.1. Thus, on objects is $(g, f) \to gf$ and on arrows, for any two natural transformations $\psi : g \Rightarrow g'$ and $\phi : f \Rightarrow f'$, $\psi * \phi : gf \Rightarrow g'f'$ is thew natural transformation given by $\psi * \phi(x) = \alpha((\psi, \phi), 1_x) = \psi\phi(1_x) = \psi\phi(x)$, i.e. the common composition of the diagram

$$\begin{tikzcd}
gf(x) \arrow[r, "g\phi(x)"] \arrow[d, "\psi f(x)"'] & gf'(x) \arrow[d, "\psi f'(x)"] \\
g'f(x) \arrow[r, "g'\phi(x)"'] & g'f'(x)
\end{tikzcd}$$

Notice that $\psi * \phi$ may be seen as the common composition $(\psi f')(g\phi) = (g'\phi)(\psi f)$. It is easy to see that the product is natural. □

Thus, for any groupoid G, the set of morphisms of the groupoid $\mathsf{END}(G) = \mathsf{GPDS}(G, G)$ is a monoid with respect to the composition just defined

$$* : \mathsf{END}(G) \times \mathsf{END}(G) \to \mathsf{END}(G).$$

Moreover, the source target and identity are homomorphisms between $\mathsf{END}(G)$ and $\mathsf{End}(G)$. To check that those compositions make $\mathsf{END}(G)$ a monoid on the category of groupoids it remains to prove the following

Proposition B.4.3 *The composition of arrows in* $\mathsf{END}(G)$

$$\mathsf{END}(G) \times_{\mathsf{End}} (G)\mathsf{END}(G) \to \mathsf{END}(G)$$

is a homomorphism with respect to the composition $*$*, i.e. , we have*

$$(\psi'\psi) * (\phi'\phi) = (\psi' * \phi')(\psi * \phi).$$

for any natural transformations $\phi : f \Rightarrow f'$*,* $\phi' : f' \Rightarrow f''$*,* $\psi : g \Rightarrow g'$ *and* $\psi' : g' \Rightarrow g''$*.*

Proof It is direct from the definition and the commutative diagram

$$\begin{tikzcd}
 & & g'f' \arrow[dr, "\psi'f'"] \arrow[drr, "\psi' * \phi'"] & & \\
gf \arrow[urr, "\psi * \phi"] \arrow[r, "\psi f"] \arrow[drr, "\psi'\psi f"'] & g'f \arrow[ur, "g'\phi"] \arrow[dr, "\psi' f"] & & g''f' \arrow[r, "g''\phi'"] & g''f'' \\
 & & g''f \arrow[ur, "g''\phi"] \arrow[urr, "g''\phi'\phi"'] & &
\end{tikzcd}$$

since the composition of the two arrows on the bottom is $(\psi'\psi) * (\phi'\phi)$. □

B.5 The symmetry groupoid and the actor of a groupoid

It can be regarded as a general expectation that the symmetry of an object of type T should in some sense be a 'group object' of type T, and so some kind of higher order structure than T itself.

When looking for a structure reflecting the symmetries of a groupoid G, it is logical to consider all invertible elements of $\mathsf{END}(G)$. Let us call $\mathsf{Aut}(G)$ the subgroup of $\mathsf{End}(G)$ of all automorphisms of the groupoid G and $\mathsf{AUT}(G)$ the full subcategory of $\mathsf{END}(G)$ having $\mathsf{Aut}(G)$ as objects. Clearly $\mathsf{AUT}(G)$ is a submonoid and a subgroupoid. Let us check that it is also a group with respect to $*$. The group-groupoid $\mathsf{AUT}(G)$ is called the *symmetry groupoid* of the groupoid G.

Proposition B.5.1 *The category* $\mathsf{AUT}(G)$ *is a group internal to groupoids*

Proof Let $\phi : f \Rightarrow f'$ be natural transformation from f to f', both being automorphisms of the groupoid G. Then the natural transformation ${f'}^{-1}\phi^{-1}f^{-1} : f^{-1} \Rightarrow {f'}^{-1}$ is the inverse of ϕ with respect to $*$. □

Now, we are going to define an equivalent structure. Let us consider the source map

$$s : \mathsf{AUT}(G) \to \mathsf{Aut}(G).$$

It is a homomorphism and the identity homomorphism is a right inverse. Thus the short exact sequence of groups and homomorphisms

$$1 \to \operatorname{Ker} s \to \mathsf{AUT}(G) \to \mathsf{Aut}(G) \to 1$$

splits, i.e there is a bijection

$$\mathsf{AUT}(G) \cong \mathsf{Aut}(G) \times \operatorname{Ker} s$$

that maps any natural transformation of automorphisms $\phi : f \Rightarrow f'$ to the pair $(f, 1_{f^{-1}} * \phi)$ where the latter is a natural transformation $1 \to f^{-1}f'$, i.e. an element in $\operatorname{Ker} s$.

This bijection is an isomorphism when we endow the cartesian product with appropriate structure. This is the semi-direct product with respect to the action of $\mathsf{Aut}(G)$ on $\operatorname{Ker} s$ on the right given by the identity and conjugation, i.e.

$$\mathsf{AUT}(G) \cong \mathsf{Aut}(G) \ltimes \operatorname{Ker} s$$

where the semidirect product $G \ltimes M$ of a group G and a G-group M is the cartesian product with the product given by $(g, m)(g', m') = (gg', m^{g'}m')$.

Thus, given $\operatorname{Ker} s$ and the action of $\mathsf{Aut}(G)$ on it, the source homomorphism is recovered directly since it is the identity on $\mathsf{Aut}(G)$ and the constant map on $\operatorname{Ker} s$ and the target homomorphism is determined once we know its restriction

$$t| : \operatorname{Ker} s \to \mathsf{Aut}(G)$$

since it is also the identity on $\mathsf{Aut}(G)$. This morphism $t|$is called the *actor* of the groupoid. We shall see that it is an example of crossed module and that it is equivalent to the group-groupoid $\mathsf{AUT}(G)$.

Let us now consider yet another group that is equivalent to the actor. For any groupoid G, we define $\mathbb{M}(G)$ the set of sections $\sigma : \mathrm{Ob}(G) \to G$ of the target map $t : G \to \mathrm{Ob}(G)$, i.e. such section σ to each object $x \in \mathrm{Ob}(G)$ corresponds an arrow $\sigma(x) : s(\sigma(x)) \to x$. Then there is a map

$$\Delta : \mathbb{M}(G) \to \mathsf{END}(G)$$

such that to any section $\sigma : \mathrm{Ob}(G) \to G$ maps the natural transformation $\Delta(\sigma) : \delta(\sigma) \to 1$ where $\delta(\sigma)$ is the functor defined on objects by $\delta(\sigma)(x) = s\sigma(x)$ and on arrows by $\delta(\sigma)(g) = \sigma(t(g))^{-1} g \sigma(s(g))$. The natural transformation $\Delta(\sigma)$ is then given by $\sigma(x)$.

It is clear from the definition that Δ is a bijection onto $\mathrm{Ker}\, t$. It is an isomorphism once we give the appropriate definition to the product of sections. For any two sections σ, τ their product $\tau * \sigma$ is defined as the section that for any $x \in \mathrm{Ob}(G)$, $\tau * \sigma(x)$ is the composition

$$s\tau(s\sigma(x))\mathrm{llabto2}\tau s\sigma(x) s\sigma(x) \xrightarrow{\sigma(x)} x.$$

It is not difficult to prove that Δ is a homomorphism with this product. Let us consider the restriction to the group of units $\mathbb{M}^*(G)$

Proposition B.5.2 *For any section $\sigma \in \mathbb{M}(G)$, the following are equivalent:*

1. *σ is a unit;*
2. *$\Delta(\sigma)$ is bijective on objects;*
3. *$\Delta(\sigma)$ is bijective on arrows;*
4. *$\Delta(\sigma)$ is an automorphism.*

Proof □

Thus, the restriction gives an isomorphism $\Delta : \mathbb{M}^*(G) \cong \mathrm{Ker}\, t \subseteq \mathsf{AUT}(G)$. Using this isomorphism, the map s becomes $\delta : \mathbb{M}^*(G) \to \mathsf{Aut}(G)$ and the action of $\mathsf{Aut}(G)$ on $\mathsf{AUT}(G)$ induces an action on $\mathbb{M}^*(G)$ given by $\sigma^f(x) = f^{-1}\sigma f(x)$. This produces another possible interpretation of the actor of a groupoid.

B.6 The case of a group

As we have seen, a group G may be regarded as a category, that is denoted also by G, with one object $*_G$ and G as set of arrows. The composition of arrows is given by the product in G. This gives a full embedding of categories

$$\mathsf{Groups} \hookrightarrow \mathsf{CAT}.$$

that is full and preserves products.

Thus the internal structure of Groups, if it has one, should correspond to that of CAT, i.e. the internal morphism between two groups G, H should be $\mathsf{CAT}(G, H) = \mathsf{GPDS}(G, H)$. We know that it is a groupoid. We shall see that, in general, this groupoid has more than one object.

The set of objects of $\mathsf{GPDS}(G, H)$ is $\mathsf{Gpds}(G, H) = \mathrm{Hom}(G, H)$, i.e. the set of homomorphisms between the two groupoids. Clearly this set has many elements in general, thus $\mathsf{GPDS}(G, H)$ lies outside the category of groups and the category of groups is not closed.

Let us see a characterisation of the arrows of $\mathsf{GPDS}(G, H)$. Recall than an arrow of Groups is just a group homomorphism $f : G \to H$. A natural transformation $\phi : f \Rightarrow f'$ is given by a unique arrow $\phi(*_G) = y \in H$ corresponding to the object $*_G$, such that for any $x \in G$ satisfies the naturality condition, i.e. the diagram

$$\begin{tikzcd} *_H \arrow[r, "f(x)"] \arrow[d, "y"'] & *_H \arrow[d, "y"] \\ *_H \arrow[r, "f'(x))"'] & *_H \end{tikzcd}$$

commutes, giving $f'(x)y = yf(x)$ for all $x \in G$. Thus f' may be recovered from f and y since $f'(x) = yf(x)y^{-1}$ for all $x \in G$. We write $y : f \Rightarrow f'$.

Let us see what is the evaluation and composition maps in this case.

Following the remark B.4.1 the evaluation map $\varepsilon_{GH} : \mathsf{GPDS}(G, H) \times G \to H$ may be easily described in the case of groups. Since both G and H have a unique object, the functor ε_{GH} is trivial on objects. To describe the action on arrows, we use the above characterisation of the elements of $\mathsf{GPDS}(G, H)$ as elements of H. Thus for any $y : f \Rightarrow f' : G \to H$ and any $x \in G$, we define $\varepsilon_{GH}(y, x) \in H$ as the common product $\varepsilon_{GH}(y, x) = yf(x) = f'(x)y$.

Following the remark B.4.2, the composition $\mathsf{GPDS}(H, K) \times \mathsf{GPDS}(G, H) \to \mathsf{GPDS}(G, K)$ for groups G, H, K can be easily described. It is direct that on objects is just the composition of homomorphisms. Let us study the arrows using the same characterisation as before. Let us consider $y : f \Rightarrow f' : G \to H$ and $z : g \Rightarrow g' : H \to K$, its composite $z * y : gf \Rightarrow g'f' : G \to K$ is the common product $z * y = zg(y) = g'(y)z$.

Thus $\mathsf{END}(G) = \mathsf{GPDS}(G, G)$ is a monoid with the product just described.

Let us study the *symmetry groupoid* of the group G. As before, $\mathsf{AUT}(G)$ is the full subcategory of $\mathsf{END}(G)$ having as set of objects $\mathsf{Aut}(G)$ the group of all automorphisms of the group G. Its elements are $x : f \Rightarrow f'$ where $x \in G$, $f, f' \in \mathsf{Aut}(G)$ and $f'(x') = xf(x')x^{-1}$ for all $x' \in G$, i.e. f' is the 'left conjugate of f by x'. As seen, it is a groupoid and a group with respect to $*$. In this case the inverse with respect to $*$ of an element $x : f \Rightarrow f'$ may be easily computed to be $f'^{-1}(x{-}1) : f^{-1} \Rightarrow f'^{-1}$.

Now, let us consider $\operatorname{Ker} s$, the kernel of the source map. Its elements are $x : 1 \Rightarrow f$, where f is left conjugation by x, i.e. $f(x') = xxx^{-1}$ for all $x' \in G$. The $*$ product in this subgroup is $x * x' = xx'$, thus $\operatorname{Ker} s$ is naturally isomorphic to G.

The action of $\mathsf{Aut}(G)$ on $\operatorname{Ker} s$ by the identity and conjugation, in this case is $x^{f'} = f'^{-1}(x)$ for any natural transformation $x : 1 \Rightarrow f$ and automorphism f'. Notice that $x^{f'} : 1 \Rightarrow f'^{-1}ff'$. With this action

$$\mathsf{AUT}(G) \cong \mathsf{Aut}(G) \ltimes \operatorname{Ker} s \cong \mathsf{Aut}(G) \ltimes G.$$

B.7 Crossed modules and quotients of groups

We start with some very basic facts on group theory.

Let N be the kernel of a homomorphism $f : G \to H$ of groups. Then N is a normal subgroup of G. This is equivalent to saying that the group G acts on the group N by conjugation in G. This is why a normal subgroup is a special case of a crossed module. We can put the emphasis slightly differently by saying that the kernel of a homomorphism of groups is a *group with action*, and in fact a special case of a crossed module.

Now a normal subgroup is closely associated with the notion of *quotient group*. The notion of quotient structure is very important in mathematics and science since it is closely associated with the idea of *classification*. In looking at insect in a rain forest, say, we do not try to list all insects, but we do try to list as many species as we can find. Similarly, in mathematics, we often want to consider sets of elements as objects in themselves, for example lines are considered as sets of points in a plane. The basic tool for this is the standard notion of equivalence relation R on a set X and the associated set X/R of equivalence classes.

In order to fit the notion of equivalence relation into the notion of quotient groups, it is convenient to use the fact that a subgroup N of a group G determines an equivalence relation $\sim_N$ on G by the rule $g \sim_N g'$ if and only if $Ng = Ng'$, for $(g, g') \in G \times G$. In general this subset $\sim_N$ of $G \times G$ is not a subgroup of $G \times G$, where the latter has its usual group structure (for example, considered as a product of categories).

Proposition B.7.1 *The equivalence relation* $\sim_N$ *is a subgroup of* $G \times G$ *if and only if the subgroup* N *is normal in* G. □

We omit the proof since this is exactly the kind of result you have to verify for yourself.

It is usual to call an equivalence relation on G which is a subgroup of $G \times G$ a *congruence* on the group G.

It was quite early observed that an equivalence relation R on a set X is a special case of a groupoid with object set X, in which the set of arrows is R and $R(x, y)$ consists of the set $\{(y, x)\}$ with multiplication $(z, y)(y, x) = (z, x)$. That is, in thinking about a groupoid H, we realise that H defines an equivalence relation on $\mathrm{Ob}(H)$ whose classes are the connected components of H. For this equivalence relation the elements of $H(x, y)$ could be thought of 'reasons why' x is equivalent to y, or as 'proofs that' x is equivalent to y. This analogy leads naturally to the consideration of higher dimensional theories, such as 'proofs of proofs', and so on. The relations of this idea with homotopy theory is steadily becoming more apparent. From this basic approach, the utility of notions of higher dimensional groupoids also becomes clear.

Thus it is natural to consider the generalisation of a congruence on a group G to some kind of groupoid on the set G. Part of the reason is that the notion of presentation of an equivalence relation is not well defined. However the notion of presentation of a groupoid (and more generally of a group-groupoid) is well defined, and so to use analogues of combinatorial group theory for equivalence relations it is convenient to widen the scope of combinatorial group theory to combinatorial groupoid theory. This also allows the discussion of presentations of group actions, by considering the corresponding covering groupoids.

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